Nhiều tác giả

TUYỂN TẬP CÁC PHƯƠNG PHÁP , KỸ THUẬT

CHỨNG MINH

BẤT ĐẲNG THỨC

Tập ba : Các tuyển tập của tác giả nước ngoài

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(a+c)^2}\right)\geqslant\frac{9}{4}$$

Hà Nội, tháng 2 năm 2015

Tuyển tập các phương pháp, kĩ thuật chứng minh

Bất Đẳng Thức

Tập ba : Các tuyển tập của tác giả nước ngoài

Lời nói đầu

Nguồn tài nguyên toán trên Internet là vô cùng phong phú. Tài liệu về Bất đẳng thức trên Internet rất nhiều và nhiều chuyên đề trong số chúng là những công cụ mạnh để giải bất đẳng thức. Việc tập hợp chúng lại thành một ấn bản lớn để tiện nghiên cứu âu có lẽ cũng là nhu cầu của nhiều người. Qua một thời gian sưu tầm và chọn lọc các tài liệu theo một vài "tiêu chí", ấn bản lớn "Tuyển tập các chuyên đề, kỹ thuật chứng minh Bất đẳng thức " đã hoàn thành. Vì dung lượng quá lớn (khoảng trên 2000 trang) thế nên ấn bản được chia làm 3 tập. Để cho các bài viết được thống nhất theo một khối chung, tôi buộc phải can thiệp, chỉnh sửa một chút tài liệu gốc, rất mong sự bỏ qua của các tác giả tài liệu trên. Một số phương pháp kinh điển như MV, GLA, ABC, UCT cũng sẽ không xuất hiện trong ấn bản này, độc giả hãy lượng thứ cho điều đó. Hi vọng ấn bản trên là một tập hợp tương đối đầy đủ về Bất đẳng thức, một lĩnh vực luôn có sự quyến rũ, cuốn hút đến không ngờ.

Mọi ý kiến đóng góp xin gửi về

Nguyễn Minh Tuấn
K62CLC Toán- Tin ĐHSPHN
Gmail : popeyenguyen94@gmail.com
Facebook : Popeye Nguyễn

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NGUYỄN MINH TUẤN (POPEYE)

Mục Lục

T. Andreescu, V. Cartoaje, G. Dospinescu, M. Lascu

Biên dịch: Dương Việt Thông

Bất Đẳng Xưa và Nay

Mục lục

Lời nói đầu

Quyển sách kết hợp những kết quả kinh điển về bất đẳng thức với những bài toán rất mới, một số bài toán được nêu chỉ vài ngày trước đây. Làm sao có thể viết được điều gì đặc biệt khi đã có quá nhiều sách về bất đẳng thức? Chúng tôi tin chắc rằng dù đề tài này rất tổng quát và thông dụng, quyển sách của chúng tôi vẫn rất khác biệt. Tất nhiên nói thì rất dễ, vậy chúng tôi nêu vài lý lẽ minh chứng. Quyển sách chứa một số lớn bài toán về bất đẳng thức, phần lớn là khó, các câu hỏi nổi tiếng trong các cuộc thi tài vì độ khó và vẻ đẹp của chúng. Và quan trọng hơn, trong cuốn sách chúng tôi đã sử dụng những lời giải của chính mình và đề xuất một số lớn bài toán độc đáo mới. Trong quyển sách có những bài toán đáng nhớ và cả những lời giải đáng nhớ. Vì thế quyển sách thích hợp với những sinh viên sử dụng thành thạo bất đẳng thức Cauchy-Schwarz và muốn cải tiến kỹ thuật và kỹ năng đại số của mình. Họ sẽ tìm thấy ở đây những bài toán khá hóc búa, những kết quả mới và cả những vấn đề có thể nghiên cứu tiếp. Các sinh viên chưa say mê trong lĩnh vực này có thể tìm được một số lớn bài toán, ý tưởng, kỹ thuật loại vừa và dễ để chuẩn bị tốt cho các kỳ thi toán. Một số bài toán chúng tôi chọn là đã biết nhưng chúng tôi đưa ra những lời giải mới để chứng tỏ sự đa dạng của những ý tưởng liên quan đến bất đẳng thức. Bất kỳ ai cũng tìm thấy ở đây việc thử thách cho những kỹ năng của mình. Nếu chúng tôi chưa thuyết phục nổi bạn, xin hãy xem những bài toán cuối cùng và hy vọng bạn sẽ đồng ý với chúng tôi.

Cuối cùng nhưng không kết thúc, chúng tôi tỏ lòng biết ơn sâu sắc những người đặt ra các bài toán có trong quyển sách này và xin lỗi vì không đưa ra đầy đủ xuất xứ dù chúng tôi đã cố gắng hết sức. Chúng tôi cũng xin cảm ơn Marian Tetiva, Dung Tran Nam, Constantin Tănăsescu, Călin Popa và Valentin Vornicu về những bài toán đẹp mà họ nêu ra cùng những bình luận quý giá, cảm ơn Cristian Babă, George Lascu và Călin Popa về việc đánh máy bản thảo và nhiều nhận xét xác đáng của họ.

Các tác giả

Chương 1. Các bài toán

1. Chứng minh rằng bất đẳng thức

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq \frac{3\sqrt{2}}{2}.$$

đúng với các số thực a, b, c bất kỳ.

Kömal

2. [Dinu Serbănescu] Cho $a, b, c \in (0, 1)$, chứng minh rằng

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<1.$$

Junior TST 2002, Romania

3. [Mircea Lascu] Cho a, b, c là các số thực dương thỏa mãn $abc = 1$. Chứng minh rằng

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq \sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Gazeta Matematiă

4. Nếu phương trình $x^4+ax^3+2x^2+bx+1 = 0$ có ít nhất một nghiệm thực, thì $a^2+b^2 \geq 8$.

Tournament of the Towns, 1993

5. Tìm giá trị lớn nhất của biểu thức $x^3 + y^3 + z^3 - 3xyz$ với $x^2 + y^2 + z^2 = 1$ và x, y, z là các số thực.

6. Cho a, b, c, x, y, z là các số thực dương thỏa mãn $x + y + z = 1$. Chứng minh rằng

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)}\leq a+b+c.$$

Ukraine, 2001

7. [Darij Grinberg] Nếu a, b, c là các số thực dương, thì

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq \frac{9}{4(a+b+c)}.$$

8. [Hojoo Lee] Cho $a, b, c \geq 0$. Chứng minh rằng

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\geq$$
$$\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ac}+c\sqrt{2c^2+ab}.$$

Gazeta Matematiă

9. Cho a, b, c là các số thực dương thỏa mãn $abc = 2$, khi đó

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Khi nào đẳng thức xảy ra?

JBMO 2002 Shorlist

10. [Ioan Tomescu] Cho $x, y, x > 0$. Chứng minh rằng

$$\frac{xyz}{(1+3x)(x+8y)(y+9z)(z+6)} \leq \frac{1}{7^4}.$$

Khi nào ta có đẳng thức?

Gazeta Matematiă

11. [Mihai Piticari, Dan Popescu] Chứng minh rằng

$$5(a^2 + b^2 + c^2 \leq 6(a^3 + b^3 + c^3) + 1,$$

với mọi $a, b, c > 0$ và $a + b + c = 1$.

12. [Mircea Lascu] Cho $x_1, x_2, ..., x_n \in \mathbb{R}, n \geq 2$ và $a > 0$ thỏa mãn

$$x_1 + x_2 + ... + x_n = a \text{ và } \ x^2{}_1 + x^2{}_2 + ... + x^2{}_n \leq \frac{a^2}{n-1}.$$

Chứng minh rằng $x_i \in [0, \dfrac{2a}{n}]$, với mọi $i \in \{1, 2, ..., n\}$.

13. [Adrian Zahariuc] Chứng minh rằng với mọi $a, b, c \in (1, 2)$ bất đẳng thức sau đây đúng

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq 1.$$

14. Cho các số thực dương a, b, c thỏa mãn $abc = 1$, chứng minh rằng

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.$$

15. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c, x, y, z là các số thực sao cho $a+x \geq b+y \geq c+z$ và $a+b+c = x+y+z$. Chứng minh rằng

$$ay + bx \geq ac + xz.$$

16. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c là các số thực thỏa mãn $abc = 1$. Chứng minh rằng

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+ac+bc}.$$

Junior TST 2003, Romania

17. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

JBMO 2002 Shorlist

18. Chứng minh rằng nếu $n > 3$ và $x_1, x_2, ..., x_n > 0$ thỏa mãn $\prod_{i=1}^{n} x_i = 1$, thì

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + ... + \frac{1}{1+x_{n-1}+x_{n-1}x_n} + \frac{1}{1+x_n+x_nx_1} > 1.$$

Russia, 2004

19. [Marian Tetiva] Cho x, y, z là các số thực dương thỏa mãn điều kiện

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Chứng minh rằng

a) $xyz \leq \frac{1}{8}$;

b) $x + y + z \leq \frac{3}{2}$;

c) $xy + xz + yz \leq \frac{3}{4} \leq x^2 + y^2 + z^2$;

d) $xy + xz + yz \leq \frac{1}{2} + 2xyz$.

20. [Marius Olteanu] Cho $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ thỏa mãn $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Chứng minh rằng

$$|\cos x_1| + |\cos x_2| + |\cos x_3| + |\cos x_4| + |\cos x_5| \geq 1.$$

Gazeta Matematiă

21. [Florina Cârlan, Marian Tetiva] Chứng minh rằng nếu $x, y, z > 0$ thỏa mãn điều kiện $x + y + z = xyz$ thì

$$xy + xz + yz \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}.$$

22. [Laurentiu Panaitopol] Chứng minh rằng

$$\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2,$$

với mọi số thực $x, y, z > -1$.

JBMO, 2003

23. Cho $a, b, c > 0$ thỏa mãn $a + b + c = 1$. Chứng minh rằng

$$\frac{a^2+b}{b+c} + \frac{b^2+c}{c+a} + \frac{c^2+a}{a+b} \geq 2.$$

24. Cho $a, b, c \geq 0$ thỏa mãn $a^4 + b^4 + c^4 \leq 2(a^2b^2 + b^2c^2 + c^2a^2)$. Chứng minh rằng

$$a^2 + b^2 + c^2 \leq 2(ab + bc + ca).$$

Kvant, 1988

25. Cho $n \geq 2$ và $x_1, x_2, ..., x_n$ là các số thực thỏa mãn

$$\frac{1}{x_1+1998} + \frac{1}{x_2+1998} + ... + \frac{1}{x_n+1998} = \frac{1}{1998}.$$

Chứng minh rằng

$$\frac{\sqrt[n]{x_1.x_2...x_n}}{n-1} \geq 1998.$$

Vietnam, 1998

26. [Marian Tetiva] Cho các số thực dương x, y, z thỏa mãn $x^2 + y^2 + z^2 = xyz$. Chứng minh các bất đẳng thức sau

a) $xyz \geq 27$;

b) $xy + xz + yz \geq 27$;

c) $x+y+z \geq 9$;

d) $xy+xz+yz \geq 2(x+y+z)+9$.

27. Cho x, y, z là các số thực dương có tổng bằng 3. Chứng minh rằng

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

Russia, 2002

28. [D.Olteanu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a+b}{b+c}.\frac{a}{2a+b+c}+\frac{b+c}{c+a}.\frac{b}{2b+c+a}+\frac{c+a}{a+b}.\frac{c}{2c+a+b} \geq \frac{3}{4}.$$

Gazeta Matematiă

29. Cho a, b, c là các số thực dương, chứng minh rằng bất đẳng thức sau đúng

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a}.$$

India, 2002

30. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ac+a^2}+\frac{c^3}{a^2-ab+b^2} \geq \frac{3(ab+bc+ca)}{a+b+c}.$$

Đề cử cho kỳ thi Olympic Toán học vùng Balkan

31. [Adrian Zahariuc] Xét các số nguyên $x_1, x_2, ..., x_n, n \geq 0$ đôi một khác nhau. Chứng minh rằng

$$x_1^2+x_2^2+...+x_n^2 \geq x_1.x_2+x_2.x_3+...+x_n.x_1+2n-3.$$

32. [Murray Klamkin] Tìm giá trị lớn nhất của biểu thức

$$x_1^2x_2+x_2^2x_3+...+x_{n-1}^2x_n+x_n^2x_1$$

với $x_1, x_2, ..., x_n \geq 0$ có tổng bằng 1 và $n>2$.

Crux Mathematicorum

33. Tìm giá trị lớn nhất của hằng số c sao cho với mọi $x_1, x_2, ..., x_n, ... > 0$ thỏa mãn $x_{k+1} \geq x_1 + x_2 + ... + x_k$ với mọi k, bất đẳng thức

$$\sqrt{x_n} + \sqrt{x_n} + ... + \sqrt{x_n} \leq c\sqrt{x_1 + x_2 + ... + x_n}$$

đúng với mọi n.

IMO Shorlist, 1986

34. Cho các số thực dương a, b, c và x, y, z thỏa mãn $a + x = b + y = c + z = 1$. Chứng minh rằng

$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \geq 3.$$

Russia, 2002

35. [Viorel Vâjâitu, Alexandru Zaharescu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq \frac{1}{4}(a+b+c).$$

Gazeta Matematiă

36. Tìm giá trị lớn nhất của biểu thức

$$a^3(b+c+d) + b^3(c+d+a) + c^3(d+a+b) + d^3(a+b+c)$$

với a, b, c, d là các số thực mà tổng bình phương của các số bằng 1.

37. [Walther Janous] Cho x, y, z là các số thực dương. Chứng minh rằng

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

Crux Mathematicorum

38. Giả sử $a_1 < a_2 < ... < a_n$ là các số thực, $n \geq 2$ là một số nguyên. Chứng minh rằng

$$a_1{a^4}_2 + a_2{a^4}_3 + ... + a_n{a^4}_1 \geq a_2{a^4}_1 + a_3{a^4}_2 + ... + a_1{a^4}_n.$$

Iran, 1999

39. [Mircea Lascu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

40. Cho $a_1, a_2, ..., a_n > 1$ là các số nguyên dương. Chứng minh rằng ít nhất một trong các số $\sqrt[a_1]{a_2}$, $\sqrt[a_2]{a_3}$, ..., $\sqrt[a_{n-1}]{a_2}$, $\sqrt[a_n]{a_1}$ nhỏ hơn hoặc bằng $\sqrt[3]{3}$.

Phỏng theo một bài toán quen biết

41. [Mircea Lascu, Marian Tetiva] Cho x, y, z là các số thực dương thỏa mãn điều kiện

$$xy + xz + yz + 2xyz = 1.$$

Chứng minh các bất đẳng thức sau

a) $xyz \leq \frac{1}{8}$;

b) $x + y + z \geq \frac{3}{2}$;

c) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 4(x+y+z)$;

d) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} - 4(x+y+z) \geq \frac{(2z-1)^2}{z(2z+1)}$, với $z = \max\{x, y, z\}$.

42. [Manlio Marangelli] Chứng minh rằng với mọi số thực dương x, y, z,

$$3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x+y+z)^3.$$

43. [Gabriel Dospinescu] Chứng minh rằng nếu a, b, c là các số thực sao cho $\max\{a, b, c\} - \min\{a, b, c\} \leq 1$, thì

$$1 + a^3 + b^3 + c^3 + 6abc \geq 3a^2b + 3b^2c + 3c^2a.$$

44. [Gabriel Dospinescu] Chứng minh rằng với bất kỳ số thực dương a, b, c ta có

$$27 + \left(2+\frac{a^2}{bc}\right)\left(2+\frac{b^2}{ca}\right)\left(2+\frac{c^2}{ab}\right) \geq 6(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

45. Cho $a_0 = \frac{1}{2}$ và $a_{k+1} = a_k + \frac{a_k^2}{n}$. Chứng minh rằng $1 - \frac{1}{n} < a_n < 1$.

TST Singapore

46. [Călin Popa] Cho a, b, c là các số thực dương, với $a, b, c \in (0, 1)$ sao cho $ab+bc+ca = 1$. Chứng minh rằng

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c}\right).$$

47. [Titu Andreescu, Gabriel Dospinescu] Cho $x, y, z \leq 1$ và $x + y + z = 1$. Chứng minh rằng

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

48. [Gabriel Dospinescu] Chứng minh rằng nếu $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, thì

$$(1-x)^2(1-y)^2(1-z)^2 \geq 2^{15}xyz(x+y)(y+z)(z+x).$$

49. Cho x, y, z là các số thực dương thỏa mãn $xyz = x + y + z + 2$. Chứng minh rằng

1) $xy + yz + zx \geq 2(x + y + z)$;

2) $\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz}$.

50. Chứng minh rằng nếu x, y, z là các số thực sao cho $x^2 + y^2 + z^2 = 2$, thì

$$x + y + z \leq xyz + 2.$$

IMO Shorlist, 1987

51. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng với mọi $x_1, x_2, ..., x_n \in (0, 1)$ và với mọi phép hoán vị σ của tập $\{1, 2, ..., n\}$, ta có bất đẳng thức

$$\sum_{i=1}^{n} \frac{1}{1-x_i} \geq \left(1 + \frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1-x_i.x_{\sigma(i)}}\right).$$

52. Cho $x_1, x_2, ..., x_n$ là các số thực dương sao cho $\sum\limits_{i=1}^{n} \frac{1}{1+x_i} = 1$. Chứng minh rằng

$$\sum_{i=1}^{n} \sqrt{x_i} \geq (n-1)\sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}.$$

Vojtech Jarnik

53. [Titu Andreescu] Cho $n > 3$ và $a_1, a_2, ..., a_n$ là các số thực thỏa mãn $a_1+a_2+...+a_n \geq n$ và $a_1^2 + a_2^2 + ... + a_n^2 \geq n^2$. Chứng rằng $\max\{a_1, a_2, ..., a_n\} \geq 2$.

USAMO, 1999

54. [Vasile Cirtoaje] Nếu a, b, c, d là các số thực dương, thì

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

55. Cho x và y là các số thực dương, chỉ ra rằng $x^y + y^x > 1$.

France, 1996

56. Chứng minh rằng nếu $a, b, c > 0$ có tích bằng 1, thì

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1).$$

MOSP, 2001

57. Chứng minh rằng với mọi $a, b, c > 0$,

$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \leq abc(ab+bc+ca).$$

58. [D.P.Mavlo] Cho $a, b, c > 0$. Chứng minh rằng

$$3 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{1+abc}.$$

Kvant, 1988

59. [Gabriel Dospinescu] Chứng minh rằng với mọi số thực dương $x_1, x_2, ..., x_n$ với tích bằng 1 ta có bất đẳng thức

$$n^n.\prod_{i=1}^{n}(x_i^n+1) \geq \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i}\right)^n.$$

60. Cho $a, b, c, d > 0$ thỏa mãn $a + b + c = 1$. Chứng minh rằng

$$a^3 + b^3 + c^3 + abcd \geq \min\left\{\frac{1}{4}, \frac{1}{9} + \frac{d}{27}\right\}.$$

Kvant, 1993

61. Chứng minh rằng với mọi số thực a, b, c ta có bất đẳng thức

$$\sum(1+a^2)^2(1+b^2)^2(a-c)^2(b-c)^2 \geq (1+a^2)(1+b^2)(1+c^2)(a-b)^2(b-c^2)(c-a^2).$$

AMM

62. [Titu Andreescu, Mircea Lascu] Cho α, x, y, z là các số thực dương thỏa mãn $xyz = 1$ và $\alpha \geq 1$. Chứng minh rằng

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

63. Chứng minh rằng với mọi số thực $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$ thỏa mãn $x_1^2+x_2^2+...+x_n^2 = y_1^2 + y_2^2 + ... + y_n^2 = 1$, ta có

$$(x_1y_2 - x_2y_1)^2 \leq 2\left(1 - \sum_{k=1}^{n} x_k y_k\right).$$

Korea, 2001

64. [Laurentiu Panaitopol] Cho $a_1, a_2, ..., a_n$ là các số nguyên dương đôi một khác nhau. Chứng minh rằng

$$a_1^2 + a_2^2 + ... + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + ... + a_n).$$

TST Romania

65. [Cawlin Popa] Cho a, b, c là các số thực dương thỏa mãn $a + b + c = 1$. Chứng minh rằng

$$\frac{b\sqrt{c}}{a(\sqrt{3c}+\sqrt{ab})} + \frac{c\sqrt{a}}{b(\sqrt{3a}+\sqrt{bc})} + \frac{a\sqrt{b}}{c(\sqrt{3b}+\sqrt{ca})} \geq \frac{3\sqrt{3}}{4}.$$

66. [Titu Andreescu, Gabriel Dospinescu] Cho a, b, c là các số thực thỏa mãn

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) = 16.$$

Chứng minh rằng

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

67. Chứng minh rằng

$$(a^2+2)(b^2+2)(c^2+2) \geq 2(ab+bc+ca)$$

với mọi a, b, c là các số thực dương.

APMO, 2004

68. [Vasile Cirtoaje] Chứng minh rằng nếu $0 < x \leq y \leq z$ và $x + y + z = xyz + 2$, thì

a) $(1 - xy)(1 - yz)(1 - xz) \geq 0$;

b) $x^2y \leq 1, x^3y^2 \leq \dfrac{32}{27}$.

69. [Titu Andreescu] Cho a, b, c là các số thực dương thỏa mãn $a + b + c \geq abc$. Chứng minh rằng có ít nhất hai trong các bất đẳng thức

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6,$$

đúng.

TST 2001, USA

70. [Gabriel Dospinescu, Marian Tetiva] Cho $x, y, z > 0$ thỏa mãn

$$x + y + z = xyz.$$

Chứng minh rằng

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

71. [Marian Tetiva] Chứng minh rằng với bất kỳ các số thực dương a, b, c ta có

$$\left|\frac{a^3 - b^3}{a + b} + \frac{b^3 - c^3}{b + c} + \frac{c^3 - a^3}{c + a}\right| \leq \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{4}.$$

Moldova TST, 2004

72. [Titu Andreescu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

USAMO, 2004

73. [Gabriel Dospinescu] Cho $n > 2$ và $x_1, x_2, ..., x_n > 0$ thỏa mãn

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = n^2 + 1.$$

Chứng minh rằng

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{x_k^2}\right) > n^2 + 4 + \frac{2}{n(n - 1)}.$$

74. [Gabriel Dospinescu, Mircea Lascu, Marian Tetiva] Chứng minh rằng với bất kỳ các số thực dương a, b, c, ta có

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1 + a)(1 + b)(1 + c).$$

75. [Titu Andreescu, Zuming Feng] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + a + c)^2}{2b^2 + (a + c)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

USAMO, 2003

76. Chứng minh rằng với mọi số thực dương x, y và với mọi số nguyên dương m, n, ta có

$$(n - 1)(m - 1)(x^{m+n} + y^{m+n}) + (m + n - 1)(x^m y^n + x^n y^m) \geq mn(x^{m+n-1} y + y^{m+n-1} x).$$

Austrian-Polish Competition, 1995

77. Cho a, b, c, d là các số thực dương thỏa mãn $abcd = 1$. Chứng minh rằng

$$\frac{a + abc}{1 + ab + abcd} + \frac{b + bcd}{1 + bc + bcde} + \frac{c + cde}{1 + cd + cdea} + \frac{d + dea}{1 + de + deab} + \frac{e + eab}{1 + ea + eabc} \geq \frac{10}{3}.$$

Crux Mathematicorum

78. [Titu Andreescu] Chứng minh rằng với mọi $a, b, c \in (0, \frac{\pi}{2})$ bất đẳng thức sau đúng

$$\frac{\sin a.\sin(a - b).\sin(a - c)}{\sin(b + c)} + \frac{\sin b.\sin(b - c).\sin(b - a)}{\sin(c + a)} + \frac{\sin c.\sin(c - a).\sin(c - b)}{\sin(a + b)} \geq 0.$$

TST 2003, USA

79. Chứng minh rằng nếu a, b, c là các số thực dương, thì

$$\sqrt{a^4 + b^4 + c^4} + \sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq \sqrt{a^3b + b^3c + c^3a} + \sqrt{ab^3 + bc^3 + ca^3}.$$

KMO Summer Program Test, 2001

80. [Gabriel Dospinescu, Mircea Lascu] Cho $n > 2$, tìm hằng số k_n nhỏ nhất có tính chất: Nếu $a_1, a_2, ..., a_n > 0$ có tích bằng 1, thì

$$\frac{a_1a_2}{(a_1^2 + a_2)(a_2^2 + a_1)} + \frac{a_2a_3}{(a_2^2 + a_3)(a_3^2 + a_2)} + ... + \frac{a_na_1}{(a_n^2 + a_1)(a_1^2 + a_n)} \leq k_n.$$

81. [Vasile Cirtoaje] Với mọi số thực a, b, c, x, y, z, chứng minh rằng bất đẳng thức sau đúng

$$ax + by + cz + \sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \geq \frac{2}{3}(a+b+c)(x+y+z).$$

Kvant, 1989

82. [Vasile Cirtoaje] Chứng minh rằng các cạnh a, b, c của một tam giác thỏa mãn bất đẳng thức

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

83. [Walther Janous] Cho $n > 2$ và $x_1, x_2, ..., x_n > 0$ có tổng bằng 1. Chứng minh rằng

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(\frac{n-x_i}{1-x_i}\right).$$

Crux Mathematicorum

84. [Vasile Cirtoaje, Gheorghe Eckstein] Xét các số thực dương $x_1, x_2, ..., x_n$ thỏa mãn $x_1x_2...x_n = 1$. Chứng minh rằng

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+...+\frac{1}{n-1+x_n} \leq 1.$$

TST 1999, Romania

85. [Titu Andreescu] Chứng minh rằng với mọi số thực không âm a, b, c thỏa mãn $a^2+b^2+c^2+abc = 4$, ta có

$$0 \leq ab+bc+ca-abc \leq 2.$$

USAMO, 2001

86. [Titu Andreescu] Chứng minh rằng với mọi số thực dương a, b, c, bất đẳng thức sau đúng

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\left\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\right\}.$$

TST 2000, USA

87. [Kiran Kedlaya] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a.\frac{a+b}{2}.\frac{a+b+c}{3}}.$$

88. Tìm hằng số k lớn nhất sao cho với mọi số nguyên dương n không là một số chính phương, ta có

$$\left|(1+\sqrt{n})\sin(\pi\sqrt{n})\right| > k.$$

Vietnamese IMO Training Camp, 1995

89. [Tran Nam Dung] Cho $x, y, z > 0$ thỏa mãn $(x+y+z)^3 = 32xyz$. Tìm giá trị nhỏ nhất và giá trị lớn nhất của biểu thức

$$\frac{x^4+y^4+z^4}{(x+y+z)^4}.$$

Vietnam, 2004

90. [George Tsintifa] Chứng minh rằng với mọi $a, b, c, d > 0$,

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

Crux Mathematicorum

91. [Titu Andreescu, Gabriel Dospinescu] Tìm giá trị lớn nhất của biểu thức

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

với a, b, c là các số thực không âm có tổng bằng 1 và n là một số nguyên dương.

92. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})}.$$

93. [Tran Nam Dung] Chứng minh rằng với mọi số thực a, b, c thỏa mãn $a^2+b^2+c^2 = 9$, ta có

$$2(a+b+c) - abc \leq 10.$$

Vietnam, 2002

94. [Vasile Cirtoaje] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)+\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)+\left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right) \geq 3.$$

95. [Gabriel Dospinescu] Cho n là số nguyên lớn hơn 2. Tìm số thực m_n lớn nhất và số thực M_n nhỏ nhất sao cho với mọi số thực dương $x_1, x_2, ..., x_n$ (với $x_n = x_0, x_{n+1} = x_1$), ta có

$$m_n \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \leq M_n.$$

96. [Vasile Cirtoaje] Nếu x, y, z là các số thực dương, thì

$$\frac{1}{x^2 + xy + y^2} + \frac{1}{y^2 + yz + z^2} + \frac{1}{z^2 + zx + x^2} \geq \frac{9}{(x + y + z)^2}.$$

Gazeta Matematiă

97. [Vasile Cirtoaje] Với mọi $a, b, c, d > 0$, chứng minh rằng

$$2(a^3 + 1)(b^3 + 1)(c^3 + 1)(d^3 + 1) \geq (1 + abcd)(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2).$$

Gazeta Matematiă

98. Chứng minh rằng với mọi số thực a, b, c, ta có

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \geq \frac{4}{7}(a^4 + b^4 + c^4).$$

Vietnam TST, 1996

99. Chứng minh rằng nếu a, b, c là các số thực dương thỏa mãn $abc = 1$, thì

$$\frac{1}{1 + a + b} + \frac{1}{1 + b + c} + \frac{1}{1 + c + a} \leq \frac{1}{2 + a} + \frac{1}{2 + b} + \frac{1}{2 + c}.$$

Bulgaria, 1997

100. [Tran Nam Dung] Tìm giá trị nhỏ nhất của biểu thức

$$\frac{1}{a} + \frac{2}{b} + \frac{3}{c}$$

trong đó a, b, c là các số thực dương thỏa mãn $21ab + 2bc + 8ca \leq 12$.

Vietnam, 2001

101. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng với mọi $x, y, z, a, b, c > 0$ thỏa mãn $xy + yz + zx = 3$, ta có

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq 3.$$

102. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(a+c-b)^2}{(a+c)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Japan, 1997

103. [Vasile Cirtoaje, Gabriel Dospinescu] Chứng minh rằng nếu $a_1, a_2, ..., a_n \geq 0$ thì

$$a_1^n + a_2^n + ... + a_n^n - na_1a_2...a_n \geq (n-1)\left(\frac{a_1 + a_2 + ... + a_{n-1}}{n-1} - a_n\right)^n$$

trong đó a_n là số nhỏ nhất trong các số $a_1, a_2, ..., a_n$.

104. [Turkevici] Chứng minh rằng với mọi số thực dương x, y, z, t, ta có

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$$

Kvant

105. Chứng minh rằng với mọi số thực $a_1, a_2, ..., a_n$, bất đẳng thức sau đúng

$$\left.\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j.$$

106. Chứng minh rằng nếu $a_1, a_2, ..., a_n, b_1, ..., b_n$ là các số thực nằm giữa 1001 và 2002, thỏa mãn

$$a_1^2 + a_2^2 + ... + a_n^2 = b_1^2 + b_2^2 + ... + b_n^2,$$

thì ta có bất đẳng thức

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + ... + \frac{a_n^3}{b_n} \leq \frac{17}{10}(a_1^2 + a_2^2 + ... + a_n^2).$$

TST Singapore

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107. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng nếu a, b, c là các số thực dương có tổng bằng 1, thì

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8(a^2b^2 + b^2c^2 + c^2a^2)^2.$$

108. [Vasile Cirtoaje] Nếu a, b, c, d là các số thực dương thỏa mãn $abcd = 1$, thì

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

Gazeta Matematiă

109. [Vasile Cirtoaje] Cho a, b, c là các sô thực dương. Chứng minh rằng

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^+b^2} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Gazeta Matematiă

110. [Gabriel Dospinescu] Cho $a_1, a_2, ..., a_n$ là các số thực và S là một tập con khác rỗng của $\{1, 2, ..., n\}$. Chứng minh rằng

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + ... + a_j)^2.$$

TST 2004, Romania

111. [Dung Tran Nam] Cho $x_1, x_2, ..., x_{2004}$ là các số thực trong đoạn $[-1, 1]$ thỏa mãn $x_1^3 + x_2^3 + ... + x_{2004}^3 = 0$. Tìm giá trị lớn nhất của $x_1 + x_2 ... + x_{2004}$.

112. [Gabriel Dospinescu, Călin Popa] Chứng minh rằng nếu $n \geq 2$ và $a_1, a_2, ..., a_n$ là các số thực có tích bằng 1, thì

$$a_1^2 + a_2^2 + ... + a_n^2 - n \geq \frac{2n}{n-1}.\sqrt[n]{n-1}(a_1 + a_2 + ... + a_n - n).$$

113. [Vasile Cirtoaje] Nếu a, b, c là cá số thực dương, thì

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \leq 3.$$

Gazeta Matematiă

114. Chứng minh rằng bất đẳng thức sau đúng với mọi số thực dương x, y, z

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}\right)\geq\frac{9}{4}.$$

Iran, 1996

115. Chứng minh rằng với mọi x, y trong đoạn $[0,1]$,

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2}\geq(1+\sqrt{5})(1-xy).$$

116. [Suranyi] Chứng minh rằng với mọi số thực dương $a_1, a_2, ..., a_n$, bất đẳng thức sau đúng

$$(n-1)(a_1^n+a_2^n+....+a_n^n)+na_1a_2...a_n\geq(a_1+a_2+...+a_n)(a_1^{n-1}+a_2^{n-1}+...+a_n^{n-1}).$$

Miklos Schweitzer Competition

117. Chứng minh rằng với mọi $x_1, x_2, ..., x_n$ có tích bằng 1,

$$\sum_{1\leq i<j\leq n}(x_i-x_j)^2\geq\sum_{i=1}^{n}x_i^2-n.$$

Dạng tổng quát của bất đẳng thức Turkevici

118. [Gabriel Dospinescu] Tìm giá trị nhỏ nhất của biểu thức

$$\sum_{i=1}^{n}\sqrt{\frac{a_1a_2...a_2}{1-(n-1)a_i}}$$

với $a_1, a_2, ..., a_n < \dfrac{1}{n-1}$ có tổng bằng 1 và $n > 2$ là một số nguyên.

119. [Vasile Cirtoaje] Cho $a_1, a_2, ..., a_n < 1$ là các số thực không âm thỏa mãn

$$a=\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}\geq\frac{\sqrt{3}}{3}.$$

Chứng minh rằng

$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+...+\frac{a_n}{1-a_n^2}\geq\frac{na}{1-a^2}.$$

120. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c, x, y, z là các số thực dương thỏa mãn

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Chứng minh rằng

$$abcxyz < \frac{1}{36}.$$

121. [Gabriel Dospinescu] Cho $n > 2$, tìm giá nhỏ nhất của hằng số k_n, sao cho nếu $x_1, x_2, ..., x_2 > 0$ có tích bằng 1, thì

$$\frac{1}{\sqrt{1+k_n x_1}} + \frac{1}{\sqrt{1+k_n x_2}} + ... + \frac{1}{\sqrt{1+k_n x_n}} \leq n-1.$$

Mathlinks Contest

122. [Vasile Cirtoaje, Gabriel Dospinescu] Cho $n > 2$, tìm giá trị lớn nhất của hằng số k_n, sao cho với bất kỳ $x_1, x_2, ..., x_n > 0$ thỏa mãn $x_1^2 + x_2^2 + ... + x_n^2 = 1$, ta có bất đẳng thức

$$(1-x_1)(1-x_2)...(1-x_n) \geq k_n x_1 x_2 ... x_n.$$

Chương 2. Các lời giải

1. Chứng minh rằng bất đẳng thức

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2} \geq \frac{3\sqrt{2}}{2}.$$

đúng với các số thực a, b, c bất kỳ.

Kömal

Lời giải 1:
Áp dụng bất đẳng thức **Minkowski** cho vế trái ta có

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2} \geq \sqrt{(a+b+c)^2+(3-a-b-c)^2}.$$

Đặt $a+b+c=x$, khi đó

$$(a+b+c)^2+(3-a-b-c)^2 = 2(x-\frac{3}{2})^2+\frac{9}{2} \geq \frac{9}{2},$$

và kết luận được chứng minh.

Lời giải 2:
Ta có bất đẳng thức

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2} \geq$$

$$\geq \frac{|a|+|1-b|}{\sqrt{2}}+\frac{|b|+|1-c|}{\sqrt{2}}+\frac{|c|+|1-a|}{\sqrt{2}}$$

và bởi vì $|x|+|1-x| \geq 1$ với mọi số thực x, đại lượng cuối cùng ít nhất bằng $\dfrac{3\sqrt{2}}{2}$.

2. [Dinu Serbănescu] Cho $a, b, c \in (0,1)$, chứng minh rằng

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)} < 1.$$

Junior TST 2002, Romania

Lời giải 1:
Dễ thấy $x^{\frac{1}{2}} < x^{\frac{1}{3}}$ với $x \in (0,1)$. Suy ra

$$\sqrt{abc} < \sqrt[3]{abc},$$

và

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}.$$

Bởi bất đẳng thức AM-GM,

$$\sqrt{abc} < \sqrt[3]{abc} \leq \frac{a+b+c}{3},$$

và

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)} \leq \frac{(1-a)+(1-b)+(1-c)}{3}.$$

Cộng hai bất đẳng thức, ta được

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \frac{a+b+c+1-a+1-b+1-c}{3} = 1,$$

điều phải chứng minh.

Lời giải 2:
Ta có

$$\sqrt{abc} + \sqrt{(1-a)(1-b(1-c)} < \sqrt{b}\sqrt{c} + \sqrt{1-b}.\sqrt{1-c} \leq 1,$$

bởi bất đẳng thức **Cauchy-Schwarz .**

Lời giải 3:
Đặt $a = \sin^2 x, b = \sin^2 y, c = \sin^2 z$, với $x, y, z \in (0, \frac{\pi}{2})$. Bất đẳng thức trở thành

$$\sin x.\sin y.\sin z + \cos x.\cos y.\cos z < 1$$

và được suy ra từ các bất đẳng thức sau

$$\sin x.\sin y.\sin z + \cos x.\cos y.\cos z < \sin x \sin y + \cos x.\cos y = \cos(x-y) \leq 1.$$

3. [Mircea Lascu] Cho a, b, c là các số thực dương thỏa mãn $abc = 1$. Chứng minh rằng

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

Gazeta Matematiă

Lời giải:

Từ bất đẳng thức **AM-GM**, ta có

$$\begin{aligned}
\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} &\geq 2\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right) \\
&= \left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ca}{b}}\right) + \left(\sqrt{\frac{ca}{b}} + \sqrt{\frac{ab}{c}}\right) + \left(\sqrt{\frac{ab}{c}} + \sqrt{\frac{bc}{a}}\right) \\
&\geq 2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \\
&\geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3\sqrt[6]{abc} \\
&= \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.
\end{aligned}$$

4. Nếu phương trình $x^4+ax^3+2x^2+bx+1=0$ có ít nhất một nghiệm thực, thì $a^2+b^2 \geq 8$.

Tournament of the Towns, 1993

Lời giải:
Cho x là nghiệm của phương trình. Sử dụng bất đẳng thức **Cauchy-Schwarz** dẫn đến

$$a^2 + b^2 \geq \frac{(x^4 + 2x^2 + 1)^2}{x^2 + x^6} \geq 8,$$

bởi vì bất đẳng thức cuối cùng tương đương với $(x^2-1)^4 \geq 0$.

5. Tìm giá trị lớn nhất của biểu thức $x^3+y^3+z^3-3xyz$ với $x^2+y^2+z^2=1$ và x, y, z là các số thực.

Lời giải:
Đặt $t = xy + yz + zx$. Dễ thấy

$$(x^3+y^3+z^3-3xyz)^2 = (x+y+z)^2(1-xy-yz-zx)^2 = (1+2t)(1-t)^2.$$

Ta cũng có $-\frac{1}{2} \leq t \leq 1$. Suy ra, ta phải tìm giá trị lớn nhất của biểu thức $(1+2t)(1-t)^2$, với $-\frac{1}{2} \leq t \leq 1$. Trong trường hợp này ta có

$$(1+2t)(1-t)^2 \leq 1 \Leftrightarrow t^2(3-2t) \geq 0,$$

vậy $|x^3+y^3+z^3-3xyz| \leq 1$. Dấu bằng xảy ra khi $x=1, y=z=0$ và giá trị lớn nhất bằng 1.

6. Cho a, b, c là các số thực dương thỏa mãn $x+y+z=1$. Chứng minh rằng

$$ax + by + cz + 2\sqrt{(xy+yz+zx)(ab+bc+ca)} \leq a+b+c.$$

Ukraine, 2001

Lời giải 1:
Ta sẽ sử dụng bất đẳng thức **Cauchy-Schwarz** hai lần. Đầu tiên, ta viết

$$ax + by + cz \leq \sqrt{a^2 + b^2 + c^2}.\sqrt{x^2 + y^2 + z^2}$$

và sau đó áp dụng bất đẳng thưc **Cauchy-Schwarz** lần nữa ta được:

$$\begin{aligned} ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)} &\leq \\ &\leq \sqrt{\sum a^2}.\sqrt{\sum x^2} + \sqrt{2\sum ab}.\sqrt{2\sum xy} \\ &\leq \sqrt{\sum x^2 + 2\sum xy}.\sqrt{\sum a^2 + 2\sum ab} \\ &= \sum a. \end{aligned}$$

Lời giải 2:
Bất đẳng thức thuần nhất với a, b, c ta có thể giả sử $a + b + c = 1$. Áp dụng bất đẳng thức **AM-GM** ta có

$$ax+by+cz+2\sqrt{(xy + yz + zx)(ab + bc + ca)} \leq ax+by+cz+xy+yz+zx+ab+bc+ca.$$

Bởi

$$xy + yz + zx + ab + bc + ca = \frac{1 - x^2 - y^2 - z^2}{2} + \frac{1 - a^2 - b^2 - c^2}{2} \leq 1 - ax - by - cz,$$

bất đẳng thức cuối cùng tương đương với $(x - a)^2 + (y - b)^2 + (z - c)^2 \geq 0$.

7. [Darij Grinberg] Nếu a, b, c là các số thực dương, thì

$$\frac{a}{(b + c)^2} + \frac{b}{(c + a)^2} + \frac{c}{(a + b)^2} \geq \frac{9}{4(a + b + c)}.$$

Lời giải 1:
Ta viết lại bất đẳng thức

$$(a + b + c)\left(\frac{a}{(b + c)^2} + \frac{b}{(c + a)^2} + \frac{c}{(a + b)^2}\right) \geq \frac{9}{4}.$$

Áp dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$(a + b + c)\left(\frac{a}{(b + c)^2} + \frac{b}{(c + a)^2} + \frac{c}{(a + b)^2}\right) \geq \left(\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}\right)^2.$$

Cuối cùng ta phải chứng minh

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2},$$

bất đẳng thức này chúng ta đã biết.

Lời giải 2:
Không mất tổng quát ta có thể giả sử $a+b+c=1$. Xét hàm

$$f:(0,1\to(0,\infty),f(x)=\frac{x}{(1-x)^2}.$$

Tính đạo hàm của f ta thấy f là hàm lồi. Vậy, ta có thể áp dụng bất đẳng thức **Jensen** và kết luận được chứng minh.

8. [Hojoo Lee] Cho $a,b,c\geq 0$. Chứng minh rằng

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\geq$$

$$\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ac}+c\sqrt{2c^2+ab}.$$

Gazeta Matematiă

Lời giải:
Ta bắt đầu từ $(a^2-b^2)^2\geq 0$. Ta viết lại bất đẳng thức như sau

$$4a^4+4a^2b^2+4b^4\geq 3a^4+6a^2b^2+3b^4,$$

suy ra

$$\sqrt{a^4+a^2b^2+b^4}\geq\frac{\sqrt{3}}{2}(a^2+b^2).$$

Từ đó, dễ thấy rằng

$$\left(\sum\sqrt{a^4+a^2b^2+b^4}\right)^2\geq 3\left(\sum a^2\right)^2.$$

Áp dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$\left(\sum a\sqrt{2a^2+bc}\right)^2\leq\left(\sum a^2\right)\left(\sum(2a^2+bc)\right)\leq 3\left(\sum a^2\right)^2$$

và bất đẳng thức được chứng minh.

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9. Cho a, b, c là các số thực dương thỏa mãn $abc = 2$, khi đó

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Khi nào đẳng thức xảy ra?

JBMO 2002 Shorlist

Lời giải 1:
Áp dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$(1) \qquad 3(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

và

$$(2) \qquad (a^2 + b^2 + c^2)^2 \leq (a + b + c)(a^3 + b^3 + c^3).$$

Kết hợp hai bất đẳng thức trên ta được

$$\begin{aligned} a^3 + b^3 + c^3 &\geq \frac{(a^2 + b^2 + c^2)(a + b + c)}{3} \\ &= \frac{(a^2 + b^2 + c^2)\ \big((a + b) + (b + c) + (c + a)\big)}{6} \\ &\geq \frac{a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}\big)^2}{6} \end{aligned}$$

Sử dụng bất đẳng thức **AM-GM** ta có

$$(3) \qquad \begin{aligned} a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b} &\geq 3\sqrt[3]{abc\sqrt{(a+b)(b+c)(c+a)}} \\ &\geq 3\sqrt[3]{abc\sqrt{8abc}} \\ &\geq 3\sqrt[3]{8} \\ &= 6. \end{aligned}$$

Vậy

$$(4) \qquad (a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b})^2 \geq 6(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}).$$

Điều phải chứng minh được suy ra từ bất đẳng thức (3) và (4).

Lời giải 2:
Ta có

$$a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b} \leq \sqrt{2(a^2 + b^2 + c^2)(a + b + c)}.$$

Sử dụng bất đẳng thức **Chebyshev**, ta có

$$\sqrt{2(a^2+b^2+c^2)(a+b+c)} \leq \sqrt{6(a^3+b^3+c^3)}$$

và cuối cùng ta chỉ cần chứng minh $a^3+b^3+c^3 \geq 3abc$, điều này đúng bởi bất đẳng thức **AM-GM** . Ta có đẳng thức nếu $a=b=c=\sqrt[3]{2}$.

10. [Ioan Tomescu] Cho $x, y, x > 0$. Chứng minh rằng

$$\frac{xyz}{(1+3x)(x+8y)(y+9z)(z+6)} \leq \frac{1}{7^4}.$$

Khi nào ta có đẳng thức?

Gazeta Matematiă

Lời giải:
Đâu tiên, chúng ta viết lại bất đẳng thức dưới dạng

$$(1+3x)\left(1+\frac{8y}{x}\right)\left(1+\frac{9z}{y}\right)\left(1+\frac{6}{z}\right) \geq 7^4.$$

Bất đẳng trên được suy ra trực tiếp từ bất đẳng thức **Huygens**. Đẳng thức xẩy ra khi $x=2, y=\dfrac{3}{2}, z=1$.

11. [Mihai Piticari, Dan Popescu] Chứng minh rằng

$$5(a^2+b^2+c^2) \leq 6(a^3+b^3+c^3)+1,$$

với mọi $a, b, c > 0$ và $a+b+c=1$.

Lời giải:
Vì $a+b+c=1$, ta có

$$a^3+b^3+c^3 = 3abc + a^2+b^2+c^2 - ab - bc - ca.$$

Bất đẳng thức trở thành

$$\begin{aligned}
5(a^2+b^2+c^2) &\leq 18abc + 6(a^2+b^2+c^2) - 6(ab+bc+ca) + 1 \\
&\Leftrightarrow 18abc + 1 - 2(ab+bc+ca) \geq 6(ab+bc+ca) \\
&\Leftrightarrow 8(ab+bc+ca) \leq 2 + 18abc \\
&\Leftrightarrow 4(ab+bc+ca) \leq 1 + 9abc \\
&\Leftrightarrow (1-2a)(1-2b)(1-2c) \leq abc \\
&\Leftrightarrow (b+c-a)(c+a-b)(a+b-c) \leq abc,
\end{aligned}$$

bất đẳng thức cuối tương đương với bất đẳng thức **Schur**.

12. [Mircea Lascu] Cho $x_1, x_2, ..., x_n \in \mathbb{R}, n \geq 2$ và $a > 0$ thỏa mãn

$$x_1 + x_2 + ... + x_n = a \text{ và } {x^2}_1 + {x^2}_2 + ... + {x^2}_n \leq \frac{a^2}{n-1}.$$

Chứng minh rằng $x_i \in [0, \frac{2a}{n}]$, với mọi $i \in \{1, 2, ..., n\}$.

Lời giải:
Sử dụng bất đẳng thức **Cauchy-Schwarz** , ta có

$$(a - x_1)^2 \leq (n-1)(x_2^2 + x_3^2 + ... + x_n^2) \leq (n-1)\left(\frac{a^2}{n-1} - x_1^2\right).$$

Suy ra

$$a^2 - 2ax_1 + x_1^2 \leq a^2 - (n-1)x_1^2 \Leftrightarrow x_1\left(x_1 - \frac{2a}{n}\right) \leq 0,$$

điều phải chứng minh.

13. [Adrian Zahariuc] Chứng minh rằng với mọi $a, b, c \in (1, 2)$ bất đẳng thức sau đây đúng

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq 1.$$

Lời giải:
Vì $a, b, c \in (1, 2)$ nên các mẫu số là dương. Khi đó

$$\begin{aligned} &\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} \geq \frac{a}{a+b+c} \\ \Leftrightarrow &b(a+b+c) \geq \sqrt{a}(4b\sqrt{c} - c\sqrt{a}) \\ \Leftrightarrow &(a+b)(b+c) \geq 4b\sqrt{ac}, \end{aligned}$$

bất đẳng thức cuối suy ra từ $a + b \geq 2\sqrt{ab}$ và $b + c \geq 2\sqrt{bc}$. Viết hai bất đẳng thức tương tự và cộng 3 bất đẳng thức lại ta được điều phải chứng minh.

14. Cho các số thực dương a, b, c thỏa mãn $abc = 1$, chứng minh rằng

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.$$

Lời giải 1:
Nếu $ab + bc + ca \leq a + b + c$, thì bất đẳng thức **Cauchy-Schwarz** giải xong bài toán:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a^2c + b^2a + c^2b}{abc} \geq \frac{\dfrac{(a+b+c)^2}{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}}{abc} \geq a + b + c.$$

Ngược lại, bất đẳng thức trên cho ta

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2c+b^2a+c^2b}{abc}\geq\frac{\frac{(ab+bc+ca)^2}{a+b+c}}{abc}\geq a+b+c$$

(ở đây ta sử dụng $abc\leq 1$).

Lời giải 2:
Thay thế a,b,c bởi ta,tb,tc với $t=\frac{1}{\sqrt[3]{abc}}$, khi đó bảo toàn giá trị vế trái của bất đẳng thức và tăng giá trị vế phải của bất đẳng thức và có $at.bt.ct=abct^3=1$. Do đó không mất tổng quát chúng ta có thể giả sử rằng $abc=1$. Khi đó tồn tại các số thực x,y,z sao cho $a=\frac{y}{x},b=\frac{z}{y},c=\frac{x}{z}$.
Bất đẳng thức **Rearrangement** cho ta

$$x^3+y^3+z^3\geq x^2y+y^2z+z^2x.$$

Suy ra

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{x^3+y^3+z^3}{xyz}\geq\frac{x^2y+y^2z+z^2x}{xyz}=a+b+c$$

điều phải chứng minh.

Lời giải 3:
Sử dụng bất đẳng thức **AM-GM** , chúng ta suy ra rằng

$$\frac{2a}{b}+\frac{b}{c}\geq 3\sqrt[3]{\frac{a^2}{bc}}\geq 3a.$$

Tương tự, $\frac{2b}{c}+\frac{c}{a}\geq 3b$ và $\frac{2c}{a}+\frac{a}{b}\geq 3c$. Cộng ba bất đẳng thức trên ta có điều phải chứng minh.

Lời giải 4:
Đặt $x=\sqrt[9]{\frac{ab^4}{c^2}},y=\sqrt[9]{\frac{ca^4}{b^2}},z=\sqrt[9]{\frac{bc^4}{a^2}}$. Do đó, $a=xy^2,b=zx^2,c=yz^2$, và $xyz\leq 1$.
Vậy, sử dụng bất đẳng thức **Rearrangement**, ta có

$$\sum\frac{a}{b}=\sum\frac{x^2}{yz}\geq xyz\sum\frac{x^2}{yz}=\sum x^3\geq\sum xy^2=\sum a.$$

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15. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c, x, y, z là các số thực dương sao cho $a + x \geq b + y \geq c + z$ và $a + b + c = x + y + z$. Chứng minh rằng

$$ay + bx \geq ac + xz.$$

Lời giải:
Ta có

$$\begin{aligned}
ay + bx - ac - xz &= a(y - c) + x(b - z) \\
&= a(a + b - x - z) + x(b - z) \\
&= a(a - x) + (a + x)(b - z) \\
&= \frac{1}{2}(a - x)^2 + \frac{1}{2}(a^2 - x^2) + (a + x)(b - z) \\
&= \frac{1}{2}(a - x)^2 + \frac{1}{2}(a + x)(a - x + 2b - 2z) \\
&= \frac{1}{2}(a - x)^2 + \frac{1}{2}(a + x)(b - c + y - z) \geq 0.
\end{aligned}$$

Dấu bằng xảy ra khi $a = x, b = z, c = y$ và $2x \geq y + z$.

16. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c là các số thực thỏa mãn $abc = 1$. Chứng minh rằng

$$1 + \frac{3}{a + b + c} \geq \frac{6}{ab + ac + bc}.$$

Junior TST 2003, Romania

Lời giải:
Ta đặt $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ và dễ thấy $xyz = 1$. Bất đẳng thức tương đương với

$$1 + \frac{3}{xy + yz + zx} \geq \frac{6}{x + y + z}.$$

Từ $(x + y + z)^2 \geq 3(xy + yz + zx)$ ta có

$$1 + \frac{3}{xy + yz + zx} \geq 1 + \frac{9}{(x + y + z)^2},$$

vậy cần phải chứng minh rằng

$$1 + \frac{9}{(x + y + z)^2} \geq \frac{6}{x + y + z}.$$

Bất đẳng thức cuối tương đương với $\left(1 - \frac{3}{x + y + z}\right)^2 \geq 0$ và điều này kết thúc chứng minh.

17. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

JBMO 2002 Shorlist

Lời giải 1:
Ta có

$$\frac{a^3}{b^2} \geq \frac{a^2}{b} + a - b \Leftrightarrow a^3 + b^3 \geq ab(a+b) \Leftrightarrow (a-b)^2(a+b) \geq 0,$$

điều này hiển nhiên đúng. Viết các bất đẳng thức tương tự và cộng lại

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + a - b + \frac{b^2}{c} + b - c + \frac{c^2}{a} + c - a = \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

Lời giải 2:
Từ bất đẳng thức **Cauchy-Schwarz** ta có

$$(a+b+c)\left(\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2}\right) \geq \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)^2,$$

vậy ta chỉ phải chứng minh rằng

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.$$

Nhưng điều này suy ra trực tiếp từ bất đẳng thức **Cauchy-Schwarz** .

18. Chứng minh rằng nếu $n > 3$ và $x_1, x_2, ..., x_n > 0$ thỏa mãn $\prod_{i=1}^{n} x_i = 1$, thì

$$\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + ... + \frac{1}{1 + x_{n-1} + x_{n-1}x_n} + \frac{1}{1 + x_n + x_nx_1} > 1.$$

Russia, 2004

Lời giải:
Ta sử dụng một dạng tương tự của phép thế cổ điển $x_1 = \frac{a_2}{a_1}, x_2 = \frac{a_3}{a_2}, ..., x_n = \frac{a_1}{a_n}$.
Trong trường hợp này bất đẳng thức trở thành

$$\frac{a_1}{a_1 + a_2 + a_3} + \frac{a_2}{a_2 + a_3 + a_4} + ... + \frac{a_n}{a_n + a_1 + a_2} > 1$$

và điều đó là hiển nhiên, bởi vì $n > 3$ và $a_i + a_{i+1} + a_{i+2} < a_1 + a_2 + ... + a_n$ với mọi i.

19. [Marian Tetiva] Cho x, y, z là các số thực dương thỏa mãn điều kiện

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Chứng minh rằng

a) $xyz \leq \frac{1}{8}$;

b) $x + y + z \leq \frac{3}{2}$;

c) $xy + xz + yz \leq \frac{3}{4} \leq x^2 + y^2 + z^2$;

d) $xy + xz + yz \leq \frac{1}{2} + 2xyz$.

Lời giải:
a) Điều này rất đơn giản. Từ bất đẳng thức **AM-GM** , ta có

$$1 = x^2 + y^2 + z^2 + 2xyz \geq 4\sqrt[4]{2x^3y^3z^3} \Longrightarrow x^3y^3z^3 \leq \frac{1}{2.4^4} \Longrightarrow xyz \leq \frac{1}{8}.$$

b) Rõ ràng, ta phải có $x, y, z \in (0, 1)$. Nếu đặt $s = x + y + z$, từ hệ thức đã cho ta có ngay

$$s^2 - 2s + 1 = 2(1 - x)(1 - y)(1 - z).$$

Sau đó, lại bởi bất đẳng thức **AM-GM** ($1 - x, 1 - y, 1 - z$ là các số dương), ta được

$$s^2 - 2s + 1 \leq 2\left(\frac{1 - x = 1 - y + 1 - z}{3}\right)^3 = 2\left(\frac{3 - s}{3}\right)^3.$$

Sau một vài bước tính ta được

$$2s^3 + 9s^2 - 27 \leq 0 \Leftrightarrow (2s - 3)(s + 3)^2 \leq 0$$

và kết luận là rõ ràng.

c) Các bất đẳng thức này là các hệ quả đơn giản của a) và b):

$$xy + xz + yz \leq \frac{(x + y + z)^3}{3} \leq \frac{1}{3}.\frac{9}{4} = \frac{3}{4}$$

và

$$x^2 + y^2 + z^= 1 - 2xyz \geq 1 - 2.\frac{1}{8} = \frac{3}{4}.$$

d) Đây là vấn đề tế nhị hơn, đầu tiên ta chú ý rằng luôn luôn có hai trong ba số lớn hơn (hoặc nhỏ hơn) $\frac{1}{2}$. Bởi tính đối xứng, ta có thể giả thiết rằng, $x, y \leq \frac{1}{2}$, hoặc $x, y \geq \frac{1}{2}$ và do đó

$$(2x-1)(2y-1) \geq 0 \Leftrightarrow x+y-2xy \leq \frac{1}{2}.$$

Mặt khác,

$$\begin{aligned} 1 &= x^2+y^2+z^2+2xyz \geq 2xy+z^2+2xyz \\ \Longrightarrow & 2xy(1+z) \leq 1-z^2 \\ \Longrightarrow & 2xy \leq 1-z. \end{aligned}$$

Bây giờ ta chỉ việc nhân vế với vế các bất đẳng thức ở trên

$$x+y-2xy \leq \frac{1}{2}$$

và

$$z \leq 1-2xy$$

để có được kết quả mong muốn:

$$xz+yz-2xyz \leq \frac{1}{2}-xy \Longleftrightarrow xy+xz+yz \leq \frac{1}{2}+2xyz.$$

Dù không quan trọng trong chứng minh, nhưng cũng chú ý rằng

$$x+y-2xy = xy\left(\frac{1}{x}+\frac{1}{y}-2\right) > 0,$$

bởi vì cả hai số $\frac{1}{x}$ và $\frac{1}{y}$ đều lớn hơn 1.

Chú ý.
1) Có thể thu được một vài bất đẳng thức khác, sử dụng

$$z+2xy \leq 1$$

và hai bất đẳng thức tương tự

$$y+2xz \leq 1, x+2yz \leq 1.$$

Ví dụ, nhân các bất đẳng thức này lần lượt với x, y, z và cộng các bất đẳng thức mới, ta có

$$x^2+y^2+z^2+6xyz \leq x+y+z,$$

hoặc

$$1 + 4xyz \leq x + y + z.$$

2) Nếu ABC là một tam giác bất kỳ, thì các số

$$x = \sin\frac{A}{2}, y = \sin\frac{B}{2}, z = \sin\frac{C}{2}$$

thoả mãn điều kiện của bài toán này; ngược lại, nếu $x, y, z > 0$ thoả mãn

$$x^2 + y^2 + z^2 + 2xyz = 1$$

thì có một tam giác ABC sao cho

$$x = \sin\frac{A}{2}, y = \sin\frac{B}{2}, z = \sin\frac{C}{2}.$$

Theo nhận xét này, các cách chứng minh mới có thể được đưa ra cho các bất đẳng thức trên.

20. [Marius Olteanu] Cho $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ thỏa mãn $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Chứng minh rằng

$$|\cos x_1| + |\cos x_2| + |\cos x_3| + |\cos x_4| + |\cos x_5| \geq 1.$$

Gazeta Matematiă

Lời giải:

Dễ dàng chứng minh được rằng

$$|\sin(x + y)| \leq \min\{|\cos x| + |\cos y|, |\sin x| + |\sin y|\}$$

và

$$|\cos(x + y)| \leq \min\{|\sin x| + |\cos y|, |\sin y| + |\cos x|\}.$$

Vậy, ta có

$$1 = |\cos \left.\sum_{i=1}^{5} x_i\right)| \leq |\cos x_1| + |\sin \left.\sum_{i=2}^{5} x_i\right)| \leq |\cos x_1| + |\cos x_2| + |\cos(x_3 + x_4 + x_5)|.$$

Nhưng bằng cách tương tự ta có thể chứng minh

$$|\cos(x_3 + x_4 + x_5)| \leq |\cos x_3| + |\cos x_4| + |\cos x_5|$$

và ta có điều phải chứng minh.

21. [Florina Cărlan, Marian Tetiva] Chứng minh rằng nếu $x, y, z > 0$ thỏa mãn điều kiện $x + y + z = xyz$ thì

$$xy + xz + yz \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}.$$

Lời giải 1:
Ta có

$$xyz = x + y + z \geq 2\sqrt{xy} + z \Longrightarrow z(\sqrt{xy})^2 - 2\sqrt{xy} - z \geq 0.$$

Bởi vì nghiệm dương của tam thức $zt^2 - 2t - z$ là

$$\frac{1+\sqrt{1+z^2}}{z},$$

từ đó ta có

$$\sqrt{xy} \geq \frac{1+\sqrt{1+z^2}}{z} \Leftrightarrow z\sqrt{xy} \geq 1 + \sqrt{1+z^2}.$$

Hiển nhiên ta có hai bất đẳng thức tương tự khác. Khi đó,

$$xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy} \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1},$$

và ta thu được chứng minh của bất đẳng thức đã cho, và một sự cải thiện nhỏ của nó

Lời giải 2:
Một sự cải thiện khác được đưa ra như sau. Bắt đầu từ

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1 \Longrightarrow x^2y^2 + y^2z^2 + z^2x^2 \geq x^2y^2z^2,$$

bất đẳng thức này tương đương với

$$(xy + yz + zx)^2 \geq 2xyz(x + y + z) + x^2y^2z^2 = 3(x + y + z)^2.$$

Hơn nữa,

$$\begin{aligned}(xy + yz + zx - 3)^2 &= (xy + yz + zx)^2 - 6(xy + yz + zx) + 9\\ &\geq 3(x + y + z)^2 - 6(xy + yz + zx) + 9\\ &= 3(x^2 + y^2 + z^2) + 9,\end{aligned}$$

suy ra

$$xy + yz + zx \geq 3 + \sqrt{3(x^2 + y^2 + z^2) + 9}.$$

Nhưng

$$\sqrt{3(x^2 + y^2 + z^2) + 9} \geq \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}$$

là một hệ quả của bất đẳng thức **Cauchy-Schwarz** và ta có sự cải thiện thứ hai và chứng minh của bất đẳng thức mong muốn:

$$xy + yz + zx \geq 3 + \sqrt{3(x^2 + y^2 + z^2) + 9} \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}.$$

22. [Laurentiu Panaitopol] Chứng minh rằng

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2,$$

với mọi số thực $x, y, z > -1$.

JBMO, 2003

Lời giải:

Ta thấy $y \leq \dfrac{1+y^2}{2}$ và $1+y+z^2>0$, vậy

$$\frac{1+x^2}{1+y+z^2}\geq \frac{1+x^2}{1+z^2+\dfrac{1+y^2}{2}}$$

và có các hệ thức tương tự. Đặt $a=1+x^2, b=1+y^2, c=1+z^2$, chỉ cần chứng minh rằng

$$\frac{a}{2c+b}+\frac{b}{2a+c}+\frac{c}{2b+a}\geq 1 \qquad (1)$$

với bất kỳ $a, b, c > 0$. Đặt $A=2c+b, B=2a+c, c=2b+a$. Khi đó

$$a=\frac{C+4B-2A}{9}, b=\frac{A+4C-2B}{9}, c=\frac{B+4A-2C}{9}$$

và bất đẳng thức (1) được viết lại

$$\frac{C}{A}+\frac{A}{B}+\frac{B}{C}+4\left(\frac{B}{A}+\frac{C}{B}+\frac{A}{C}\right)\geq 15.$$

Bởi vì $A, B, C > 0$, từ bất đẳng thức **AM-GM** ta có

$$\frac{C}{A}+\frac{A}{B}+\frac{B}{C}\geq 3\sqrt[3]{\frac{C}{A}.\frac{A}{B}.\frac{B}{C}}=3$$

và

$$\frac{B}{A}+\frac{C}{B}+\frac{A}{C}\geq 3$$

và ta có điều phải chứng minh.

Một cách chứng minh khác cho (1) bởi sử dụng bất đẳng thức **Cauchy-Schwarz** :

$$\frac{a}{2c+b}+\frac{b}{2a+c}+\frac{c}{2b+a}=\frac{a^2}{2ac+ab}+\frac{b^2}{2ab+cb}+\frac{c^2}{2bc+ac}\geq\frac{(a+b+c)^2}{3(ab+bc+ca)}\geq 1.$$

23. Cho $a, b, c > 0$ thỏa mãn $a+b+c=1$. Chứng minh rằng

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b} \geq 2.$$

Lời giải:
Sử dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$\sum \frac{a^2+b}{b+c} \geq \frac{\left(\sum a^2+1\right)^2}{\sum a^2(b+c)+\sum a^2+\sum ab}.$$

Vậy ta chỉ cần chứng minh rằng

$$\frac{\left(\sum a^2+1\right)^2}{\sum a^2(b+c)+\sum a^2+\sum ab} \geq 2 \Leftrightarrow 1+\left(\sum a^2\right)^2 \geq 2\sum a^2(b+c)+2\sum ab.$$

Bất đẳng thức cuối cùng có thể biến đổi như sau

$$\begin{aligned}
&1+\left(\sum a^2\right)^2 \geq 2\sum a^2(b+c)+2\sum ab\\
\Leftrightarrow &1+\left(\sum a^2\right)^2 \geq 2\sum a^2-2\sum a^3+2\sum ab\\
\Leftrightarrow &\left(\sum a^2\right)^2+2\sum a^3 \geq \sum a^2,
\end{aligned}$$

và điều này đúng, bởi vì

$$\sum a^3 \geq \frac{\sum a^2}{3} (\text{ bất đẳng thức } \textbf{Chebyshev} \ )$$

và

$$\left(\sum a^2\right)^2 \geq \frac{\sum a^2}{3}.$$

24. Cho $a, b, c \geq 0$ thỏa mãn $a^4+b^4+c^4 \leq 2(a^2b^2+b^2c^2+c^2a^2)$. Chứng minh rằng

$$a^2+b^2+c^2 \leq 2(ab+bc+ca).$$

Kvant, 1988

Lời giải:
Điều kiện

$$\sum a^4 \leq 2\sum a^2b^2$$

tương đương với

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) \geq 0.$$

Trong bất kỳ trường hợp nào mà $a=b+c$ hoặc $b=c+a$ hoặc $c=a+b$, thì bất đẳng thức

$$\sum a^2 \leq 2\sum ab$$

là hiển nhiên. Vậy, giả sử $a+b \neq c, b+c \neq a, c+a \neq b$. Bởi vì nhiều nhất một trong các số $b+c-a, c+a-b, a+b-c$ là âm và tích của các số là không âm, nên tất cả các số là dương. Suy ra, ta có thể giả thiết rằng

$$a^2 < ab+ac, b^2 < bc+ba, c^2 < ca+cb$$

và kết luận được chứng minh.

25. Cho $n \geq 2$ và $x_1, x_2, ..., x_n$ là các số thực dương thỏa mãn

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+...+\frac{1}{x_n+1998}=\frac{1}{1998}.$$

Chứng minh rằng

$$\frac{\sqrt[n]{x_1.x_2...x_n}}{n-1} \geq 1998.$$

Vietnam, 1998

Lời giải:
Đặt $\dfrac{1998}{1998+x_i}=a_i$. Bài toán quy về chứng minh rằng với mọi số thực dương $a_1, a_2, ..., a_n$ thỏa mãn $a_1+a_2+...+a_n=1$ ta có bất đẳng thức

$$\prod_{i=1}^{n}\left(\frac{1}{a_i}-1\right) \geq (n-1)^n.$$

Bất đẳng thức này có thể thu được bởi nhân các bất đẳng thức

$$\frac{1}{a_i}-1=\frac{a_1+...+a_{i-1}+a_{i+1}+...+a_n}{a_i} \geq (n-1)\sqrt[n-1]{\frac{a_1...a_{i-1}a_{i+1}...a_n}{a_i^{n-1}}}.$$

26. [Marian Tetiva] Cho các số thực dương x, y, z thỏa mãn $x^2+y^2+z^2=xyz$. Chứng minh các bất đẳng thức sau

a) $xyz \geq 27$;

b) $xy + xz + yz \geq 27$;

c) $x + y + z \geq 9$;

d) $xy + xz + yz \geq 2(x + y + z) + 9$.

Lời giải:
a), b), c) Sử dụng các bất đẳng thức đã biết, chúng ta có

$$xyz = x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2} \Longrightarrow (xyz)^3 \geq 27(xyz)^2,$$

điều này cho $xyz \geq 27$. Khi đó

$$xy + xz + yz \geq 3\sqrt[3]{(xyz)^2} \geq 3\sqrt[3]{27^2} = 27$$

và

$$x + y + z \geq 3\sqrt[3]{xyz} \geq 3\sqrt[3]{27} = 9.$$

d) Ta chú ý rằng $x^2 < xyz \Longrightarrow x < yz$ và các hệ thức tương tự. Suy ra

$$xy < yz.xz \Longrightarrow 1 < z^2 \Longrightarrow z > 1.$$

Do đó cả ba số đều lớn hơn 1. Đặt

$$a = x - 1, b = y - 1, c = z - 1.$$

Ta có $a > 0, b > 0, c > 0$ và

$$x = a + 1, y = b + 1, z = c + 1.$$

Thay vào giả thiết đã cho ta có

$$(a + 1)^2 + (b + 1)^2 + (c + 1)^2 \geq (a + 1)(b + 1)(c + 1),$$

điều này tương đương với

$$a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + ac + bc.$$

Nếu đặt $q = ab + ac + bc$, ta có

$$q \leq a^2 + b^2 + c^2, \sqrt{3q} \leq a + b + c$$

và

$$abc \leq \left(\frac{q}{3}\right)^{\frac{3}{2}} = \frac{(3q)^{\frac{3}{2}}}{27}.$$

Suy ra

$$q + \sqrt{3q} + 2 \le a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + ac + bc \le \frac{(\sqrt{3q})^3}{27} + q$$

và đặt $x = \sqrt{3q}$, ta có

$$x + 2 \le \frac{x^3}{27} \Longleftrightarrow (x-6)(x+3)^2 \ge 0.$$

Cuối cùng,

$$\sqrt{3q} = x \ge 6 \Longrightarrow q = ab + ac + bc \ge 12.$$

Bây giờ, nhắc lại rằng $a = x - 1, b = y - 1, c = z - 1$, vậy ta có

$$\begin{aligned} &(x-1)(y-1) + (x-1)(y-1) + (y-1)(z-1) \ge 12 \\ \Longrightarrow &xy + xz + yz \ge 2(x+y+z) + 9; \end{aligned}$$

và ta có điều phải chứng minh.
Ta có thể chứng minh một bất đẳng thức mạnh hơn

$$xy + xz + yz \ge 4(x+y+z) - 9.$$

Hãy thử đi!

27. Cho x, y, z là các số thực dương có tổng bằng 3. Chứng minh rằng

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \ge xy + yz + zx.$$

Russia, 2002

Lời giải:
Viết lại bất đẳng thức dưới dạng

$$\begin{aligned} &x^2 + 2\sqrt{x} + y^2 + 2\sqrt{y} + z^2 + 2\sqrt{z} \ge x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \\ \Longleftrightarrow &x^2 + 2\sqrt{x} + y^2 + 2\sqrt{y} + z^2 + 2\sqrt{z} \ge 9. \end{aligned}$$

Bây giờ, từ bất đẳng thức **AM-GM** , ta có

$$\begin{aligned} &x^2 + 2\sqrt{x} = x^2 + \sqrt{x} + \sqrt{x} \ge 3\sqrt[3]{x^2.x} = 3x, \\ &y^2 + 2\sqrt{y} \ge 3y, z^2 + 2\sqrt{z} \ge 3z, \end{aligned}$$

suy ra

$$x^2 + y^2 + z^2 + 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) \ge 3(x+y+z) \ge 9.$$

28. [D.Olteanu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a+b}{b+c}.\frac{a}{2a+b+c}+\frac{b+c}{c+a}.\frac{b}{2b+c+a}+\frac{c+a}{a+b}.\frac{c}{2c+a+b}\geq\frac{3}{4}.$$

Gazeta Matematiă

Lời giải:
Đặt $x=a+b, y=b+c, z=a+c$ và sau một vài bước tính toán ta thu được bất đẳng thức tương đương dạng

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x}\geq\frac{9}{2}.$$

Nhưng sử dụng bất đẳng thức **Cauchy-Schwarz** ,

$$\begin{aligned}\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{x}{x+y}+\frac{y}{y+z}+\frac{z}{z+x}&\geq\frac{(x+y+z)^2}{xy+yz+zx}+\frac{(x+y+z)^2}{xy+yz+zx+x^2+y^2+z^2}\\&=\frac{2(x+y+z)^4}{2(xy+yz+zx)(xy+yz+zx+x^2+y^2+z^2)}\\&\geq\frac{8(x+y+z)^4}{\left(xy+yz+zx+(x+y+z)^2\right)^2}\end{aligned}$$

ta có điều phải chứng minh.

29. Cho a, b, c là các số thực dương, chứng minh rằng bất đẳng thức sau đúng

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a}.$$

India, 2002

Lời giải:
Ta đặt $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$. Dễ thấy rằng

$$\frac{a+c}{b+c}=\frac{1+xy}{1+y}=x+\frac{1-x}{1+y}.$$

Thiết lập những hệ thức tương tự, bài toán quy về chứng minh rằng nếu $xyz=1$, thì

$$\begin{aligned}&\frac{x-1}{y+1}+\frac{y-1}{z+1}+\frac{z-1}{x+1}\geq 0\\\Longleftrightarrow&(x^2-1)(z+1)+(y^2-1)(x+1)+(z^2-1)(y+1)\geq 0\\\Longleftrightarrow&x^2z+z^2y+y^2x+x^2+y^2+z^2\geq x+y+z+3.\end{aligned}$$

Nhưng đây là bất đẳng thức rất dễ chứng minh. Thật vậy, sử dụng bất đẳng thức **AM-GM** ta có

$$x^2z + z^2y + y^2x \geq 3,$$

và chỉ còn phải chứng minh

$$x^2 + y^2 + z^2 \geq x + y + z,$$

điều này được suy ra từ bất đẳng thức

$$x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3} \geq x + y + z.$$

30. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

Đề cử cho kỳ thi Olympic Toán học vùng Balkan

Lời giải 1:
Vì $a + b + c \geq \dfrac{3(ab + bc + ca)}{a + b + c}$, nên chỉ cần chứng minh rằng:

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c.$$

Từ bất đẳng thức **Cauchy-Schwarz** , ta có

$$\sum \frac{a^3}{b^2 - bc + c^2} \geq \frac{(\sum a^2)^2}{\sum a(b^2 - bc + c^2)}.$$

Vậy chúng ta phải chỉ ra rằng

$$(a^2 + b^2 + c^2)^2 \geq (a + b + c). \sum a(b^2 - bc + c^2).$$

Bất đẳng thức này tương đương với

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

đây chính là bất đẳng thức **Schur**.

Chú ý.
Bất đẳng thức

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c$$

đã được đề nghị bởi Vasile Caarrtoaje trong **Gazeta Matematiă** như là một trường hợp đặc biệt ($n = 3$) của bất đẳng thức tổng quát

$$\frac{2a^n - b^n - c^n}{b^2 - bc + c^2} + \frac{2b^n - c^n - a^n}{c^2 - ca + a^2} + \frac{2c^n - a^n - b^n}{a^2 - ab + b^2} \geq 0.$$

Lời giải 2:
Ta viết lại bất đẳng thức

$$\sum \frac{(b+c)a^3}{b^3 + c^3} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

Nhưng bất đẳng thức này được suy ra từ một kết quả tổng quát:
Nếu $a, b, c, x, y, z > 0$ thì

$$\sum \frac{a(y+z)}{b+c} \geq 3\frac{\sum xy}{\sum x}.$$

Bất đẳng thức cuối cùng là một hệ quả trực tiếp (và yếu hơn) của kết quả từ bài toán 101.

Lời giải 3:
Đặt $A = \sum \dfrac{a^3}{b^2 - bc + c^2}$ và

$$B = \sum \frac{b^3 + c^3}{b^2 - bc + c^2} = \sum \frac{(b+c)(b^2 - bc + c^2)}{b^2 - bc + c^2} = 2\sum a.$$

Vậy ta có

$$\begin{aligned}
A + B &= \left(\sum a^3\right)\left(\sum \frac{1}{b^2 - bc + c^2}\right) \\
&= \frac{1}{2}\left(\sum (b+c)(b^2 - bc + c^2)\right)\left(\sum \frac{1}{b^2 - bc + c^2}\right) \\
&\geq \frac{1}{2}\left(\sum \sqrt{b+c}\right)^2,
\end{aligned}$$

điều này suy từ bất đẳng thức **Cauchy-Schwarz** . Do đó

$$A \geq \frac{1}{2}\left(\sum \sqrt{b+c}\right)^2 - 2\sum a = \sum \sqrt{a+b}.\sqrt{b+c} - \sum a.$$

Ta kí hiệu

$$A_a = \sqrt{c+a}.\sqrt{b+a} - a = \frac{ab + bc + ca}{\sqrt{ab + bc + ca + a^2} + a}$$

và thiết lập các hệ thức tương tự với A_b and A_c. Vậy $A \geq A_a + A_b + A_c$. Nhưng bởi vì $\left(\sum a\right)^2 \geq 3\left(\sum ab\right)$ ta có

$$\begin{aligned} A_a &\geq \frac{ab+bc+ca}{\sqrt{\dfrac{(a+b+c)^2}{3}+a^2}+a} \\ &= 3\frac{ab+bc+ca}{(a+b+c)^2}.\quad \sqrt{\frac{(a+b+c)^2}{3}+a^2}-a\Bigg) \end{aligned}$$

và các bất đẳng thức tương tự cũng đúng. Vậy chúng ta chỉ cần chứng minh

$$\sum\left(\sqrt{\frac{(a+b+c)^2}{3}+a^2}-a\right) \geq a+b+c \Longleftrightarrow \sum\sqrt{\frac{1}{3}+\left(\frac{a}{a+b+c}\right)^2} \geq 2.$$

Xét hàm lồi $f(t) = \sqrt{\dfrac{1}{3}+t^2}$. Sử dụng bất đẳng thức **Jensen** cuối cùng ta suy ra

$$\sum\sqrt{\frac{1}{3}+\left(\frac{a}{a+b+c}\right)^2} \geq 2.$$

Dấu bằng xảy ra khi và chỉ khi $a = b = c$.

31. [Adrian Zahariuc] Xét các số nguyên $x_1, x_2, ..., x_n, n \geq 0$ đôi một khác nhau. Chứng minh rằng

$$x_1^2 + x_2^2 + ... + x_n^2 \geq x_1.x_2 + x_2.x_3 + ... + x_n.x_1 + 2n - 3.$$

Lời giải:
Bất đẳng thức có thể viết lại như sau

$$\sum_{i=1}^{n-1}(x_i - x_{i+1})^2 + (x_n - x_1)^2 \geq 2(2n-3).$$

Đặt $x_m = \min\{x_1, x_2, ..., x_n\}$ và $x_M = \max\{x_1, x_2, ..., x_n\}$. Không mất tính tổng quát ta có thể giả sử $m < M$. Đặt

$$S_1 = (x_m - x_{m+1})^2 + ... + (x_{M-1} - x_M)^2$$

và

$$S_2 = (x_1 - x_2)^2 + ... + (x_{m-1} - x_m)^2 + (x_M - x_{M+1})^2 + ... + (x_n - x_1)^2.$$

Bất đẳng thức $\sum\limits_{i=1}^{k} a_i^2 \geq \dfrac{1}{k}\left(\sum\limits_{i=1}^{k} a_i\right)^2$ (suy ra từ bất đẳng thức **Cauchy-Schwarz**) dẫn đến

$$S_1 \geq \frac{(x_M - x_m)^2}{M-m}$$

và

$$S_2 \geq \frac{(x_M - x_m)^2}{n-(M-m)}.$$

Vậy

$$\begin{aligned}\sum (x_i - x_{i+1})^2 &= S_1 + S_2\\ &\geq (x_M - x_m)^2\left(\frac{1}{M-m} + \frac{1}{n-(M-m)}\right)\\ &\geq (n-1)^2\frac{4}{n}\\ &= 4n - 8 + \frac{4}{n}\\ &> 4n-8.\end{aligned}$$

Nhưng

$$\sum_{i=1}^{n}(x_i - x_{i+1})^2 = \sum_{i=1}^{n} x_i - x_{i+1} = 0 \mod 2.$$

Vậy

$$\sum_{i=1}^{n}(x_i - x_{i+1})^2 \geq 4n - 6$$

và bài toán đã được giải xong.

32. [Murray Klamkin] Tìm giá trị lớn nhất của biểu thức

$$x_1^2x_2 + x_2^2x_3 + ... + x_{n-1}^2x_n + x_n^2x_1$$

với $x_1, x_2, ..., x_n \geq 0$ có tổng bằng 1 và $n > 2$.

Crux Mathematicorum

Lời giải:

Đầu tiên, hiển nhiên là giá trị lớn nhất ít nhất là $\frac{4}{27}$, bởi vì giá trị này có thể đạt được với $x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = ... = x_n = 0$. Bây giờ, ta sẽ chứng minh bằng quy nạp bất đẳng thức

$$x_1^2x_2 + x_2^2x_3 + ... + x_{n-1}^2x_n + x_n^2x_1 \leq \frac{4}{27}$$

với mọi $x_1, x_2, ..., x_{n-1}, x_n \geq 0$ có tổng bằng 1. Đầu tiên ta chứng minh bước quy nạp. Giả sử bất đẳng thức đúng cho n và ta sẽ chứng minh nó đúng cho $n+1$. Ta có thể giả sử rằng $x_2 = \min\{x_1, x_2, ..., x_{n+1}\}$. Điều này suy ra rằng

$$x_1^2x_2 + x_2^2x_3 + ... + x_n^2x_{n+1} + x_{n+1}^2.x_1 \leq (x_1+x_2)^2x_3 + ... + x_n^2x_{n+1} + x_{n+1}^2(x_1+x_2).$$

Từ giả thiết quy nạp ta có

$$(x_1+x_2)^2x_3+x_3^2x_4+...+x_n^2x_{n+1}+x_{n+1}^2(x_1+x_2)\le\frac{4}{27}$$

và bước quy nạp được chứng minh. Vậy, ta còn phải chứng minh $a^2b+b^2c+c^2a\le\dfrac{4}{27}$ nếu $a+b+c=1$. Ta có thể giả sử rằng a là số lớn nhất trong các số a,b,c. Trong trường hợp này bất đẳng thức $a^2b+b^2c+c^2a\le\left(a+\dfrac{c}{2}\right)^2.\left(b+\dfrac{c}{2}\right)$ trực tiếp suy ra từ $abc\ge b^2c, \dfrac{a^2c}{2}\ge\dfrac{ac^2}{2}$. Bởi vì

$$1=\frac{a+\frac{c}{2}}{2}+\frac{a+\frac{c}{2}}{2}+b+\frac{c}{2}\ge 3\sqrt[3]{\frac{\left(b+\frac{c}{2}\right)\left(a+\frac{c}{2}\right)^2}{4}},$$

ta đã chứng minh được $a^2b+b^2c+c^2a\le\dfrac{4}{27}$ và điều này chỉ ra rằng giá trị lớn nhất bằng $\dfrac{4}{27}$.

33. Tìm giá trị lớn nhất của hằng số c sao cho với mọi $x_1,x_2,...,x_n,...>0$ thỏa mãn $x_{k+1}\ge x_1+x_2+...+x_k$ với mọi k, bất đẳng thức

$$\sqrt{x_n}+\sqrt{x_n}+...+\sqrt{x_n}\le c\sqrt{x_1+x_2+...+x_n}$$

đúng với mọi n.

IMO Shorlist, 1986

Lời giải:
Đầu tiên, ta xem điều gì xảy ra nếu x_{k+1} và $x_1+x_2+...+x_k$ gần nhau với mọi k. Ví dụ, ta có thể lấy $x_k=2^k$, bởi vì trong trường hợp này ta có $x_1+x_2+...+x_k=x_{k+1}-2$. Vậy, ta thấy rằng

$$c\ge\frac{\sum\limits_{k=1}^{n}\sqrt{2^k}}{\sqrt{\sum\limits_{k=1}^{n}2^k}}=\frac{1}{\sqrt{2}-1}.\frac{(\sqrt{2})^n-1}{\sqrt{2^n-1}}=\frac{1}{\sqrt{2}-1}.\frac{1-\frac{1}{\sqrt{2}^n}}{\sqrt{1-\frac{1}{2^n}}}$$

với mọi n. Lấy giới hạn, ta được $c\ge1+\sqrt{2}$. Bây giờ, ta chứng minh rằng $1+\sqrt{2}$ là hằng số cần tìm. Ta sẽ chứng minh bất đẳng thức

$$\sqrt{x_1}+\sqrt{x_2}+...+\sqrt{x_n}\le(1+\sqrt{2})\sqrt{x_1+x_2+...+x_n}$$

bằng quy nạp. Với $n = 1$ hoặc $n = 2$ bất đẳng thức là hiển nhiên. Giả sử bất đẳng thức đúng với n và ta sẽ chứng minh rằng

$$\sqrt{x_1} + \sqrt{x_2} + ... + \sqrt{x_n} + \sqrt{x_{n+1}} \le (1 + \sqrt{2})\sqrt{x_1 + x_2 + ... + x_n + x_{n+1}}.$$

Tất nhiên, chỉ cần chứng minh rằng

$$\sqrt{x_{n+1}} \le (1 + \sqrt{2})(\sqrt{x_1 + x_2 + ... + x_n + x_{n+1}} - \sqrt{x_1 + x_2 + ... + x_n})$$

bất đẳng thức này tương đương với

$$\sqrt{x_1 + x_2 + ... + x_n + x_{n+1}} + \sqrt{x_1 + x_2 + ... + x_n} \le (1 + \sqrt{2})\sqrt{x_{n+1}}.$$

Bất đẳng thức trên đúng bởi vì

$$x_1 + x_2 + ... + x_n \le x_{n+1}.$$

34. Cho các số thực dương a, b, c và x, y, z thỏa mãn $a + x = b + y = c + z = 1$. Chứng minh rằng

$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \ge 3.$$

Russia, 2002

Lời giải:
Chúng ta thấy rằng $abc + xyz = (1 - b)(1 - c) + ac + ab - a$. Vậy

$$\frac{1-c}{a} + \frac{c}{1-b} - 1 = \frac{abc + xyz}{a(1-b)}.$$

Sử dụng các đồng nhất thức này ta trực tiếp suy ra

$$3 + (xyz + abc)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) = \frac{a}{1-c} + \frac{b}{1-a} + \frac{c}{1-b} + \frac{1-c}{a} + \frac{1-a}{b} + \frac{1-b}{c}.$$

Bây giờ, ta chỉ còn phải áp dụng bất đẳng thức **AM-GM**

$$\frac{a}{1-c} + \frac{b}{1-a} + \frac{c}{1-b} + \frac{1-c}{a} + \frac{1-a}{b} + \frac{1-b}{c} \ge 6.$$

35. [Viorel Vâjâitu, Alexandru Zaharescu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \le \frac{1}{4}(a+b+c).$$

Lời giải 1:
Ta có chuỗi bất đẳng thức dưới đây

$$\sum \frac{ab}{a+b+2c} = \sum \frac{ab}{a+c+b+c} \geq \sum \frac{ab}{4}\left(\frac{1}{a+c}+\frac{1}{b+c}\right) = \frac{a+b+c}{4}.$$

Lời giải 2:
Bởi vì bất đẳng thức là thuần nhất, không mất tính tổng quát ta có thể xem rằng $a+b+c=1$ và do đó bất đẳng thức tương đương với

$$\sum \frac{1}{a(a+1)} \leq \frac{1}{4abc}.$$

Ta có $\dfrac{1}{t(t+1)} = \dfrac{1}{t} - \dfrac{1}{t+1}$, vậy bất đẳng thức tương đương với

$$\sum \frac{1}{a} \leq \sum \frac{1}{a+1} + \frac{1}{4abc}.$$

Bây giờ ta sẽ chứng minh sự xen vào giữa sau:

$$\sum \frac{1}{a} \leq \frac{9}{4} + \frac{1}{4abc} \leq \sum \frac{1}{a+1} + \frac{1}{4abc}.$$

Bất đẳng thức bên phải suy ra từ bất đẳng thức **Cauchy-Schwarz** :

$$\left(\sum \frac{1}{a+1}\right)\left(\sum (a+1)\right) \geq 9$$

và đồng nhất thức $\sum(a+1)=4$. Bất đẳng thức bên trái có thể viết lại thành $\sum ab \leq \dfrac{1+9abc}{4}$, nó chính xác là bất đẳng thức **Schur**.

36. Tìm giá trị lớn nhất của biểu thức

$$a^3(b+c+d)+b^3(c+d+a)+c^3(d+a+b)+d^3(a+b+c)$$

với a, b, c, d là các số thực mà tổng bình phương của các số bằng 1.

Lời giải:
Ý tưởng là

$$\begin{aligned} &a^3(b+c+d)+b^3(c+d+a)+c^3(d+a+b)+d^3(a+b+c)\\ =&ab(a^2+b^2)+ac(a^2+c^2)+ad(a^2+d^2)+bc(b^2+c^2)+bd(b^2+d^2)+cd(c^2+d^2). \end{aligned}$$

Bây giờ, bởi vì biểu thức $ab(a^2+b^2)$ xuất hiện khi viết $(a-b)^4$, ta hay xem biểu thức ban đầu có thể viết lại như thế nào:

$$\begin{aligned}\sum ab(a^2+b^2) &= \sum \frac{a^4+b^4+6a^2b^2-(a-b)^4}{4}\\ &= \frac{3\sum a^4+6\sum a^2b^2-\sum(a-b)^4}{4}\\ &= \frac{3-\sum(a-b)^4}{4}\\ &\geq \frac{3}{4}.\end{aligned}$$

Giá trị lớn nhất đạt được với $a=b=c=d=\dfrac{1}{2}$

37. [Walther Janous] Cho x, y, z là các số thực dương. Chứng minh rằng

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

Crux Mathematicorum

Lời giải 1:
Ta có $(x+y)(x+z)=xy+(x^2+yz)+xz \geq xy+2x\sqrt{yz}+xz=(\sqrt{xy}+\sqrt{xz})^2$.
Từ đó

$$\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} \leq \sum \frac{x}{x+\sqrt{xy}+\sqrt{xz}}.$$

Nhưng

$$\sum \frac{x}{x+\sqrt{xy}+\sqrt{xz}}=\sum \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}=1$$

và điều này đã giải quyết bài toán.

Lời giải 2:
Từ bất đẳng thức **Huygens** ta có $\sqrt{(x+y)(x+z)} \geq x+\sqrt{yz}$ và sử dụng bất đẳng thức này cho các bất đẳng thức tương tự ta có

$$\begin{aligned}&\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\\ &\leq \frac{x}{2x+\sqrt{yz}}+\frac{y}{2y+\sqrt{zx}}+\frac{z}{2z+\sqrt{xy}}.\end{aligned}$$

Bây giờ, ta kí hiệu $a = \frac{\sqrt{yz}}{x}, b = \frac{\sqrt{zx}}{y}, c = \frac{\sqrt{xy}}{z}$ và bất đẳng thức trở thành

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1.$$

Từ kí hiệu ở trên ta có thể thấy rằng $abc = 1$, vậy sau khi quy đồng mẫu số bất đẳng thức cuối cùng trở thành $ab + bc + ca \geq 3$, mà bất đẳng thức này suy ra từ bất đẳng thức **AM-GM** .

38. Giả sử $a_1 < a_2 < ... < a_n$ là các số thực, $n \geq 2$ là một số nguyên. Chứng minh rằng

$$a_1{a^4}_2 + a_2{a^4}_3 + ... + a_n{a^4}_1 \geq a_2{a^4}_1 + a_3{a^4}_2 + ... + a_1{a^4}_n.$$

Iran, 1999

Lời giải:
Thoạt nhìn ta thấy ngay rằng sau khi chứng minh bất đẳng thức cho $n = 3$, bất đẳng thức sẽ được chứng minh bằng quy nạp cho n lớn hơn. Vậy, ta phải chứng minh rằng với mọi $a < b < c$ ta có

$$ab(b^3 - a^3) + bc(c^3 - b^3) \geq ca(c^3 - a^3)$$
$$\iff (c^3 - b^3)(ac - bc) \leq (b^3 - a^3)(ab - ac).$$

Bởi vì $a < b < c$, bất đẳng thức cuối cùng quy về

$$a(b^2 + ab + a^2) \leq c(c^2 + bc + b^2).$$

Và bất đẳng thức cuối tương đương với

$$(c - a)(a^2 + b^2 + c^2 + ab + bc + ca) \geq 0,$$

bất đẳng thức này là hiển nhiên.

39. [Mircea Lascu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Lời giải:
Sử dụng bất đẳng thức $\frac{1}{x+y} \leq \frac{1}{4x} + \frac{1}{4y}$ ta suy ra

$$\frac{4a}{b+c} \leq \frac{a}{b} + \frac{a}{c}, \frac{4b}{a+c} \leq \frac{b}{a} + \frac{b}{c} \text{ và } \frac{4c}{a+b} \leq \frac{c}{a} + \frac{c}{b}.$$

Cộng 3 bất đẳng thức trên, ta có điều phải chứng minh.

40. Cho $a_1, a_2, ..., a_n > 1$ là các số nguyên dương. Chứng minh rằng ít nhất một trong các số $\sqrt[a_1]{a_2}$, $\sqrt[a_2]{a_3}, ...,$ $\sqrt[a_{n-1}]{a_2}$, $\sqrt[a_n]{a_1}$ nhỏ hơn hoặc bằng $\sqrt[3]{3}$.

Phỏng theo một bài toán quen biết

Lời giải:
Giả sử ta có $a_{i+1}^{\frac{1}{a_i}} > 3^{\frac{1}{3}}$ với mọi i. Đầu tiền, ta sẽ chứng minh rằng $n^{\frac{1}{n}} \leq 3^{\frac{1}{3}}$ với mọi số tự nhiên n. Với $n = 1, 2, 3, 4$ điều này là hiển nhiên. Giả sử bất đẳng thức đúng cho $n > 3$ và ta chứng minh nó đúng cho $n + 1$. Điều này suy ra từ hệ thức

$$1 + \frac{1}{n} \leq 1 + \frac{1}{4} < \sqrt[3]{3} \Longrightarrow 3^{\frac{n+1}{3}} = \sqrt[3]{3}.3^{\frac{n}{3}} \geq \frac{n+1}{n}.n = n + 1.$$

Vậy, sử dụng nhận xét trên, ta được $a_{i+1}^{\frac{1}{a_i}} > 3^{\frac{1}{3}} \geq a^{\frac{1}{a_{i+1}}} \Longrightarrow a_{i+1} > a_i$ với mọi i, điều này có nghĩa là $a_1 < a_2 < ...a_{n-1} < a_n < a_1$, mâu thuẫn.

41. [Mircea Lascu, Marian Tetiva] Cho x, y, z là các số thực dương thỏa mãn điều kiện

$$xy + xz + yz + 2xyz = 1.$$

Chứng minh các bất đẳng thức sau

a) $xyz \leq \dfrac{1}{8}$;

b) $x + y + z \geq \dfrac{3}{2}$;

c) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 4(x + y + z)$;

d) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 4(x + y + z) \geq \dfrac{(2z-1)^2}{z(2z+1)}$, với $z = \max\{x, y, z\}$.

Lời giải:
a) Đặt $t^3 = xyz$, theo bất đẳng thức **AM-GM** ta có

$$1 = xy + xz + yz + 2xyz \geq 3t^2 + 2t^3 \Longleftrightarrow (2t-1)(t+1)^2 \leq 0,$$

do đó $2t - 1 \leq 0 \Longleftrightarrow t \leq \dfrac{1}{2}$, điều này có nghĩa là $xyz \leq \dfrac{1}{8}$.

b) Cũng kí hiệu $s = x + y + z$; các bất đẳng thức sau đều quen biết

$$(x + y + z)^2 \geq 3(xy + xz + yz)$$

và

$$(x+y+z)^3 \geq 27xyz;$$

khi đó ta có $2s^3 \geq 54xyz = 27 - 27(xy+xz+yz) \geq 27 - 9s^2$, nghĩa là

$$2s^3 + 9s^2 - 27 \geq 0 \iff (2s-3)(s+3)^2 \geq 0,$$

từ đó $2s - 3 \geq 0 \iff s \geq \frac{3}{2}$.
Hoặc, bởi vì $p \leq \frac{1}{8}$, ta có

$$s^2 \geq 3q = 3(1-2p) \geq 3(1-\frac{2}{8}) = \frac{9}{4};$$

nếu ta đặt $q = xy + xz + yz, p = xyz$.
Bây giờ, ta có thể thấy bất đẳng thức sau cũng đúng

$$q = xy + xz + yz \geq \frac{3}{4}.$$

c) Ba số x, y, z không thể tất cả nhỏ hơn $\frac{1}{2}$, bởi vì trong trường hợp này ta gặp mâu thuẫn

$$xy + xz + yz + 2xyz < \frac{3}{4} + 2.\frac{1}{8} = 1;$$

do tính đối xứng ta có thể giả sử $z \geq \frac{1}{2}$.
Ta có $1 = (2z+1)xy + z(x+y) \geq (2z+1)xy + 2z\sqrt{xy}$, mà cũng có thể viết dưới dạng

$$(2z+1)\sqrt{xy} - 1)(\sqrt{xy} + 1) \leq 0;$$

và điều này cho bất đẳng thức

$$xy \leq \frac{1}{(2z+1)^2}.$$

Ta cũng có $1 = (2z+1)xy + z(x+y) \leq (2z+1)\frac{(x+y)^2}{4} + z(x+y)$, do đó

$$2z+1)(x+y) - 2\big)(x+y+z) \geq 0,$$

bất đẳng thức này cho ta

$$x + y \geq \frac{2}{2z+1}.$$

Bất đẳng thức cần chứng minh

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x+y+z)$$

có thể được sắp xếp lại dưới dạng

$$(x+y)\left(\frac{1}{xy}-4\right) \geq \frac{4z^2-1}{z} = \frac{(2z-1)(2z+1)}{z}.$$

Từ những tính toán ở trên ta suy ra rằng

$$(x+y)\left(\frac{1}{xy}-4\right) \geq \frac{2}{2z+1}((2z+1)^2-4) = \frac{2(2z-1)(2z+3)}{2z+1}$$

(giả thiết $z \geq \dfrac{1}{2}$ cho phép nhân vế với vế của các bất đẳng thức), và điều này có nghĩa là bài toán được giải nếu ta chứng minh được

$$\frac{2(2z+3)}{2z+1} \geq \frac{2z+1}{z} \iff 4z^2+6z \geq 4z^2+4z+1;$$

nhưng bất đẳng thức đúng với $z \geq \dfrac{1}{2}$ và ta có điều phải chứng minh.

d) Hiển nhiên, nếu z là số lớn nhất trong các số x, y, z, thì $z \geq \dfrac{1}{2}$; ta thấy rằng

$$\begin{aligned}\frac{1}{x}+\frac{1}{y}-4(x+y) &= (x+y)\left(\frac{1}{xy}-4\right)\\ &\geq \frac{2}{2z+1}((2z+1)^2-4)\\ &= \frac{2(2z-1)(2z+3)}{2z+1}\\ &= 4z-\frac{1}{z}+\frac{(2z-1)^2}{z(2z+1)},\end{aligned}$$

từ đó ta có bất đẳng thức cuối cùng

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-4(x+y+z) \geq \frac{(2z-1)^2}{z(2z+1)}.$$

Hiển nhiên, trong vế phải z có thể được thay thế bởi bất kỳ một trong ba số mà lớn hơn hay bằng $\dfrac{1}{2}$ (hai số còn lại cũng vạy, chắc chắn phải có một số).

Chú ý.
Dễ thấy rằng điều kiện đã cho suy ra tồn tại các số dương a, b, c sao cho $x = \dfrac{a}{b+c}, y = \dfrac{b}{c+a}, z = \dfrac{c}{a+b}$. Và bây giờ a), b) và c) trực tiếp quy về các bất đẳng thức quen biết! Hãy thử chứng minh d) sử dụng sự thay thế này.

42. [Manlio Marangelli] Chứng minh rằng với mọi số thực dương x, y, z,

$$3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x + y + z)^3.$$

Lời giải:
Sử dụng bất đẳng thức **AM-GM** , ta có

$$\frac{1}{3} + \frac{y^2z}{y^2z + z^2x + x^2y} + \frac{xy^2}{yz^2 + zx^2 + xy^2} \geq \frac{3y\sqrt[3]{xyz}}{\sqrt[3]{3.(y^2z + z^2x + x^2y)(yz^2 + zx^2 + xy^2)}}$$

và hai bất đẳng thức tương tự

$$\frac{1}{3} + \frac{z^2x}{y^2z + z^2x + x^2y} + \frac{yz^2}{yz^+zx^2 + xy^2} \geq \frac{3z\sqrt[3]{xyz}}{\sqrt[3]{3.(y^2z + z^2x + x^2y)(yz^2 + zx^2 + xy^2)}}$$

$$\frac{1}{3} + \frac{x^2y}{y^2z + z^2x + x^2y} + \frac{zx^2}{yz^2 + zx^2 + xy^2} \geq \frac{3x\sqrt[3]{xyz}}{\sqrt[3]{3.(y^2z + z^2x + x^2y)(yz^2 + zx^2 + xy^2)}}$$

Khi đó, cộng ba bất đẳng thức lại, ta có chính xác bất đẳng thức cần chứng minh.

43. [Gabriel Dospinescu] Chứng minh rằng nếu a, b, c là các số thực sao cho $\max\{a, b, c\} - \min\{a, b, c\} \leq 1$, thì

$$1 + a^3 + b^3 + c^3 + 6abc \geq 3a^2b + 3b^2c + 3c^2a.$$

Lời giải:
Hiển nhiên ta có thể giả sử $a = \min\{a, b, c\}$ và ta có thể viết $b = a + x, c = a + y$, với $x, y \in [0, 1]$. Dễ thấy rằng

$$a^3 + b^3 + c^3 - 3abc = 3a(x^2 - xy + y^2) + x^3 + y^3$$

và

$$a^2b + b^2c + c^2a - 3abc = a(x^2 - xy + y^2) + x^2y.$$

Vậy, bất đẳng thức trở thành $1 + x^3 + y^3 \geq 3x^2y$. Nhưng điều này suy ra từ hệ thức $1 + x^3 + y^3 \geq 3xy \geq 3x^2y$, bởi vì $0 \leq x, y \leq 1$.

44. [Gabriel Dospinescu] Chứng minh rằng với bất kỳ các số thực dương a, b, c ta có

$$27 + \left(2 + \frac{a^2}{bc}\right)\left(2 + \frac{b^2}{ca}\right)\left(2 + \frac{c^2}{ab}\right) \geq 6(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Lời giải:
Bởi khai triển hai vế, bất đẳng thức tương đương với

$$2abc(a^3 + b^3 + c^3 + 3abc - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b) +$$
$$+(a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 - a^3b^2c - a^3bc^2 - ab^3c^2 - ab^2c^3 - a^2b^3c - a^2bc^3) \geq 0.$$

Nhưng bất đẳng thức này đúng nhờ áp dụng bất đẳng thức **Schur** với a, b, c và ab, bc, ca.

45. Cho $a_0 = \dfrac{1}{2}$ và $a_{k+1} = a_k + \dfrac{a_k^2}{n}$. Chứng minh rằng $1 - \dfrac{1}{n} < a_n < 1$.

TST Singapore

Lời giải:

Ta có

$$\frac{1}{a_{k+1}} = \frac{1}{a_k} - \frac{1}{a_k + n}$$

và do đó

$$\sum_{k=0}^{n-1} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) = \sum_{k=0}^{n-1} \frac{1}{a_k + n} < 1 \Longrightarrow 2 - \frac{1}{a_n} < 1 \Longrightarrow a_n < 1.$$

Bởi vì dãy số này tăng nên ta có

$$2 - \frac{1}{a_n} = \sum_{k=0}^{n-1} \frac{1}{a_k + n} > \frac{n}{n+1}.$$

Từ bất đẳng thức này và bất đẳng thức trước ta trực tiếp suy ra kết luận

$$1 - \frac{1}{n} < a_n < 1.$$

46. [Călin Popa] Cho a, b, c là các số thực dương, với $a, b, c \in (0, 1)$ sao cho $ab+bc+ca = 1$. Chứng minh rằng

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4} \left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c} \right).$$

Lời giải:

Ta đã biết với mọi tam giác ABC đồng nhất thức $\sum \tan \dfrac{A}{2} \tan \dfrac{B}{2} = 1$ đúng và bởi vì tan là toàn ánh trên $[0, \dfrac{\pi}{2})$ chúng ta có thể đặt $a = \tan \dfrac{A}{2}, b = \tan \dfrac{B}{2}, c = \tan \dfrac{C}{2}$. Điều kiện $a, b, c \in (0, 1)$ cho ta các góc của tam giác ABC là góc nhọn. Với sự thay thế này, bất đẳng thức trở thành

$$\sum \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \geq 3 \sum \frac{1 - \tan^2 \frac{A}{2}}{2 \tan \frac{A}{2}}$$

$\Longleftrightarrow \sum \tan A \geq 3 \sum \dfrac{1}{\tan A}$

$\Longleftrightarrow \tan A \tan B \tan C (\tan A + \tan B + \tan C) \geq 3(\tan A \tan B + \tan B \tan C + \tan C \tan A)$

$\Longleftrightarrow (\tan A + \tan B + \tan C)^2 \geq 3(\tan A \tan B + \tan B \tan C + \tan C \tan A)$,

điều này hiển nhiên đúng bởi vì $\tan A, \tan B, \tan C > 0$.

47. [Titu Andreescu, Gabriel Dospinescu] Cho $x, y, z \leq 1$ và $x + y + z = 1$. Chứng minh rằng

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Lời giải:
Sử dụng hệ thức $(4-3t)(1-3t)^2 \geq 0, t \leq 1$, ta thấy rằng

$$\frac{1}{1+x^2} \leq \frac{27}{50}(2-x).$$

Viết hai biểu thức tương tự cho y và z và cộng chúng lại, ta có được bất đẳng thức cần chứng minh.

Chú ý.
Dù có vẻ quá dễ, bài toán này giúp chúng ta chứng minh bất đẳng thức khó sau đây

$$\sum \frac{(b+c-a)^2}{a^2+(b+c)^2} \geq \frac{3}{5}.$$

Thực ra, bài toán này tương đương với bài toán khó trên đây. Hãy thử chứng minh điều này!

48. [Gabriel Dospinescu] Chứng minh rằng nếu $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, thì

$$(1-x)^2(1-y)^2(1-z)^2 \geq 2^{15}xyz(x+y)(y+z)(z+x).$$

Lời giải:
Đặt $a = \sqrt{x}, b = \sqrt{y}$ và $c = \sqrt{z}$. Khi đó

$$1 - x = 1 - a^2 = (a+b+c)^2 - a^2 = (b+c)(2a+b+c).$$

Bây giờ chúng ta phải chứng minh rằng

$$\begin{aligned}&\big((a+b)(b+c)(c+a)(2a+b+c)(a+2b+c)(a+b+2c)\big)^2 \geq \\ &\geq 2^{15}a^2b^2c^2(a^2+b^2)(b^2+c^2)(c^2+a^2).\end{aligned}$$

Nhưng bất đẳng thức này đúng vì nó suy ra từ các bất đẳng thức đúng sau

$$ab(a^2+b^2) \leq \frac{(a+b)^4}{8}$$

(bất đẳng thức này tương đương với $(a-b)^4 \geq 0$) và

$$(2a+b+c)(a+2b+c)(a+b+2c) \geq 8(b+c)(c+a)(a+b).$$

49. Cho x, y, z là các số thực dương thỏa mãn $xyz = x + y + z + 2$. Chứng minh rằng

1) $xy + yz + zx \geq 2(x + y + z)$;

2) $\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz}$.

Lời giải:
Điều kiện ban đầu $xyz = x + y + z + 2$ có thể viết lại như sau

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1.$$

Bây giờ, giả sử

$$\frac{1}{1+x} = a, \frac{1}{1+y} = b, \frac{1}{1+z} = c.$$

Khi đó

$$x = \frac{1-a}{a} = \frac{b+c}{a}, y = \frac{c+a}{b}, z = \frac{a+b}{c}.$$

(1) Ta có

$$\begin{aligned}
&xy + yz + zx \geq 2(x + y + z) \\
\Longleftrightarrow &\frac{b+c}{a}.\frac{c+a}{b} + \frac{c+a}{b}.\frac{a+b}{c} + \frac{a+b}{c}.\frac{b+c}{a} \geq 2\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) \\
\Longleftrightarrow &a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a) \\
\Longleftrightarrow &\sum a(a-b)(a-c) \geq 0,
\end{aligned}$$

đây chính là bất đẳng thức **Schur**.

(2) Ở đây ta có

$$\begin{aligned}
&\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz} \\
\Longleftrightarrow &\sqrt{\frac{a}{b+c}.\frac{b}{c+a}} + \sqrt{\frac{b}{c+a}.\frac{c}{a+b}} + \sqrt{\frac{c}{a+b}.\frac{a}{b+c}} \leq \frac{3}{2}.
\end{aligned}$$

Bất đẳng thức này có thể chứng minh bởi cộng bất đẳng thức

$$\sqrt{\frac{a}{b+c}.\frac{b}{c+a}} = \sqrt{\frac{b}{b+c}.\frac{a}{c+a}} \leq \frac{1}{2}\left(\frac{a}{a+c} + \frac{b}{b+c}\right),$$

với các bất đẳng thức tương tự.

50. Chứng minh rằng nếu x, y, z là các số thực $x^2 + y^2 + z^2 = 2$, thì

$$x + y + z \leq xyz + 2.$$

IMO Shorlist, 1987

Lời giải 1:
Nếu một trong các số x, y, z là âm, chẳng hạn là x, thì

$$2 + xyz - x - y - z = (2 - y - z) - x(1 - yz) \geq 0,$$

bởi vì $y + z \leq \sqrt{2(y^2 + z^2)} \leq 2$ và $zy \leq \dfrac{z^2 + y^2}{2} \leq 1$. Vậy ta có thể giả sử rằng $0 < x \leq y \leq z$. Nếu $z \leq 1$ thì

$$2 + xyz - x - y - z = (1 - z)(1 - xy) + (1 - x)(1 - y) \geq 0.$$

Bây giờ, nếu $z > 1$ thì ta có

$$z + (x + y) \leq \sqrt{2(z^2 + (x + y)^2)} = 2\sqrt{1 + xy} \leq 2 + xy \leq 2 + xyz.$$

Điều này kết thúc chứng minh.

Lời giải 2:
Sử dụng bất đẳng thức **Cauchy-Schwarz** , ta thấy rằng

$$x + y + z - xyz = x(1 - yz) + y + z \leq \sqrt{(x^2 + (y + z)^2).(1 + (1 - yz)^2)}.$$

Vậy, ta chỉ cần chứng minh rằng vế phải không lớn hơn 2. Điều này tương đương với

$$(2 + 2yz)(2 - yz + (yz)^2) \leq 4 \Longleftrightarrow 2(yz)^3 \leq 2(yz)^2,$$

điều nay hiển nhiên đúng, bởi vì $2 \geq y^2 + z^2 \geq 2yz$.

51. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng với mọi $x_1, x_2, ..., x_n \in (0, 1)$ và với mọi phép hoán vị σ của tập $\{1, 2, ..., n\}$, ta có bất đẳng thức

$$\sum_{i=1}^{n} \frac{1}{1 - x_i} \geq \left(1 + \frac{\sum\limits_{i=1}^{n} x_i}{n}\right) . \left(\sum_{i=1}^{n} \frac{1}{1 - x_i . x_{\sigma(i)}}\right).$$

Lời giải:
Sử dụng bất đẳng thức **AM-GM** và hệ thức $\dfrac{1}{x+y} \geq \dfrac{1}{4x} + \dfrac{1}{4y}$, ta có thể viết chuỗi bất đẳng thức dưới đây

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{1 - x_i.y_i} &\leq \sum_{i=1}^{n} \frac{1}{1 - \dfrac{x_i^2 + y_i^2}{2}} \\ &= 2\sum_{i=1}^{n} \frac{1}{1 - x_i^2 + 1 - y_i^2} \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{2(1 - x_i^2)} + \frac{1}{2(1 - y_i^2)} \right),\end{aligned}$$

suy ra

$$\sum_{i=1}^{n} \frac{1}{1 - x_i.x_{\sigma(i)}} \leq \sum_{i=1}^{n} \frac{1}{1 - x_i^2}.$$

Vậy, ta chỉ cần chứng minh bất đẳng thức

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{1 - x_i} \geq \left(1 + \frac{\sum\limits_{i=1}^{n} x_i}{n} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1 - x_i^2} \right) \\ \Longleftrightarrow &\sum_{i=1}^{n} \frac{x_i}{1 - x_i^2} \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1 - x_i^2} \right).\end{aligned}$$

Đó là bất đẳng thức **Chebyshev** cho hệ $(x_1, x_2, ..., x_n)$ và

$$\left(\frac{1}{1 - x_1^2}, \frac{1}{1 - x_2^2}, ..., \frac{1}{1 - x_n^2} \right).$$

52. Cho $x_1, x_2, ..., x_n$ là các số thực dương sao cho $\sum\limits_{i=1}^{n} \dfrac{1}{1 + x_i} = 1$. Chứng minh rằng

$$\sum_{i=1}^{n} \sqrt{x_i} \geq (n - 1) \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}.$$

Vojtech Jarnik

Lời giải 1:
Đặt $\dfrac{1}{1+x_i} = a_i$, bất đẳng thức trở thành

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \geq (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$
$$\Longleftrightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{a_i(1-a_i)}} \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$
$$\Longleftrightarrow n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \leq \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} \frac{1}{\sqrt{a_i(1-a_i)}}\right).$$

Nhưng bất đẳng thức cuối cùng là hệ quả của bất đẳng thức **Chebyshev** cho các bộ n số $(a_1, a_2, ..., a_n)$ và

$$\frac{1}{\sqrt{a_1(1-a_1)}}, \frac{1}{\sqrt{a_2(1-a_2)}}, .., \frac{1}{\sqrt{a_n(1-a_n)}}.$$

Lời giải 2:
Với cùng kí hiệu trên, ta phải chứng minh

$$(n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{a_1 + a_2 + ... + a_{i-1} + a_{i+1} + ... + a_n}} \leq$$
$$\leq \sum_{i=1}^{n} \sqrt{\frac{a_1 + a_2 + ... + a_{i-1} + a_{i+1} + ... + a_n}{a_i}}.$$

Nhưng sử dụng bất đẳng thức **Cauchy-Schwarz** và bất đẳng thức **AG-GM**, ta suy ra rằng

$$\sum_{i=1}^{n} \sqrt{\frac{a_1 + a_2 + ... + a_{i-1} + a_{i+1} + ... + a_n}{a_i}}$$
$$\geq \sum_{i=1}^{n} \frac{\sqrt{a_1} + \sqrt{a_2} + ... + \sqrt{a_{i-1}} + \sqrt{a_{i+1}} + ... + \sqrt{a_n}}{\sqrt{n-1}\sqrt{a_i}}$$
$$= \sum_{i=1}^{n} \frac{\sqrt{a_i}}{\sqrt{n-1}} \left(\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + ... + \frac{1}{\sqrt{a_{i-1}}} + \frac{1}{\sqrt{a_{i+1}}} + ... + \frac{1}{\sqrt{a_n}}\right)$$
$$\geq \sum_{i=1}^{n} \frac{(n-1)\sqrt{n-1}\sqrt{a_i}}{\sqrt{a_1} + \sqrt{a_2} + ... + \sqrt{a_{i-1}} + \sqrt{a_{i+1}} + ... + \sqrt{a_n}}$$
$$\geq \sum_{i=1}^{n} (n-1) \sqrt{\frac{a_i}{a_1 + a_2 + ... + a_{i-1} + a_{i+1} + ... + a_n}}$$

và ta có điều phải chứng minh.

53. [Titu Andreescu] Cho $n > 3$ và $a_1, a_2, ..., a_n$ là các số thực thỏa mãn $a_1+a_2+...+a_n \geq n$ và $a_1^2 + a_2^2 + ... + a_n^2 \geq n^2$. Chứng rằng $\max\{a_1, a_2, ..., a_n\} \geq 2$.

USAMO, 1999

Lời giải:
Ý tưởng tự nhiên nhất là giả sử rằng $a_i < 2$ với mọi i và thay thế $x_i = 2 - a_i > 0$. Khi đó ta có $\sum\limits_{i=1}^{n}(2 - x_i) \geq n \Longrightarrow \sum\limits_{i=1}^{n} x_i \leq n$ và cũng có

$$n^2 \leq \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n}(2 - x_i)^2 = 4n - 4\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2.$$

Bây giờ, sử dụng giả thiết rằng $x_i > 0$, ta được $\sum\limits_{i=1}^{n} x_i^2 < \left(\sum\limits_{i=1}^{n} x_i\right)^2$, kết hợp với các bất đẳng thức ở trên ta có

$$n^2 < 4n - 4\sum_{i=1}^{n} x_i + \left(\sum_{i=1}^{n} x_i\right)^2 < 4n + (n-4)\sum_{i=1}^{n} x_i$$

(ta sử dụng hệ thức $\sum\limits_{i=1}^{n} x_i \leq n$). Do đó, ta có $(n-4)\left(\sum\limits_{i=1}^{n} x_i - n\right) > 0$, bất đẳng thức này không xảy ra vì $n \geq 4$ và $\sum\limits_{i=1}^{n} x_i \leq n$. Vậy, giả sử của chúng ta là sai và do đó $\max\{a_1, a_2, ..., a_n\} \geq 2$.

54. [Vasile Cirtoaje] Nếu a, b, c, d là các số thực dương, thì

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

Lời giải:
Ta có

$$\begin{aligned}
&\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \\
=&\frac{a+c}{b+c} + \frac{b+d}{c+d} + \frac{c+a}{d+a} + \frac{b+d}{a+b} - 4 \\
=&(a+c)\left(\frac{1}{b+c} + \frac{1}{d+a}\right) + (b+d)\left(\frac{1}{c+d} + \frac{1}{a+b}\right) - 4.
\end{aligned}$$

Từ

$$\frac{1}{b+c}+\frac{1}{d+a}\geq\frac{4}{(b+c)+(d+a)},$$
$$\frac{1}{c+d}+\frac{1}{a+b}\geq\frac{4}{(c+d)+(a+b)},$$

ta có

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq\frac{4(a+c)}{(b+c)+(d+a)}+\frac{4(b+d)}{(c+d)+(a+b)}-4=0.$$

Dấu bằng xảy ra khi $a=c$ và $b=d$.

Giải thuyết (Vasile Cirtoaje)
Nếu a,b,c,d,e là các số thực dương, thì

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\geq 0.$$

55. Cho x và y là các số thực dương, chỉ ra rằng $x^y+y^x>1$.

France, 1996

Lời giải:
Ta sẽ chứng minh rằng $a^b\geq\dfrac{a}{a+b-ab}$ với mọi $a,b\in(0,1)$. Thật vậy, từ bất đẳng thức **Bernoulli** suy ra rằng

$$a^{1-b}=(1+a-1)^{1-b}\leq 1+(a-1)(1-b)=a+b-ab$$

và suy ra kết luận. Bây giờ, nếu x hoặc y lớn hơn 1, ta có điều phải chứng minh. Ngược lại, giả sử $0<x,y<1$. Trong trường hợp này ta áp dụng sự bất đẳng thức ở trên và thấy rằng

$$x^y+y^x\geq\frac{x}{x+y-xy}+\frac{y}{x+y-xy}>\frac{x}{x+y}+\frac{y}{x+y}=1.$$

56. Chứng minh rằng nếu $a,b,c>0$ có tích bằng 1, thì

$$(a+b)(b+c)(c+a)\geq 4(a+b+c-1).$$

MOSP, 2001

Lời giải 1:
Sử dụng đồng nhất thức $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - 1$ ta quy bài toán thành
$$ab + bc + ca + \frac{3}{a+b+c} \geq 4.$$
Bây giờ, ta có thể áp dụng bất đẳng thức **AM-GM** cho dạng sau
$$ab + bc + ca + \frac{3}{a+b+c} \geq 4\sqrt[4]{\frac{(ab+bc+ca)^3}{9(a+b+c)}}.$$
Vậy ta chỉ cần chứng minh
$$(ab+bc+ca)^3 \geq 9(a+b+c).$$
Nhưng bất đẳng thức này dễ, bởi vì ta có $ab+bc+ca \geq 3$ và $(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$.

Lời giải 2:
Ta sẽ sử dụng hệ thức
$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca).$$
Vậy ta chỉ cần chứng minh $\frac{2}{9}(ab+bc+ca) + \frac{1}{a+b+c} \geq 1$. Sử dụng bất đẳng thức **AM-GM** , ta có thể viết
$$\frac{2}{9}(ab+bc+ca) + \frac{1}{a+b+c} \geq 3\sqrt[3]{\frac{(ab+bc+ca)^2}{81(a+b+c)}} \geq 1,$$
bởi vì
$$(ab+bc+ca)^2 \geq 3abc(a+b+c).$$

57. Chứng minh rằng với mọi $a, b, c > 0$,
$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \leq abc(ab+bc+ca).$$

Lời giải:
Rõ ràng, nếu một trong các thừa số ở vế trái là âm, ta có điều phải chứng minh. Vậy, ta có thể giả thiết rằng a, b, c là các cạnh của tam giác ABC. Với kí hiệu thông thường của một tam giác, bất đẳng thức trở thành
$$(a^2+b^2+c^2).\frac{16K^2}{a+b+c} \leq abc(ab+bc+ca)$$
$$\iff (a+b+c)(ab+bc+ca)R^2 \geq abc(a^2+b^2+c^2).$$

Nhưng bất đẳng thức trên suy ra từ hệ thức

$$(a+b+c)(ab+bc+ca) \geq 9abc$$

và

$$0 \leq OH^2 = 9R^2 - a^2 - b^2 - c^2.$$

58. [D.P.Mavlo] Cho $a, b, c > 0$. Chứng minh rằng

$$3 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{1+abc}.$$

Kvant, 1988

Lời giải:
Bất đẳng thức tương đương với bất đẳng thức dưới đây

$$\sum a + \sum \frac{1}{a} + \sum \frac{a}{b} \geq 3\frac{\sum ab + \sum a}{abc+1}$$

hoặc

$$abc\sum a + \sum \frac{1}{a} + \sum a^2c + \sum \frac{a}{b} \geq 2\left(\sum a + \sum ab\right).$$

Nhưng bất đẳng thức này suy ra từ các bất đẳng thức

$$a^2bc + \frac{b}{c} \geq 2ab, b^2ca + \frac{c}{a} \geq 2bc, c^2ab + \frac{a}{b} \geq 2ca$$

và

$$a^2c + \frac{1}{c} \geq 2a, b^2a + \frac{1}{a} \geq 2b, c^2b + \frac{1}{b} \geq 2c.$$

59. [Gabriel Dospinescu] Chứng minh rằng với mọi số thực dương $x_1, x_2, ..., x_n$ với tích bằng 1 ta có bất đẳng thức

$$n^n.\prod_{i=1}^{n}(x_i^n + 1) \geq \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{1}{x_i}\right)^n.$$

Lời giải:
Sử dụng bất đẳng thức **AM-GM** , ta suy ra rằng

$$\frac{x_1^n}{1+x_1^n} + \frac{x_2^n}{1+x_2^n} + ... + \frac{x_{n-1}^n}{1+x_{n-1}^n} + \frac{1}{1+x_n^n} \geq \frac{n}{x_n.\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}}$$

và
$$\frac{1}{1+x_1^n}+\frac{1}{1+x_2^n}+...+\frac{1}{1+x_{n-1}^n}+\frac{x_n^n}{1+x_n^n}\geq\frac{n.x_n}{\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}}.$$
Vậy, ta phải có
$$\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}\geq x_n+\frac{1}{x_n}.$$
Hiển nhiên, bất đẳng thức này cũng đúng cho các biến khác, vậy ta có thể cộng tất cả các bất đẳng thức lại để thu được
$$n.\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}\geq\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{1}{x_i},$$
đây là bất đẳng thức cần chứng minh.

60. Cho $a,b,c,d>0$ thỏa mãn $a+b+c=1$. Chứng minh rằng
$$a^3+b^3+c^3+abcd\geq\min\left\{\frac{1}{4},\frac{1}{9}+\frac{d}{27}\right\}.$$

Kvant, 1993

Lời giải:
Giả sử rằng bất đẳng thức không đúng. Khi đó ta có, do $abc\leq\dfrac{1}{27}$, bất đẳng thức $d(\dfrac{1}{27}-abc)>a^3+b^3+c^3-\dfrac{1}{9}$. Ta có thể giả sử rằng $abc<\dfrac{1}{27}$. Bây giờ, ta sẽ chỉ ra sự mâu thuẫn bằng chứng minh $a^3+b^3+c^3+abcd\geq\dfrac{1}{4}$. Ta chỉ cần chứng minh rằng
$$\frac{a^3+b^3+c^3-\frac{1}{9}}{\frac{1}{27}-abc}.abc+a^3+b^3+c^3\geq\frac{1}{4}.$$
Nhưng bất đẳng thức này tương đương với $4\sum a^3+15abc\geq 1$. Ta sử dụng đồng nhất thức $\sum a^3=3abc+1-3\sum ab$ và quy bài toán về chứng minh $\sum ab\leq\dfrac{1+9abc}{4}$, đây là bất đẳng thức **Schur**.

61. Chứng minh rằng với mọi số thực a,b,c ta có bất đẳng thức
$$\sum(1+a^2)^2(1+b^2)^2(a-c)^2(b-c)^2\geq(1+a^2)(1+b^2)(1+c^2)(a-b)^2(b-c^2)(c-a^2).$$

Lời giải:
Bất đẳng thức có thể được viết dưới dạng $\sum \frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} \geq 1$ (tất nhiên, ta có thể giả thiết a, b, c khác nhau). Bây giờ, cộng các bất đẳng thức

$$\frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} + \frac{(1+b^2)(1+c^2)}{(1+a^2)(b-c)^2} \geq 2\frac{1+b^2}{|a-b||c-b|}$$

(nhận được bởi sử dụng bất đẳng thức **AM-GM**) ta suy ra rằng

$$\sum \frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} \geq \sum \frac{1+b^2}{|(b-a)(b-c)|},$$

vậy ta chỉ cần chứng minh rằng hệ thức cuối ít nhất bằng 1. Nhưng điều này suy ra từ

$$\sum \frac{1+b^2}{|(b-a)(b-c)|} \geq \left|\sum \frac{1+b^2}{(b-a)(b-c)}\right| = 1$$

và bài toán đã giải xong.

62. [Titu Andreescu, Mircea Lascu] Cho α, x, y, z là các số thực dương thỏa mãn $xyz = 1$ và $\alpha \geq 1$. Chứng minh rằng

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

Lời giải 1:
Ta có thể giả sử rằng $x \geq y \geq z$. Khi đó ta có

$$\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}$$

và $x^{\alpha-1} \geq y^{\alpha-1} \geq z^{\alpha-1}$. Sử dụng bất đẳng thức **Chebyshev** ta suy ra

$$\sum \frac{x^\alpha}{y+z} \geq \frac{1}{3}.\left(\sum x^{\alpha-1}\right).\left(\sum \frac{x}{y+z}\right).$$

Bây giờ, ta chỉ cần để ý rằng kết luận suy ra từ các bất đẳng thức $\sum x^{\alpha-1} \geq 3$ (từ bất đẳng thức **AM-GM**) và $\sum \frac{x}{y+z} \geq \frac{3}{2}$.

Lời giải 2:
Theo bất đẳng thức **Cauchy-Schwarz** , ta có:

$$\big[x(y+z)+y(z+x)+z(x+y)\big]\left(\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y}\right) \geq \big(x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}}\big)^2.$$

Vậy ta cần chỉ ra rằng

$$\left(x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}}\right)^2 \geq 3(xy + yz + zx).$$

Từ $(x+y+z)^2 \geq 3(xy+yz+zx)$, suy ra chỉ cần chứng minh

$$x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}} \geq x + y + z.$$

Từ bất đẳng thức **Bernoulli**, ta có

$$x^{\frac{1+\alpha}{2}} = [1 + (x-1)]^{\frac{1+\alpha}{2}} \geq 1 + \frac{1+\alpha}{2}(x-1) = \frac{1-\alpha}{2} + \frac{1+\alpha}{2}x$$

và tương tự,

$$y^{\frac{1+\alpha}{2}} \geq \frac{1-\alpha}{2} + \frac{1+\alpha}{2}y, \quad z^{\frac{1+\alpha}{2}} \geq \frac{1-\alpha}{2} + \frac{1+\alpha}{2}z.$$

Suy ra

$$\begin{aligned} x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}} - (x+y+z) &\geq \frac{3(1-\alpha)}{2} + \frac{1+\alpha}{2}(x+y+z) - (x+y+z) \\ &= \frac{\alpha-1}{2}(x+y+z-3) \\ &\geq \frac{\alpha-1}{2}(3\sqrt[3]{xyz} - 3) \\ &= 0. \end{aligned}$$

Dấu bằng xảy ra khi $x = y = z = 1$.

Chú ý:
Thay $\beta = \alpha + 1 (\beta \geq 2)$ và $x = \dfrac{1}{a}, y = \dfrac{1}{b}, z = \dfrac{1}{c} (abc = 1)$ bất đẳng thức trở thành

$$\frac{1}{a^\beta(b+c)} + \frac{1}{b^\beta(c+a)} + \frac{1}{c^\beta(a+b)} \geq \frac{3}{2}.$$

Cho $\beta = 3$, ta thu được bài toán IMO 1995 (đươc đề nghị bởi Russia).

63. Chứng minh rằng với mọi số thực $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$ thỏa mãn $x_1^2 + x_2^2 + ... + x_n^2 = y_1^2 + y_2^2 + ... + y_n^2 = 1$, ta có

$$(x_1y_2 - x_2y_1)^2 \leq 2\left(1 - \sum_{k=1}^{n} x_k y_k\right).$$

Korea, 2001

Lời giải:
Hiển nhiên ta có bất đẳng thức

$$
\begin{aligned}
(x_1.y_2 - x_2.y_1)^2 &\le \sum_{1\le i<j\le n} (x_iy_j - x_jy_i)^2 \\
&= \left. \sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right) - \left(\sum_{i-1}^{n} x_iy_i \right)^2 \\
&= \left. 1 - \sum_{i=1}^{n} x_iy_i \right) \left(1 + \sum_{i=1}^{n} x_iy_i \right).
\end{aligned}
$$

Bởi vì ta cũng có $\left|\sum\limits_{i=1}^{n} x_iy_i\right| \le 1$, ta thấy ngay rằng

$$\left. 1 - \sum_{i=1}^{n} x_iy_i \right) \left(1 + \sum_{i=1}^{n} x_iy_i \right) \le 2 \quad \left. 1 - \sum_{i=1}^{n} x_iy_i \right)$$

và bài toán đã giải xong.

64. [Laurentiu Panaitopol] Cho $a_1, a_2, ..., a_n$ là các số nguyên dương đôi một khác nhau. Chứng minh rằng

$$a_1^2 + a_2^2 + ... + a_n^2 \ge \frac{2n+1}{3}(a_1 + a_2 + ... + a_n).$$

TST Romania

Lời giải:
Không mất tổng quát, ta có thể giả sử rằng $a_1 < a_2 < ... < a_n$ và suy ra $a_i \ge i$ với mọi i. Vậy, ta có thể đặt $b_i = a_i - i \ge 0$ và bất đẳng thức trở thành

$$\sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} ib_i + \frac{n(n+1)(2n+1)}{6} \ge \frac{2n+1}{3}.\sum_{i=1}^{n} b_i + \frac{n(n+1)(2n+1)}{6}.$$

Bây giờ, sử dụng hệ thức $a_{i+1} > a_i$ ta suy ra $b_1 \le b_2 \le ... \le b_n$ và từ bất đẳng thức **Chebyshev** ta suy ra rằng

$$2\sum_{i=1}^{n} ib_i \ge (n+1)\sum_{i=1}^{n} b_i \ge \frac{2n+1}{3}\sum_{i=1}^{n} b_i$$

và trực tiếp suy ra kết luận. Và cũng từ các hệ thức ở trên ta có thể trực tiếp suy ra rằng dấu bằng xảy ra khi và chỉ khi $a_1, a_2, ..., a_n$ là một hoán bị của các số $1, 2, ...n$.

65. [Cawlin Popa] Cho a, b, c là các số thực dương thỏa mãn $a+b+c=1$. Chứng minh rằng

$$\frac{b\sqrt{c}}{a(\sqrt{3c}+\sqrt{ab})}+\frac{c\sqrt{a}}{b(\sqrt{3a}+\sqrt{bc})}+\frac{a\sqrt{b}}{c(\sqrt{3b}+\sqrt{ca})}\geq\frac{3\sqrt{3}}{4}.$$

Lời giải:
Viết lại bất đẳng thức dưới dạng

$$\sum\frac{\sqrt{\dfrac{bc}{a}}}{\sqrt{\dfrac{3ca}{b}+a}}\geq\frac{3\sqrt{3}}{4}.$$

Với sự thay thế $x=\sqrt{\dfrac{bc}{a}}, y=\sqrt{\dfrac{ca}{b}}, z=\sqrt{\dfrac{ab}{c}}$, điều kiện $a+b+c=1$ trở thành $xy+yz+zx=1$ và bất đẳng thức trở thành

$$\sum\frac{x}{\sqrt{3}y+yz}\geq\frac{3\sqrt{3}}{4}.$$

Nhưng, bởi áp dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$\sum\frac{x^2}{\sqrt{3}xy+xyz}\geq\frac{(\sum x)^2}{\sqrt{3}+3xyz}\geq\frac{3\sum xy}{\sqrt{3}+\dfrac{1}{\sqrt{3}}}=\frac{3\sqrt{3}}{4},$$

ở đây chúng ta sử dụng các bất đẳng thức

$$\left(\sum x\right)^2\geq 3\left(\sum xy\right) \text{ và } xyz\leq\frac{1}{3\sqrt{3}}.$$

66. [Titu Andreescu, Gabriel Dospinescu] Cho a, b, c là các số thực thỏa mãn

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2)=16.$$

Chứng minh rằng

$$-3\leq ab+bc+cd+da+ac+bd-abcd\leq 5.$$

Lời giải:
Ta viết lại điều kiện đã cho dưới dạng $16=\prod(i+a).\prod(a-i)$. Sử dụng các tổng đối xứng, ta có thể viết lại biểu thức trên dưới dạng

$$16=\left(1-i\sum a-\sum ab+i\sum abc+abcd\right)\left(1+i\sum a-\sum ab-i\sum abc+abcd\right).$$

Vậy ta có đồng nhất thức

$$16 = \left(1 - \sum ab + abcd\right)^2 + \left(\sum a - \sum abc\right)^2.$$

Điều này có nghĩa là

$$\left|1 - \sum ab + abcd\right| \leq 4$$

và từ đây ta có điều phải chứng minh.

67. Chứng minh rằng

$$(a^2+2)(b^2+2)(c^2+2) \geq 2(ab+bc+ca)$$

với mọi a, b, c là các số thực dương.

APMO, 2004

Lời giải 1:
Ta sẽ chứng minh bất đẳng thức mạnh hơn: $(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2$. Bởi vì $(a+b+c)^2 \leq (|a|+|b|+|c|)^2$, ta có thể giả sử rằng a, b, c là không âm. Ta sẽ sử dụng một sự kiện là nếu x và y cùng dấu thì $(1+x)(1+y) \geq 1+x+y$. Vậy, ta viết bất đẳng thức dưới dạng

$$\prod \left(\frac{a^2-1}{3} + 1\right) \geq \frac{(a+b+c)^2}{9}$$

và ta có các trường hợp
i) Nếu a, b, c ít nhất bằng 1 thì $\prod \left(\frac{a^2-1}{3} + 1\right) \geq 1 + \sum \frac{a^2-1}{3} \geq \frac{(\sum a)^2}{9}$.
ii) Nếu có hai trong ba số ít nhất bằng 1, giả sử chúng là a và b, khi đó ta có

$$\begin{aligned}\prod \left(\frac{a^2-1}{3} + 1\right) &\geq \left(1 + \frac{a^2-1}{3} + \frac{b^2-1}{3}\right)\left(\frac{c^2+2}{3}\right) \\ &= \frac{a^2+b^2+1}{3} . \frac{1^2+1^2+c^2}{3} \\ &\geq \frac{(a+b+c)^2}{9}\end{aligned}$$

bởi bất đẳng thức **Cauchy-Schwarz** .
iii) Nếu cả ba số nhiều nhất bằng 1, thì bởi bất đẳng thức **Bernoulli** ta có

$$\prod \left(\frac{a^2-1}{3} + 1\right) \geq 1 + \sum \frac{a^2-1}{3} \geq \frac{\left(\sum a\right)^2}{9}$$

và hoàn thành chứng minh.

Lời giải 2:
Khai triển mọi số hạng, ta quy bài toán về chứng minh

$$(abc)^2 + 2\sum a^2b^2 + 4\sum a^2 + 8 \geq 9\sum ab.$$

Bởi vì $3\sum a^2 \geq 3\sum ab$ và $2\sum a^2b^2 + 6 \geq 4\sum ab$, ta còn lại bất đẳng thức $(abc)^2 + \sum a^2 + 2 \geq 2\sum ab$. Tất nhiên, ta có thể giả sử rằng a, b, c là các số không âm và ta có thể viết $a = x^2, b = y^2, c = z^2$. Trong trường hợp này

$$2\sum ab - \sum a^2 = (x+y+z)(x+y-z)(y+z-x)(z+x-y).$$

Hiển nhiên rằng nếu x, y, z không là độ dài các cạnh của một tam giác, thì bất đẳng thức là tầm thường. Ngược lại, ta có thể đặt $x = u+v, y = v+w, z = w+u$ và bất đẳng thức được quy về

$$[(u+v)(v+w)(w+u)]^4 + 2 \geq 16(u+v+w)uvw.$$

Ta có $[(u+v)(v+w)(w+u)]^4 + 1 + 1 \geq 3\sqrt[3]{(u+v)^4(v+w)^4(u+w)^4}$ và ta chỉ còn phải chứng minh rằng vế phải bất đẳng thức cuối ít nhất là $16(u+v+w)uvw$. Điều này dẫn đến

$$(u+v)^4(v+w)^4(w+u)^4 \geq \frac{16^3}{3^3}(uvw)^3(u+v+w)^3.$$

Nhưng bất đẳng thức này suy ra từ các bất đẳng thức đã biết

$$\begin{aligned}
&(u+v)(v+w)(w+u) \geq \frac{8}{9}(u+v+w)(uv+vw+wu),\\
&(uv+vw+wu)^4 \geq 3^4(uvw)^{\frac{8}{3}},\\
&u+v+w \geq 3\sqrt[3]{uvw}.
\end{aligned}$$

Lời giải 3:
Tương tự như trong **Lời giải 2**, ta quy bài toán về chứng minh rằng

$$(abc)^2 + 2 \geq 2\sum ab - \sum a^2.$$

Bây giờ, sử dụng bất đẳng thức **Schur**, chúng ta suy ra rằng

$$2\sum ab - \sum a^2 \leq \frac{9abc}{a+b+c}$$

và như là một hệ quả trực tiếp của bất đẳng thức **AM-GM** ta có

$$\frac{9abc}{a+b+c} \leq 3\sqrt[3]{(abc)^2}.$$

Điều này chỉ ra rằng khi ta chứng minh

$$(abc)^2 + 2 \geq 3\sqrt[3]{(abc)^2},$$

thì bài toán được giải xong. Nhưng khẳng định cuối cùng được suy từ bất đẳng thức **AM-GM** .

68. [Vasile Cirtoaje] Chứng minh rằng nếu $0 < x \leq y \leq z$ và $x + y + z = xyz + 2$, thì

a) $(1 - xy)(1 - yz)(1 - xz) \geq 0$;

b) $x^2y \leq 1, x^3y^2 \leq \dfrac{32}{27}$.

Lời giải:
a) Ta có

$$\begin{aligned}(1 - xy)(1 - yz) &= 1 - xy - yz + xy^2z \\ &= 1 - xy - yz + y(x + y + z - 2) \\ &= (y - 1)^2 \geq 0,\end{aligned}$$

và tương tự

$$(1 - yz)(1 - zx) = (1 - z)^2 \geq 0, \quad (1 - zx)(1 - xy) = (1 - x)^2 \geq 0.$$

Vậy các biểu thức $1 - xy, 1 - yz$, và $1 - zx$ có cùng dấu.

b) Ta viết lại hệ thức

$$x + y + z = xyz + 2$$

như sau

$$(1 - x)(1 - y) + (1 - z)(1 - xy) = 0.$$

Nếu $x > 1$ thì $z \geq y \geq x > 1$ và do đó $(1 - x)(1 - y) + (1 - z)(1 - xy) > 0$, điều này không thể xảy ra. Vậy ta có $x \leq 1$. Tiếp theo ta phân biệt hai trường hợp 1) $xy \leq 1$; 2) $xy > 1$.

1) $xy \leq 1$. Ta có $x^2y \leq x \leq 1$ và $x^3y^2 \leq x \leq 1 < \dfrac{32}{27}$.

2) $xy > 1$. Từ $y \geq \sqrt{xy}$ ta có $y > 1$. Tiếp theo ta viết lại hệ thức $x + y + z = xyz + 2$ thành $x + y - 2 = (xy - 1)z$. Bởi vì $z \geq y$ cho ta

$$x + y - 2 \geq (xy - 1)y \Longrightarrow (y - 1)(2 - x - xy) \geq 0,$$

vậy $2 \geq x(1 + y)$. Sử dụng bất đẳng thức **AM-GM** , ta có

$$1 + y \geq 2\sqrt{y} \text{ và } 1 + y = 1 + \frac{y}{2} + \frac{y}{2} \geq 3\sqrt[3]{1.\frac{y}{2}.\frac{y}{2}}.$$

Do đó

$$2 \geq 2x\sqrt{y} \text{ và } 2 \geq 3x\sqrt[3]{\frac{y^2}{4}},$$

Điều này có nghĩa là $x^2y \leq 1$ và $x^3y^2 \leq \frac{32}{27}$. Đẳng thức $x^2y = 1$ xảy ra khi $x = y = 1$ và đẳng thức $x^3y^2 = \frac{32}{27}$ xảy ra khi $x = \frac{2}{3}, y = z = 2$.

69. [Titu Andreescu] Cho a, b, c là các số thực dương thỏa mãn $a + b + c \geq abc$. Chứng minh rằng có ít nhất hai trong các bất đẳng thức

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

đúng.

TST 2001, USA

Lời giải:

Ý tưởng tự nhiên là thay thế $\frac{1}{a} = x, \frac{1}{b} = y, \frac{1}{c} = z$. Khi đó, ta có $x, y, z > 0$ và $xy + yz + zx \geq 1$ và ta phải chứng minh ít nhất hai trong các bất đẳng thức

$$2x + 3y + 6z \geq 6, \quad 2y + 3z + 6x \geq 6, \quad 2z + 3x + 6y \geq 6$$

đúng. Giả sử trường hợp này không xảy ra. Khi đó ta có thể giả sử rằng

$$2x + 3y + 6z < 6 \text{ và } 2z + 3x + 6y < 6.$$

Cộng lại, ta được $5x + 9y + 8z < 12$. Nhưng chúng ta có $x \geq \frac{1 - yz}{y + z}$. Suy ra $12 > \frac{5 - 5yz}{y + z} + 9y + 8z$, bất đẳng thức này chính là

$$12(y + z) > 5 + 9y^2 + 8z^2 + 12yz \Longleftrightarrow (2z - 1)^2 + (3y + 2z - 2)^2 < 0,$$

Điều này hiển nhiên không xảy ra. Do đó ta có điều phải chứng minh.

70. [Gabriel Dospinescu, Marian Tetiva] Cho $x, y, z > 0$ thỏa mãn

$$x + y + z = xyz.$$

Chứng minh rằng

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

Lời giải 1:
Bởi vì $x < xyz \Longrightarrow yz > 1$ (và các hệ thức tương tự $xz > 1, xy > 1$) nên có nhiều nhất một trong trong ba số nhỏ hơn 1. Trong bất kì trường hợp nào ($x \leq 1, y \geq 1, z \geq 1$ hoặc các trường hợp tương tự) bất đẳng thức cần chứng minh hiển nhiên đúng. Còn duy nhất một trường hợp ta phải phân tích là khi $x \geq 1, y \geq 1$ và $z \geq 1$.
Trong trường hợp này ta kí hiệu

$$x - 1 = a, y - 1 = b, z - 1 = c.$$

Khi đó a, b, c là các số thực không âm, và bởi vì

$$x = a + 1, y = b + 1, z = c + 1$$

thỏa mãn

$$a + 1 + b + 1 + c + 1 = (a + 1)(b + 1)(c + 1),$$

điều này có nghĩa là

$$abc + ab + ac + bc = 2.$$

Đặt $x = \sqrt[3]{xyz}$; ta có

$$ab + bc + ca \geq 3\sqrt[3]{abbcca} = 3x^2,$$

khi đó ta có

$$\begin{aligned}
&x^3 + 3x^2 \leq 2 \\
\Longleftrightarrow &(x + 1)(x^2 + 2x - 2) \leq 0 \\
\Longleftrightarrow &(x + 1)(x + 1 + \sqrt{3})(x + 1 - \sqrt{3}) \leq 0.
\end{aligned}$$

Do $x \geq 0$, điều này suy ra

$$\sqrt[3]{abc} = x \leq \sqrt{3} - 1,$$

hoặc, tương đương

$$abc \leq (\sqrt{3} - 1)^3,$$

bất đẳng thức trên chính xác là

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

Chứng minh được hoàn thành.

Lời giải 2:
Giống như trong lời giải đầu tiên (và bởi vì tính đối xứng) ta có thể giả sử rằng $x \geq 1, y \geq 1$; thậm chí ta có thể giả sử $x > 1, y > 1$ (với $x = 1$ bất đẳng thức là rõ ràng). Khi đó ta có $xy > 1$ và từ điều kiện đã cho ta có

$$z = \frac{x + y}{xy - 1}.$$

Hệ thức phải chứng minh là

$$(x-1)(y-1)(z-1) \le 6\sqrt{3}-10$$
$$\Longleftrightarrow 2xyz-(xy+xz+yz) \le 6\sqrt{3}-9,$$

hoặc, với biểu thức trên đây của z,

$$2xy.\frac{x+y}{xy-1}-xy-(x+y)\frac{x+y}{xy-1} \le 6\sqrt{3}-9$$
$$\Longleftrightarrow (xy-x-y)^2+(6\sqrt{3}-10)xy \le 6\sqrt{3}-9,$$

sau một vài bước tính toán.
Bây giờ, ta đặt $x=a+1, y=b+1$ và biến đổi thành

$$a^2b^2+(6\sqrt{3}-10)(a+b+ab)-2ab \ge 0.$$

Nhưng

$$a+b \ge 2\sqrt{ab}$$

và $6\sqrt{3}-10>0$, vậy ta chỉ cần chỉ ra rằng

$$a^2b^2+(6\sqrt{3}-10)(2\sqrt{ab}+ab)-2ab \ge 0.$$

Thay $t=\sqrt{ab} \ge 0$, bất đẳng thức trở thành

$$t^4+(6\sqrt{3-12})t^2+2(6\sqrt{3}-10)t \ge 0,$$

hoặc

$$t^3+(6\sqrt{3-12})t+2(6\sqrt{3}-10) \ge 0.$$

Đạo hàm của hàm số

$$f(t)=t^3+(6\sqrt{3}-12)t+2(6\sqrt{3}-10), t \ge 0$$

là

$$f'(t)=3\left(t^2-(\sqrt{3}-1)^2\right)$$

và có duy nhất một nghiệm dương của phương trình $f'(t)=0$. Đó là $\sqrt{3}-1$ và dễ thấy rằng $\sqrt{3}-1$ là một điểm cực tiểu của f trong khoảng $[0,\infty)$. Do đó

$$f(t) \ge f(\sqrt{3}-1)=0,$$

và ta có điều phải chứng minh.

Nhận xét cuối cùng: thực ra ta có

$$f(t)=(t-\sqrt{3}+1)^2(t+2\sqrt{3}-2),$$

điều này chỉ ra rằng $f(t) \ge 0$ với $t \ge 0$.

71. [Marian Tetiva] Chứng minh rằng với bất kỳ các số thực dương a, b, c ta có

$$\left|\frac{a^3-b^3}{a+b}+\frac{b^3-c^3}{b+c}+\frac{c^3-a^3}{c+a}\right| \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{4}.$$

Moldova TST, 2004

Lời giải 1:
Đầu tiên, ta thấy rằng vế trái có thể được biến đổi thành

$$\begin{aligned}\sum \frac{a^3-b^3}{a+b} &= (a^3-b^3)\left(\frac{1}{a+b}-\frac{1}{a+c}\right)+(b^3-c^3)\left(\frac{1}{b+c}-\frac{1}{a+c}\right)\\ &= \frac{(a-b)(c-b)(a-c)(\sum ab)}{(a+b)(b+c)(c+a)},\end{aligned}$$

vì vậy ta phải chứng minh bất đẳng thức

$$\frac{|(a-b)(b-c)(c-a)|(ab+bc+ca)}{(a+b)(b+c)(c+a)} \le \frac{1}{2}\left(\sum a^2-\sum ab\right).$$

Dễ dàng chứng minh rằng

$$(a+b)(b+c)(c+a) \ge \frac{8}{9}(a+b+c)(ab+bc+ca)$$

do đó ta còn phải chứng minh

$$\frac{2}{9}.\sum a\left(\sum (a-b)^2\right) \ge \left|\prod (a-b)\right|.$$

Sử dụng bất đẳng thức **AM-GM** , ta cần chứng minh bất đẳng thức sau để thu được bât đẳng thức trên

$$\frac{8}{27}\left(\sum a\right)^3 \ge \left|\prod (a-b)\right|.$$

Bất đẳng thức này dễ chứng minh. Chỉ cần để ý rằng ta có thể giả sử $a \ge b \ge c$ và trong trường hợp này bất đẳng thức trở thành

$$(a-b)(a-c)(b-c) \le \frac{8}{27}(a+b+c)^3$$

và bất đẳng thức này suy ra từ bất đẳng thức **AM-GM** .

Lời giải 2 (bởi Marian Tetiva):

Dễ thấy rằng bất đẳng thức không chỉ tuần hoàn mà còn đối xứng. Vì thế chúng ta có thể giả sử rằng $a \geq b \geq c > 0$. Ý tưởng là sử dụng bất đẳng thức

$$x + \frac{y}{2} \geq \frac{x^2 + xy + y^2}{x + y} \geq y + \frac{x}{2},$$

bất đẳng thức đúng nếu $x \geq y > 0$. Chứng minh bất đẳng thức dễ và ta sẽ bỏ qua. Bây giờ, bởi vì $a \geq b \geq c > 0$, ta có ba bất đẳng thức

$$a + \frac{b}{2} \geq \frac{a^2 + ab + b^2}{a + b} \geq b + \frac{a}{2}, b + \frac{c}{2} \geq \frac{b^2 + bc + c^2}{b + c} \geq c + \frac{b}{2}$$

và hiển nhiên

$$a + \frac{c}{2} \geq \frac{a^2 + ac + c^2}{a + c} \geq c + \frac{a}{2}.$$

Do đó

$$\begin{aligned}
\sum \frac{a^3 - b^3}{a + b} &= (a - b)\frac{a^2 + ab + b^2}{a + b} + (b - c)\frac{b^2 + bc + c^2}{b + c} - (a - c).\frac{a^2 + ac + c^2}{a + c} \\
&\geq (a - b)\left(b + \frac{a}{2}\right) + (b - c)\left(c + \frac{b}{2}\right) - (a - c)\left(a + \frac{c}{2}\right) \\
&= -\sum \frac{(a - b)^2}{4}.
\end{aligned}$$

Bằng cách tương tự ta có thể chứng minh rằng

$$\sum \frac{a^3 - b^3}{a + b} \leq \sum \frac{(a - b)^2}{4}$$

và ta có điều phải chứng minh.

72. [Titu Andreescu] Cho a, b, c là các số thực dương. Chứng minh rằng

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

USAMO, 2004

Lời giải:
Ta bắt đầu với bất đẳng thức $a^5 - a^2 + 3 \geq a^3 + 2 \Longleftrightarrow (a^2 - 1)(a^3 - 1) \geq 0$. Vậy, còn phải chỉ ra rằng

$$\prod (a^3 + 2) \geq \left(\sum a\right)^3.$$

Sử dụng bất đẳng thức **AM-GM** , ta có

$$\frac{a^3}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}\geq\frac{3a}{\sqrt[3]{\prod(a^3+2)}}.$$

Ta viết hai bất đẳng thức tương tự và cộng các bất đẳng thức lại. Ta thấy rằng

$$\prod(a^3+2)\geq\left(\sum a\right)^3,$$

đây là bất đẳng thức mong muốn.

73. [Gabriel Dospinescu] Cho $n>2$ và $x_1,x_2,...,x_n>0$ thỏa mãn

$$\left.\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)=n^2+1.$$

Chứng minh rằng

$$\left.\sum_{k=1}^{n}x_k^2\right)\left(\sum_{k=1}^{n}\frac{1}{x_k^2}\right)>n^2+4+\frac{2}{n(n-1)}.$$

Lời giải:
Trong bài toán này, sự kết hợp giữa các đồng nhất thức và bất đẳng thức **Cauchy-Schwarz** là cách để chứng minh. Vậy, ta bắt đầu với biểu thức

$$\sum_{1\leq i<j\leq n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2.$$

Ta có thể thấy ngay rằng

$$\begin{aligned}\sum_{1\leq i<j\leq n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2 &= \sum_{1\leq i<j\leq n}\left(\frac{x_i^2}{x_j^2}+\frac{x_j^2}{x_i^2}-4.\frac{x_i}{x_j}-4.\frac{x_j}{x_i}+6\right)\\ &= \left.\sum_{i=1}^{n}x_i^2\right)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)-4\left.\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)+3n^2.\end{aligned}$$

Do đó ta có thể suy ra từ bất đẳng thức

$$\sum_{1\leq i<j\leq n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2\geq 0$$

là
$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right) \geq n^2 + 4.$$
Tiếc rằng, điều này chưa đủ để kết luận. Vậy, ta hãy cố gắng cực tiểu hóa biểu thức
$$\sum_{1\leq i<j\leq n} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} - 2\right)^2.$$
Điều này có thể làm bởi sử dụng bất đẳng thức **Cauchy-Schwarz** :
$$\sum_{1\leq i<j\leq n} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} - 2\right)^2 \geq \frac{\left(\sum_{1\leq i<j\leq n} (\frac{x_i}{x_j} + \frac{x_j}{x_i} - 2)\right)^2}{\binom{n}{2}}.$$
Bởi vì $\sum_{1\leq i<j\leq n} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} - 2\right) = 1$, ta suy ra rằng
$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right) \geq n^2 + 4 + \frac{2}{n(n-1)},$$
đó là điều ta muốn chứng minh. Tất nhiên, ta cần chứng minh là đẳng thức không xảy ra. Nhưng đẳng thức sẽ dẫn đến $x_1 = x_2 = ... = x_n$, nó mâu thuẫn với giả thiết
$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) = n^2 + 1.$$

74. [Gabriel Dospinescu, Mircea Lascu, Marian Tetiva] Chứng minh rằng với bất kỳ số các số thực dương a, b, c, ta có
$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1+a)(1+b)(1+c).$$

Lời giải 1:
Đặt $f(a,b,c) = a^2+b^2+c^2+abc+2-a-b-c-ab-bc-ca$. Ta phải chứng minh rằng tất cả các giá trị của f đều không âm. Nếu $a,b,c > 3$, thì ta có $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1$, điều này có nghĩa là $f(a,b,c) > a^2+b^2+c^2-a-b-c > 0$. Vậy, ta có thể giả sử rằng $a \leq 3$ và đặt $m = \frac{b+c}{2}$. Dễ tính toán để chỉ ra rằng $f(a,b,c)-f(a,m,m) = \frac{(3-a)(b-c)^2}{4} \geq 0$ và chỉ còn phải chứng minh $f(a,m,m) \geq 0$, điều này chính là
$$(a+1)m^2 - 2(a+1)m + a^2 - a + 2 \geq 0.$$

Bất đẳng thức này hiển nhiên đúng, bởi vì biệt thức của phương trình bậc hai là $-4(a+1)(a-1)^2 \le 0$.

Lời giải 2:
Nhắc lại bất đẳng thức **Turkevici**

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \ge x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2$$

với mọi số thực x, y, z, t. Đặt $t = 1, a = x^2, b = y^2, x = z^2$ và sử dụng hệ thức $2\sqrt{abc} \le abc + 1$, ta có được bất đẳng thức cần chứng minh.

75. [Titu Andreescu, Zuming Feng] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+a+c)^2}{2b^2+(a+c)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

USAMO, 2003

Lời giải 1:
Bởi vì bất đẳng thức là thuần nhất, ta có thể giả sử $a+b+c=3$. Khi đó

$$\begin{aligned}\frac{(2a+b+c)^2}{2a^2+(b+c)^2} &= \frac{a^2+6a+9}{3a^2-6a+9}\\ &= \frac{1}{3}\left(1+2.\frac{4a+3}{2+(a-1)^2}\right)\\ &\le \frac{1}{3}\left(1+2.\frac{4a+3}{2}\right)\\ &= \frac{4a+4}{3}.\end{aligned}$$

Suy ra

$$\sum \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \ge \frac{1}{3}\sum(4a+4) = 8.$$

Lời giải 2:
Kí hiệu $x = \frac{b+c}{2}, y = \frac{c+a}{b}, z = \frac{a+b}{c}$. Ta phải chứng minh rằng

$$\sum \frac{(x+2)^2}{x^2+2} \le 8 \Longleftrightarrow \sum \frac{2x+1}{x^2+2} \le \frac{5}{2} \Longleftrightarrow \sum \frac{(x-1)^2}{x^2+2} \ge \frac{1}{2}.$$

Nhưng, từ bất đẳng thức **Cauchy-Schwarz** , ta có

$$\sum \frac{(x-1)^2}{x^2+2} \ge \frac{(x+y+z-3)^2}{x^2+y^2+z^2+6}.$$

Ta còn phải chứng minh rằng

$$2(x^2+y^2+z^2+2xy+2yz+2zx-6x-6y-6z+9) \geq x^2+y^2+z^2+6$$
$$\Longleftrightarrow x^2+y^2+z^2+4(xy+yz+zx)-12(x+y+z)+12 \geq 0.$$

Bây giờ $xy+yz+zx \geq 3\sqrt[3]{x^2y^2z^2} \geq 12$ (bởi vì $xyz \geq 8$), vậy ta vẫn còn phải chứng minh rằng

$$(x+y+z)^2+24-12(x+y+z)+12 \geq 0,$$

Bất đẳng thức này tương đương với $(x+y+z-6)^2 \geq 0$, điều này hiển nhiên đúng.

76. Chứng minh rằng với mọi số thực dương x, y và với mọi số nguyên dương m, n, ta có

$$(n-1)(m-1)(x^{m+n}+y^{m+n})+(m+n-1)(x^m y^n+x^n y^m) \geq mn(x^{m+n-1}y+y^{m+n-1}x).$$

Austrian-Polish Competition, 1995

Lời giải:
Ta biến đổi bất đẳng thức như sau:

$$mn(x-y)(x^{m+n-1}-y^{m+n-1}) \geq (m+n-1)(x^m-y^m)(x^n-y^n)$$
$$\Longleftrightarrow \frac{x^{m+n-1}-y^{m+n-1}}{(m+n-1)(x-y)} \geq \frac{x^m-y^m}{m(x-y)}.\frac{x^n-y^n}{n(x-y)}$$

(ta giả sử rằng $x > y$). Hệ thức cuối cũng có thể viết lại thành

$$(x-y)\int_y^x t^{m+n-2}dt \geq \int_y^x t^{m-1}dt.\int_y^x t^{n-1}dt$$

và bất đẳng thức này suy ra từ bất đẳng thức **Chebyshev** cho tích phân.

77. Cho a, b, c, d, e là các số thực dương thỏa mãn $abcde = 1$. Chứng minh rằng

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc} \geq \frac{10}{3}.$$

Crux Mathematicorum

Lời giải:
Xét phép thế

$$a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{t}, d=\frac{t}{u}, e=\frac{u}{x}$$

với $x, y, z, t, u > 0$. Ta có

$$\frac{a + abc}{1 + ab + abcd} = \frac{\frac{1}{y} + \frac{1}{t}}{\frac{1}{x} + \frac{1}{z} + \frac{1}{u}}.$$

Viết các hệ thức khác tương tự, và kí hiệu $\frac{1}{x} = a_1, \frac{1}{y} = a_2, \frac{1}{z} = a_3, \frac{1}{t} = a_4, \frac{1}{u} = a_5$, $a_i > 0$, ta có

$$\sum \frac{a_2 + a_4}{a_1 + a_3 + a_5} \geq \frac{10}{3}.$$

Sử dụng bất đẳng thức **Cauchy-Schwarz** , ta làm nhỏ vế trái bởi

$$\frac{4S^2}{2S^2 - (a_2 + a_4)^2 - (a_1 + a_4)^2 - (a_3 + a_5)^2 - (a_2 + a_5)^2 - (a_1 + a_3)^2},$$

với $S = \sum_{i=1}^{5} a_i$. Bởi áp dụng bất đẳng thức **Cauchy-Schwarz** một lần nữa cho mẫu số của phân số, ta thu được kết luận.

78. [Titu Andreescu] Chứng minh rằng với mọi $a, b, c \in (0, \frac{\pi}{2})$ bất đẳng thức sau đúng

$$\frac{\sin a. \sin(a - b). \sin(a - c)}{\sin(b + c)} + \frac{\sin b. \sin(b - c). \sin(b - a)}{\sin(c + a)} + \frac{\sin c. \sin(c - a). \sin(c - b)}{\sin(a + b)} \geq 0.$$

TST 2003, USA

Lời giải:
Đặt $x = \sin a, y = \sin b, z = \sin c$. Khi đó ta có $x, y, z > 0$. Dễ thấy rằng hệ thức sau là đúng:

$$\sin a. \sin(a - b). \sin(a - c). \sin(a + b). \sin(a + c) = x(x^2 - y^2)(x^2 - z^2)$$

Sử dụng những hệ thức tương tự cho những số hạng khác, ta phải chứng minh rằng:

$$\sum x(x^2 - y^2)(x^2 - z^2) \geq 0.$$

Với sự thay thế $x = \sqrt{u}.y = \sqrt{v}, z = \sqrt{w}$ bất đẳng thức trở thành $\sum \sqrt{u}(u-v)(u-w) \geq 0$. Nhưng bất đẳng thức này suy ra từ bất đẳng thức **Schur.**

79. Chứng minh rằng nếu a, b, c là các số thực dương, thì

$$\sqrt{a^4 + b^4 + c^4} + \sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq \sqrt{a^3b + b^3c + c^3a} + \sqrt{ab^3 + bc^3 + ca^3}.$$

KMO Summer Program Test, 2001

Lời giải:
Hiển nhiên ta chỉ cần chứng minh các bất đẳng thức

$$\sum a^4 + \sum a^2b^2 \geq \sum a^3b + \sum ab^3$$

và

$$\left(\sum a^4\right)\left(\sum a^2b^2\right) \geq \left(\sum a^3b\right)\left(\sum ab^3\right).$$

Bất đẳng thức đầu suy ra từ bất đẳng thức **Schur**

$$\sum a^4 + abc\sum a \geq \sum a^3b + \sum ab^3$$

và hệ thức

$$\sum a^2b^2 \geq abc\sum a.$$

Bất đẳng thức thứ hai là một hệ quả đơn giản của bất đẳng thức **Cauchy-Schwarz** :

$$(a^3b + b^3c + c^3a)^2 \leq (a^2b^2 + b^2c^2 + c^2a^2)(a^4 + b^4 + c^4)$$

$$(ab^3 + bc^3 + ca^3)^2 \leq (a^2b^2 + b^2c^2 + c^2a^2)(a^4 + b^4 + c^4).$$

80. [Gabriel Dospinescu, Mircea Lascu] Cho $n > 2$, tìm hằng số k_n nhỏ nhất có tính chất: Nếu $a_1, a_2, ..., a_n > 0$ có tích bằng 1, thì

$$\frac{a_1a_2}{(a_1^2 + a_2)(a_2^2 + a_1)} + \frac{a_2a_3}{(a_2^2 + a_3)(a_3^2 + a_2)} + ... + \frac{a_na_1}{(a_n^2 + a_1)(a_1^2 + a_n)} \leq k_n.$$

Lời giải:
Đầu tiên ta đặt $a_1 = a_2 = ... = a_{n-1} = x, a_n = \dfrac{1}{x^{n-1}}$. Ta suy ra

$$k_n \geq \frac{2x^{2n-1}}{(x^{n+1} + 1)(x^{2n-1} + 1)} + \frac{n-2}{(1+x)^2} > \frac{n-2}{(1+x)^2}$$

với mọi $x > 0$. Rõ ràng, điều này suy ra $k_n \geq n - 2$. Ta chứng minh rằng $n - 2$ là một hằng số tốt và bài toán sẽ được giải.
Đầu tiên, ta sẽ chứng minh rằng $(x^2 + y)(y^2 + x) \geq xy(1 + x)(1 + y)$. Thật vậy, bất đẳng thức chính là $(x + y)(x - y)^2 \geq 0$. Vậy, chỉ cần chứng minh rằng

$$\frac{1}{(1 + a_1)(1 + a_2)} + \frac{1}{(1 + a_2)(1 + a_3)} + ... + \frac{1}{(1 + a_n)(1 + a_1)} \leq n - 2.$$

Bây giờ, ta đặt $a_1 = \dfrac{x_1}{x_2}, ..., a_n = \dfrac{x_n}{x_1}$ và bất đẳng thức ở trên trở thành

$$\sum_{k=1}^{n}\left(1 - \frac{x_{k+1}x_{k+2}}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})}\right) \geq 2,$$

bất đẳng thức này cũng có thể được viết dưới dạng

$$\sum_{k=1}^{n} \frac{x_k}{x_k + x_{k+1}} + \frac{x_{k+1}^2}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})} \geq 2.$$

Rõ ràng,

$$\sum \frac{x_k}{x_k + x_{k+1}} \geq \sum_{k=1}^{n} \frac{x_k}{x_1 + x_2 + ... + x_n} = 1.$$

Vậy, ta phải chứng minh bất đẳng thức

$$\frac{x_{k+1}^2}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})} \geq 1.$$

Sử dụng bất đẳng thức **Cauchy-Schwarz**, ta suy ra rằng

$$\frac{x_{k+1}^2}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})} \geq \frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n}(x_k + x_{k+1})(x_{k+1} + x_{k+2})}$$

và ta chỉ cần chứng minh rằng

$$\left(\sum_{k=1}^{n} x_k\right)^2 \geq \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}.$$

Nhưng bất đẳng thức này tương đương với

$$2\sum_{1 \leq i < j \leq n} x_i x_j \geq 2\sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}$$

và điều này là hiển nhiên đúng. Vậy, $k_n = n - 2$.

81. [Vasile Cirtoaje] Với mọi số thực a, b, c, x, y, z, chứng minh rằng bất đẳng thức sau đúng

$$ax + by + cz + \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} \geq \frac{2}{3}(a + b + c)(x + y + z).$$

Kvant, 1989

Lời giải:

Ta kí hiệu $t = \sqrt{\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}}$. Sử dụng sự thay thế $x = tp, y = tq$ và $z = tr$, điều này suy ra

$$a^2+b^2+c^2 = p^2+q^2+r^2.$$

Bất đẳng thức đã cho trở thành

$$ap+bq+cr+a^2+b^2+c^2 \geq \frac{2}{3}(a+b+c)(p+q+r),$$

$$(a+p)^2+(b+q)^2+(c+r)^2 \geq \frac{4}{3}(a+b+c)(p+q+r).$$

Từ

$$4(a+b+c)(p+q+r) \leq [(a+b+c)+(p+q+r)]^2,$$

suy ra chỉ cần chứng minh rằng

$$(a+p)^2+(b+q)^2+(c+r)^2 \geq \frac{1}{3}[(a+p)+(b+q)+(c+r)]^2.$$

Bất đẳng thức này hiển nhiên là đúng.

82. [Vasile Cirtoaje] Chứng minh rằng các cạnh a, b, c của một tam giác thỏa mãn bất đẳng thức

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

Lời giải 1:

Ta có thể giả sử rằng c là số nhỏ nhất trong các số a, b, c. Khi đó đặt $x = b - \dfrac{a+c}{2}$. Sau một vài bước tính toán, bất đẳng thức trở thành

$$(3a-2c)x^2+\left(x+c-\frac{a}{4}\right)(a-c)^2 \geq 0$$
$$\Longleftrightarrow (3a-2c)(2b-a-c)^2+(4b+2c-3a)(a-c)^2 \geq 0.$$

Điều này trực tiếp suy ra từ các bất đẳng thức $3a \geq 2c, 4b+2c-3a = 3(b+c-a)+b-c > 0$.

Lời giải 2:

Thực hiện một phép thay thế $a = y+z, b = z+x, c = x+y$ và quy đồng mẫu số. Bài toán quy về chứng minh rằng

$$x^3+y^3+z^3+2(x^2y+y^2z+z^2x) \geq 3(xy^2+yz^2+zx^2).$$

Tất nhiên ta có thể giả sử rằng x là số nhỏ nhất trong các số x, y, z. Khi đó ta có thể viết $y = x + m, z = x + n$ với m và n các số không âm. Một sự tính toán ngắn chỉ ra rằng bất đẳng thức quy về

$$2x(m^2 - mn + n^2) + m^3 + n^3 + 2m^2n - 3n^2m \geq 0.$$

Ta chỉ còn phải chứng minh rằng

$$m^3 + n^3 + 2m^2n \geq 3n^2m \Longleftrightarrow (n-m)^3 - (n-m)m^2 + m^3 \geq 0$$

và bất đẳng thức này trực tiếp suy ra từ bất đẳng thức $t^3 - t + 1 \geq 0$, đúng với $t \geq -1$.

83. [Walther Janous] Cho $n > 2$ và $x_1, x_2, ..., x_n > 0$ có tổng bằng 1. Chứng minh rằng

$$\prod_{i=1}^{n}\left(1 + \frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(\frac{n - x_i}{1 - x_i}\right).$$

Crux Mathematicorum

Lời giải 1:
Ý tưởng tự nhiên là sử dụng hệ thức

$$\frac{n - x_i}{1 - x_i} = 1 + \frac{n-1}{x_1 + x_2 + ... + x_{i-1} + x_{i+1} + ... + x_n}.$$

Vậy ta có

$$\prod_{j \neq i}\left(\frac{n - x_i}{1 - x_i}\right) \leq \prod_{i=1}^{n}\left(1 + \frac{1}{\sqrt[n-1]{x_1.x_2....x_{i-1}.x_{i+1}...x_n}}\right)$$

và ta phải chứng minh bất đẳng thức

$$\prod_{i=1}^{n}\left(1 + \frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(1 + \frac{1}{\sqrt[n-1]{x_1.x_2....x_{i-1}.x_{i+1}...x_n}}\right).$$

Nhưng bất đẳng thức này không quá khó, bởi vì bất đẳng thức trên suy ra trực tiếp bởi phép nhân các bất đẳng thức

$$\prod_{i=1}^{n}\left(1 + \frac{1}{x_j}\right) \geq \left(1 + \sqrt[n-1]{\prod_{j \neq i}\frac{1}{x_j}}\right)^{n-1}$$

thu được từ bất đẳng thức **Huygens.**

Lời giải 2:
Ta sẽ chứng minh bất đẳng thức mạnh hơn là

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)\geq\left(\frac{n^2-1}{n}\right)^n.\prod_{i=1}^{n}\frac{1}{1-x_i}.$$

Rõ ràng là bất đẳng thức này mạnh hơn bất đẳng thức ban đầu. Đầu tiên, ta chứng minh rằng

$$\prod_{i=1}^{n}\frac{1+x_i}{1-x_i}\geq\left(\frac{n+1}{n-1}\right)^n.$$

Bất đẳng thức này suy ra từ bất đẳng thức **Jensen** với hàm lồi $f(x)=\ln(1+x)-\ln(1-x)$. Vậy, chỉ cần chứng minh rằng

$$\frac{\left(\frac{n+1}{n-1}\right)^n}{\prod\limits_{i=1}^{n}x_i}.\prod_{i=1}^{n}(1-x_i)^2\geq\left(\frac{n^2-1}{n}\right)^n.$$

Nhưng ta thấy ngay rằng đây chính là bất đẳng thức được chứng minh trong lời giải của bài toán 121.

84. [Vasile Cirtoaje, Gheorghe Eckstein] Xét các số thực dương $x_1,x_2,...,x_n$ thỏa mãn $x_1x_2...x_n=1$. Chứng minh rằng

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+...+\frac{1}{n-1+x_n}\leq 1.$$

TST 1999, Romania

Lời giải 1:
Giả sử bất đẳng thức không đúng với một hệ n số nào đó. Khi đó ta có thể tìm được một số $k>1$ và n số có tổng bằng 1, giả sử các số đó là a_i, sao cho $\dfrac{1}{n-1+x_i}=ka_i$. Khi đó ta có

$$1=\prod_{i=1}^{n}\left(\frac{1}{ka_i}-n+1\right)<\prod_{i=1}^{n}\left(\frac{1}{a_i}-n+1\right).$$

Ở đây ta sử dụng hệ thức $a_i<\dfrac{1}{n-1}$. Bây giờ, ta viết $1-(n-1)a_k=b_k$ và ta thấy $\sum\limits_{k=1}^{n}b_k=1$ và cũng có $\prod\limits_{i=1}^{n}\left(1-b_k\right)<(n-1)^nb_1...b_n$. Nhưng điều này mâu thuẫn với sự kiện là với mỗi j ta có

$$1-b_j=b_1+b_2+...+b_{j-1}+b_{j+1}+...+b_n\geq(n-1)\sqrt[n-1]{b_1...b_{j-1}b_{j+1}...b_n}.$$

Lời giải 2:
Ta viết bất đẳng thức viết dưới dạng

$$
\begin{aligned}
&-\frac{n-1}{n-1+x_1}-\frac{n-1}{n-1+x_2}-...-\frac{n-1}{n-1+x_n}\geq -(n-1)\\
\Longleftrightarrow&(1-\frac{n-1}{n-1+x_1})+(1-\frac{n-1}{n-1+x_2})+...+(1-\frac{n-1}{n-1+x_n})\geq n-(n-1)\\
\Longleftrightarrow&\frac{x_1}{n-1+x_1}+\frac{x_2}{n-1+x_2}+...+\frac{x_n}{n-1+x_n}\geq 1.
\end{aligned}
$$

Bất đẳng thức cuối suy ra từ tổng của các bất đẳng thức dưới đây

$$\frac{x_1}{n-1+x_1}\geq\frac{x_1^{1-\frac{1}{n}}}{x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+...+x_n^{1-\frac{1}{n}}},...,\frac{x_n}{n-1+x_n}\geq\frac{x_n^{1-\frac{1}{n}}}{x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+...+x_n^{1-\frac{1}{n}}}.$$

Bất đẳng thức đầu tiên trong các bất đẳng thức trên tương đương với

$$x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+...+x_n^{1-\frac{1}{n}}\geq (n-1)x_1^{-\frac{1}{n}}$$

được suy từ bất đẳng thức **AM-GM** .

Chú ý.
Thay thế các số $x_1, x_2, ..., x_n$ lần lượt bằng các số $\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}$, bất đẳng thức trở thành

$$\frac{1}{1+(n-1)x_1}+\frac{1}{1+(n-1)x_2}+...+\frac{1}{1+(n-1)x_n}\geq 1.$$

85. [Titu Andreescu] Chứng minh rằng với mọi số thực không âm a, b, c thỏa mãn $a^2+b^2+c^2+abc=4$, ta có

$$0\leq ab+bc+ca-abc\leq 2.$$

USAMO, 2001

Lời giải 1 (bởi Richard Stong):
Cận dưới không khó. Thật vậy, ta có $ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2}$ và chỉ cần chứng minh rằng $abc\leq\sqrt[3]{a^2b^2c^2}$, Điều này suy ra từ hệ thức $abc\leq 4$. Ngược lại, cận trên là khó. Đầu tiên ta thấy rằng có hai trong ba số cùng lớn hơn hoặc bằng 1 hoặc cùng nhỏ hơn hoặc bằng 1. Giả sử hai số đó là b và c. Khi đó ta có

$$4\geq 2bc+a^2+abc\Longrightarrow(2-a)(2+a)\geq bc(2+a)\Longrightarrow bc\leq 2-a.$$

Vậy

$$ab+bc+ca-abc\leq ab+2-a+ac-abc$$

và chỉ cần chứng minh bất đẳng thức

$$ab + 2 - a + ac - abc \leq 2 \Longleftrightarrow b + c - bc \leq 1 \Longleftrightarrow (b-1)(c-1) \geq 0,$$

bất đẳng thức này đúng do cách chọn của chúng ta.

Lời giải 2:
Ta không chứng minh lại cận dưới, vì đây là một bài toán dễ. Ta tập trung vào cận trên. Giả sử $a \geq b \geq c$ và giả sử $a = x + y, b = x - y$. Giả thiết trở thành

$$x^2(2+c) + y^2(2-c) = 4 - c^2$$

và ta phải chứng minh rằng

$$(x^2 - y^2)(1-c) \leq 2(1 - xc).$$

Bởi vì $y^2 = 2 + c - \dfrac{2+c}{2-c}x^2$, bài toán yêu cầu chứng minh bất đẳng thức

$$\frac{4x^2 - (4-c^2)}{2-c}(1-c) \leq 2(1-cx).$$

Rõ ràng, ta có $c \leq 1$ và

$$0 \leq y^2 = 2 + c - \frac{2+c}{2-c}x^2 \Longrightarrow x^2 \leq 2 - c \Longrightarrow x \leq \sqrt{2-c}$$

(ta sử dụng sự kiện là $a \geq b \Longrightarrow y \geq 0$ and $b \geq 0 \Longrightarrow x \geq y \geq 0$). Bây giờ, xét hàm số

$$f : [0, \sqrt{2-c}] \to \mathbb{R}, f(x) = 2(1-cx) - \frac{4x^2 - (4-c^2)}{2-c}(1-c).$$

Ta có $f'(x) = -2c - 8x\dfrac{1-c}{2-c} \leq 0$, suy ra f là hàm giảm và $f(x) \geq f(\sqrt{2} - c)$. Vậy, ta phải chứng minh rằng

$$\begin{aligned}
&f(\sqrt{2-c}) \geq 0\\
\Longleftrightarrow &2(1 - c\sqrt{2-c}) \geq (2-c)(1-c)\\
\Longleftrightarrow &3 \geq c + 2\sqrt{2-c}\\
\Longleftrightarrow &(1 - \sqrt{2-c})^2 \geq 0
\end{aligned}$$

điều này hiển nhiên đúng. Vậy, bài toán đã giải xong.

86. [Titu Andreescu] Chứng minh rằng với mọi số thực dương a, b, c, bất đẳng thức sau đúng

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\left\{(\sqrt{a} - \sqrt{b})^2, (\sqrt{b} - \sqrt{c})^2, (\sqrt{c} - \sqrt{a})^2\right\}.$$

TST 2000, USA

Lời giải 1:
Một ý tưởng tự nhiên là phải giả sử ngược lại, điều này có nghĩa là

$$\frac{a+b+c}{3} - \sqrt[3]{abc} > a + b - 2\sqrt{ab}$$

$$\frac{a+b+c}{3} - \sqrt[3]{abc} > b + c - 2\sqrt{bc}$$

$$\frac{a+b+c}{3} - \sqrt[3]{abc} > c + a - 2\sqrt{ca}.$$

Cộng các bất đẳng thức lại, ta thấy rằng

$$a + b + c - 3\sqrt[3]{abc} > 2(a + b + c - \sqrt{ab} - \sqrt{bc} - \sqrt{ca}).$$

Bây giờ, ta sẽ chứng minh rằng $a + b + c - 3\sqrt[3]{abc} \leq 2(a + b + c - \sqrt{ab} - \sqrt{bc} - \sqrt{ca})$ và bài toán sẽ được giải. Vì bất đẳng thức trên là thuần nhất, ta có thể giả sử rằng $abc = 1$. Khi đó, bất đẳng thức trở thành

$$2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} - a - b - c \leq 3.$$

Bây giờ, sử dụng bất đẳng thức **Schur**, ta thấy rằng với bất kì các số thực dương x, y, z, ta có

$$2xy + 2yz + 2zx - x^2 - y^2 - z^2 \leq \frac{9xyz}{x + y + z} \leq 3\sqrt[3]{x^2y^2z^2}.$$

Ta chỉ còn phải đặt $x = \sqrt{a}, y = \sqrt{b}, z = \sqrt{c}$ trong bất đẳng thức trên.

87. [Kiran Kedlaya] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a.\frac{a+b}{2}.\frac{a+b+c}{3}}.$$

Lời giải (bởi Anh Cường):
Ta có

$$a + \sqrt{ab} + \sqrt[3]{abc} \leq a + \sqrt[3]{ab\frac{a+b}{2}} + \sqrt[3]{abc}.$$

Bây giờ ta sẽ chứng minh

$$a + \sqrt[3]{ab\frac{a+b}{2}} + \sqrt[3]{abc} \leq \sqrt[3]{a.\frac{a+b}{2}.\frac{a+b+c}{3}}.$$

Bởi bất đẳng thức **AM-GM** , ta có

$$\sqrt[3]{1.\frac{2a}{a+b}.\frac{3a}{a+b+c}} \le \frac{1+\dfrac{2a}{a+b}+\dfrac{3a}{a+b+c}}{3},$$

$$\sqrt[3]{1.1.\frac{3b}{a+b+c}} \le \frac{2+\dfrac{3b}{a+b+c}}{3},$$

$$\sqrt[3]{1.\frac{2b}{a+b}.\frac{3c}{a+b+c}} \le \frac{1+\dfrac{2b}{a+b}+\dfrac{3c}{a+b+c}}{3}.$$

Bây giờ, chỉ cần cộng các bất đẳng thức lại, ta có bất đẳng thức cần chứng minh. Dấu bằng xảy ra khi $a=b=c$.

88. Tìm hằng số k lớn nhất sao cho với mọi số nguyên dương n không là môt số chính phương, ta có

$$\left|(1+\sqrt{n})\sin(\pi\sqrt{n})\right| > k.$$

Vietnamese IMO Training Camp, 1995

Lời giải:
Ta sẽ chứng minh rằng $\dfrac{\pi}{2}$ là hằng số tốt nhất. Hiển nhiên ta có

$$k < (1+\sqrt{i^2+1})\left|\sin(\pi\sqrt{i^2+1})\right|$$

với mọi số nguyên dương i. Bởi vì $\left|\sin(\pi\sqrt{i^2+1})\right| = \sin\dfrac{\pi}{i+\sqrt{i^2+1}}$, ta suy ra rằng

$$\frac{\pi}{i+\sqrt{i^2+1}} \ge \sin\frac{\pi}{i+\sqrt{i^2+1}} > \frac{k}{1+\sqrt{i^2+1}},$$

từ đây suy ra $k \le \dfrac{\pi}{2}$. Bây giờ ta chứng minh rằng $\dfrac{\pi}{2}$ là hằng số tốt. Rõ ràng, bất đẳng thức có thể viết

$$\sin(\pi\{\sqrt{n}\}) > \frac{\pi}{2(1+\sqrt{n})}.$$

Ta có hai trường hợp
i) Trường hợp thứ nhất: khi $\{\sqrt{n}\} \le \dfrac{1}{2}$. Hiển nhiên,

$$\{\sqrt{n}\} \ge \sqrt{n}-\sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}},$$

và bởi vì $\sin x \geq x - \dfrac{x^3}{6}$ ta thấy rằng

$$\sin(\pi\{\sqrt{n}\}) \geq \sin\frac{\pi}{\sqrt{n-1}+\sqrt{n}} \geq \left(\frac{\pi}{\sqrt{n-1}+\sqrt{n}}\right) - \frac{1}{6}.\left(\frac{\pi}{\sqrt{n-1}+\sqrt{n}}\right)^3.$$

Ta chứng minh đẳng thức cuối cùng ít nhất bằng $\dfrac{\pi}{2(1+\sqrt{n})}$. Điều này dẫn đến

$$\frac{2+\sqrt{n}-\sqrt{n-1}}{1+\sqrt{n}} > \frac{\pi^2}{3(\sqrt{n}+\sqrt{n-1})^2},$$

hoặc $6(\sqrt{n}+\sqrt{n-1})^2+3(\sqrt{n}+\sqrt{n-1}) > \pi^2(1+\sqrt{n})$ và điều này là hiển nhiên đúng.
ii) Trường hợp thứ hai: khi $\{\sqrt{n}\} > \dfrac{1}{2}$. Đặt $x = 1-\{\sqrt{n}\} < \dfrac{1}{2}$ và $n = k^2+p, 1 \leq p \leq 2k$. Bởi vì $\{\sqrt{n}\} > \dfrac{1}{2} \Longrightarrow p \geq k+1$. Khi đó dễ thấy rằng $x \geq \dfrac{1}{k+1+\sqrt{k^2+2k}}$ và ta chỉ cần chứng minh rằng

$$\sin\frac{\pi}{k+1+\sqrt{k^2+2k}} \geq \frac{\pi}{2(1+\sqrt{k^2+k})}.$$

Sử dụng một lần nữa bất đẳng thức $\sin x \geq x - \dfrac{x^3}{6}$, ta suy ra

$$\frac{2\sqrt{k^2+k}-\sqrt{k^2+2k}-k+1}{1+\sqrt{k^2+k}} > \frac{\pi^2}{3(1+k+\sqrt{k^2+2k})^2}.$$

Nhưng từ bất đẳng thức **Cauchy-Schwarz** ta có $2\sqrt{k^2+k}-\sqrt{k^2+2k}-k \geq 0$. Bởi vì bất đẳng thức $\left(1+k+\sqrt{k^2+2k}\right)^2 > \dfrac{\pi^2}{3}\left(1+\sqrt{k^2+k}\right)$ đúng, trường hợp này cũng được giải quyết.

89. [Tran Nam Dung] Cho $x, y, z > 0$ thỏa mãn $(x+y+z)^3 = 32xyz$. Tìm giá trị nhỏ nhất và giá trị lớn nhất của biểu thức

$$\frac{x^4+y^4+z^4}{(x+y+z)^4}.$$

Vietnam, 2004

Lời giải 1 (bởi Tran Nam Dung):
Ta có thể giả sử rằng $x+y+z = 4$ và $xyz = 2$. Suy ra, ta phải tìm các giá trị cực trị

của $\dfrac{x^4+y^4+z^4}{4^4}$. Bây giờ ta có

$$\begin{aligned} x^4+y^4+z^4 &= (x^2+y^2+z^2)^2 - 2\sum x^2y^2 \\ &= \left(16-2\sum xy\right)^2 - 2\left(\sum xy\right)^2 + 4xyz(x+y+z) \\ &= 2a^2-64a+288, \end{aligned}$$

với $a = xy+yz+zx$. Bởi vì $y+z = 4-x$ và $yz = \dfrac{2}{x}$, ta phải có $(4-x)^2 \geq \dfrac{8}{x}$, điều này suy ra rằng $3-\sqrt{5} \leq x \leq 2$. Do tính đối xứng, ta có $x, y, z \in [3-\sqrt{5}, 2]$. Điều này có nghĩa là $(x-2)(y-2)(z-2) \leq 0$ và

$$(x-3+\sqrt{5})(y-3+\sqrt{5})(z-3+\sqrt{5}) \geq 0.$$

Hiển nhiên từ các biểu thức trong ngoặc, ta suy ra rằng

$$a \in \left[5, \frac{5\sqrt{5}-1}{2}\right].$$

Nhưng bởi vì $\dfrac{x^4+y^4+z^4}{4^4} = \dfrac{(a-16)^2-112}{128}$, ta tìm thấy các giá trị cực trị của biểu thức là $\dfrac{383-165\sqrt{5}}{256}, \dfrac{9}{128}$, đạt được với các bộ ba số tương ứng $\left(3-\sqrt{5}, \dfrac{1+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$, $(2, 1, 1)$.

Lời giải 2:
Như trong lời giải trên, ta phải tìm các giá trị cực trị của $x^2+y^2+z^2$ khi $x+y+z = 1, xyz = \dfrac{1}{32}$, bởi vì sau đó các giá trị cực trị của biểu thức $x^4+y^4+z^4$ có thể được tìm thấy trực tiếp. Ta sử dụng sự thay thế $x = \dfrac{a}{4}, y = \dfrac{b}{4}, z = \dfrac{c}{2}$, với $abc = 1, a+b+2c = 4$. Khi đó $x^2+y^2+z^2 = \dfrac{a^2+b^2+4c^2}{16}$ và do đó ta phải tìm các giá trị cực trị của $a^2+b^2+4c^2$. Bây giờ, $a^2+b^2+4c^2 = (4-2c)^2 - \dfrac{2}{c} + 4c^2$ và bài toán quy về tìm giá trị lớn nhất và giá trị nhỏ nhất của $4c^2-8c-\dfrac{1}{c}$ với a, b, c là các số thực dương thỏa mãn $abc = 1, a+b+2c = 4$. Hiển nhiên, điều này dẫn đến $(4-2c)^2 \geq \dfrac{4}{c}$, hoặc $c \in \left[\dfrac{3-\sqrt{5}}{2}, 1\right]$. Nhưng điều này quy về việc xét hàm số $f(x) = 4x^2-8x-\dfrac{1}{x}$ xác định trên $\left[\dfrac{3-\sqrt{5}}{2}, 1\right]$, đây là một bài toán dễ.

90. [George Tsintifa] Chứng minh rằng với mọi $a, b, c, d > 0$,

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

Crux Mathematicorum

Lời giải:
Ta áp dụng bất đẳng thức **Mac-Laurin** cho

$$x = abc, y = bcd, z = cda, t = dab.$$

Ta sẽ thấy rằng

$$\left(\frac{\sum abc}{4}\right)^3 \geq \frac{\sum abc.bcd.cda}{4} = \frac{a^2b^2c^2d^2 \sum a}{4}.$$

Vậy, ta chỉ cần chứng minh bất đẳng thức mạnh hơn

$$(a+b)(b+c)(c+d)(d+a) \geq (a+b+c+d)(abc+bcd+cda+dab).$$

Bây giờ, ta nhận xét rằng

$$\begin{aligned}(a+b)(b+c)(c+d)(d+a) &= (ac+bd+ad+bc)(ac+bd+ab+cd)\\ &= (ac+bd)^2 + \sum a^2(bc+bd+cd)\\ &\geq 4abcd + \sum a^2(bc+bd+cd)\\ &= (a+b+c+d)(abc+bcd+cda+dab).\end{aligned}$$

và bài toán đã giải xong.

91. [Titu Andreescu, Gabriel Dospinescu] Tìm giá trị lớn nhất của biểu thức

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

với a, b, c là các số thực không âm mà có tổng bằng 1 và n là một số nguyên dương.

Lời giải:
Đầu tiên, ta sẽ giải quyết trường hợp $n > 1$. Ta sẽ chứng minh giá trị lớn nhất là $\frac{1}{3.4^{n-1}}$. Rõ ràng $ab, bc, ca \leq \frac{1}{4}$ và do đó

$$\frac{(ab)^n}{1-ab} + \frac{(ab)^n}{1-ab} + \frac{(ab)^n}{1-ab} \leq \frac{4}{3}\bigg((ab)^n + (bc)^n + (ca)^n\bigg).$$

Suy ra, chúng ta phải chứng minh rằng $(ab)^n + (bc)^n + (ca)^n \le \dfrac{1}{4^n}$. Giả sử a là số lớn nhất trong các số a, b, c. Khi đó ta có

$$\frac{1}{4^n} \ge a^n(1-a)^n = a^n(b+c)^n \ge a^n b^n + a^n c^n + n a^n b^{n-1} c \ge a^n b^n + b^n c^n + c^n a^n.$$

Vậy, ta chứng minh được rằng trong trường hợp này giá trị lớn nhất nhiều nhất bằng $\dfrac{1}{3.4^{n-1}}$. Nhưng với $a = b = \dfrac{1}{2}, c = 0$ giá trị này đạt được và điều này chỉ ra rằng giá trị lớn nhất là $\dfrac{1}{3.4^{n-1}}$ với $n > 1$. Bây giờ, giả sử rằng $n = 1$. Trong trường hợp này ta có

$$\sum \frac{ab}{1-ab} = \sum \frac{1}{1-ab} - 3.$$

Sử dụng giả thiết $a + b + c = 1$, không khó để chứng minh rằng

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} = \frac{3 - \sum ab + abc}{1 - \sum ab + abc - a^2b^2c^2}.$$

Chúng ta sẽ chứng minh rằng $\sum \dfrac{ab}{1-ab} \le \dfrac{3}{8}$. Với sự nhận xét ở trên, vấn đề này quy về

$$\sum ab \le \frac{3 - 27(abc)^2 + 19abc}{11}.$$

Nhưng sử dụng bất đẳng thức **Schur** ta suy ra rằng $\sum ab \le \dfrac{1+9abc}{4}$ và vậy đủ để chỉ ra rằng

$$\frac{9abc+1}{4} \le \frac{3 - 27(abc)^2 + 19abc}{11} \Longleftrightarrow 108(abc)^2 + 23abc \le 1,$$

điều này đúng bởi vì $abc \le \dfrac{1}{27}$. Do đó với $n = 1$, giá trị lớn nhất là $\dfrac{3}{8}$ đạt được với $a = b = c$.

92. Cho a, b, c là các số thực dương. Chứng minh rằng

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \ge \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})}.$$

Lời giải:
Sự quan sát sau đây là chủ yếu

$$\begin{aligned}(1+abc)\left(\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)}\right)+3 &= \sum \frac{1+abc+a+ab}{a(1+b)} \\ &= \sum \frac{1+a}{a(1+b)} + \sum \frac{b(c+1)}{1+b}.\end{aligned}$$

Bây giờ sử dụng hai lần bất đẳng thức **AM-GM** ta được

$$\sum \frac{1+a}{a(1+b)} + \sum \frac{b(c+1)}{1+b} \geq \frac{3}{\sqrt[3]{abc}} + 3\sqrt[3]{abc}.$$

Vậy ta còn lại với bất đẳng thức

$$\frac{\frac{3}{\sqrt[3]{abc}} + 3\sqrt[3]{abc} - 3}{1+abc} \geq \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})},$$

thực tế đây là một đồng nhất thức!

93. [Tran Nam Dung] Chứng minh rằng với mọi số thực a, b, c thỏa mãn $a^2+b^2+c^2 = 9$, ta có

$$2(a+b+c) - abc \leq 10.$$

Vietnam, 2002

Lời giải 1 (bởi Gheorghe Eckstein):
Bởi vì $\max\{a,b,c\} \leq 3$ và $|abc| \leq 10$, ta chỉ cần xét các trường hợp khi $a, b, c \geq 0$ hoặc có đúng một trong ba số là âm. Đầu tiên, ta giả sử rằng a, b, c là không âm. Nếu $abc \geq 1$, thì ta có điều phải chứng minh, bởi vì

$$2(a+b+c) - abc \leq 2\sqrt{3(a^2+b^2+c^2)} - 1 < 10.$$

Ngược lại, ta có thể giả sử rằng $a < 1$. Trong trường hợp này ta có

$$2(a+b+c) - abc \leq 2\left(a + 2\sqrt{\frac{b^2+c^2}{2}}\right) = 2a + 2\sqrt{18-2a^2} \leq 10.$$

Bây giờ, giả sử rằng chỉ có một số âm và giả sử $c < 0$. Suy ra, bài toán quy về chứng minh rằng với mọi x, y, z không âm có tổng bình phương bằng 9, ta có

$$4(x+y-z) + 2xyz \leq 20.$$

Nhưng ta có thể viết như là

$$(x-2)^2 + (y-2)^2 + (z-1)^2 \geq 2xyz - 6z - 2.$$

Bởi vì

$$2xyz - 6z - 2 \leq x(y^2+z^2) - 6z - 2 = -z^3 + 3z - 2 = -(z-1)^2(z+2),$$

bất đẳng thức đúng.

Lời giải 2:
Hiển nhiên, ta có $|a|, |b|, |c| \leq 3$ và $|a+b+c|, |abc| \leq 3\sqrt{3}$. Ta cũng có thể giả sử rằng a, b, c khác không và $a \leq b \leq c$. Nếu $c < 0$ thì ta có

$$2(a+b+c) - abc < -abc \leq 3\sqrt{3} < 10.$$

Cũng vậy, nếu $a \leq b < 0 < c$ thì ta có

$$2(a+b+c) < 2c \leq 6 < 10 + abc$$

bởi vì $abc > 0$. Nếu $a < 0 < b \leq c$, sử dụng bất đẳng thức **Cauchy-Schwarz** ta được $2b + 2c - a \leq 9$. Suy ra

$$2(a+b+c) = 2b + 2c - a + 3a \leq 9 + 3a$$

và chỉ còn phải chứng minh rằng $3a - 1 \leq abc$. Nhưng $a < 0$ và $2bc \leq 9 - a^2$, vậy chỉ cần chứng minh

$$\frac{9a - a^3}{2} \geq 3a - 1 \Longleftrightarrow (a+1)^2(a-2) \leq 0,$$

bất đẳng thức này đúng. Do đó, ta chỉ còn trường hợp $0 < a \leq b \leq c$. Trong trường hợp này $2b + 2c + a \leq 9$ và $2(a+b+c) \leq 9 + a$. Vậy, ta cần chứng minh $a \leq 1 + abc$. Bất dẳng thức này hiển nhiên đúng nếu $a < 1$ còn nếu $a > 1$ ta có $b, c > 1$ và bất đẳng thức vẫn đúng. Vậy, bài toán đã được giải xong.

94. [Vasile Cirtoaje] Cho a, b, c là các số thực dương. Chứng minh rằng

$$\left(a + \frac{1}{b} - 1\right)\left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right)\left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right)\left(a + \frac{1}{b} - 1\right) \geq 3.$$

Lời giải 1:
Với các kí hiệu $x = a + \dfrac{1}{b} - 1, y = b + \dfrac{1}{c} - 1, z = c + \dfrac{1}{a} - 1$, bất đẳng thức trở thành

$$xy + yz + zx \geq 3.$$

Không mất tính tổng quát ta giả sử rằng $x = \max\{x, y, z\}$. Từ

$$\begin{aligned}(x+1)(y+1)(z+1) &= abc + \frac{1}{abc} + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ &\geq 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ &= 5 + x + y + z,\end{aligned}$$

ta có
$$xyz + xy + yz + zx \geq 4,$$
với dấu bằng xảy ra khi v chỉ khi $abc = 1$. Bởi vì $y + z = \dfrac{1}{a} + b + \dfrac{(c-1)^2}{c} > 0$ ta phân biệt hai trường hợp a) $x > 0, yz \leq 0$; b) $x > 0, y > 0, z > 0$.
a) $x > 0, yz \leq 0$. Ta có $xyz \leq 0$ v từ $xyz + xy + zx \geq 4$ ta có $xy + yz + zx \geq 4 > 3$.
b) $x > 0, y > 0, z > 0$. Ta kí hiệu $xy + yz + zx = 3d^2$ với $d > 0$. Từ các bất đẳng thức trung bình, ta có
$$xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2},$$
từ đó ta suy ra rằng $xyz \leq d^3$. Dựa trên kết quả cơ bản n y, từ bất đẳng thức $xyz + xy + yz + zx \geq 4$, ta được
$$d^3 + 3d^2 \geq 4, (d-1)(d+2)^2 \geq 0, d \geq 1,$$
vậy $xy + yz + zx \geq 3$. Với điều n y bất đẳng thức đã cho được chứng minh. Ta có dấu bằng xảy ra trong trường hợp $a = b = c = 1$.

Lời giải 2:
Đặt $u = x + 1, v = y + 1, w = z + 1$. Khi đó ta có
$$uvw = u + v + w + abc + \frac{1}{abc} \geq u + v + w + 2.$$
Bây giờ, xét h m số $f(t) = 2t^3 + t^2(u+v+w) - uvw$. Bởi vì $\lim\limits_{t\to\infty} f(t) = \infty$ v $f(1) \leq 0$, ta có thể tìm được một số thực $r \geq 1$ sao cho $f(r) = 0$. Xét các số $m = \dfrac{u}{r}, n = \dfrac{v}{r}, p = \dfrac{w}{r}$.
Chúng thỏa mãn $mnp = m + n + p + 2$, từ b i toán 49 ta suy ra rằng
$$mn + np + pm \geq 2(m + n + p) \Longrightarrow uv + vw + wu \geq 2r(u + v + w) \geq 2(u + v + w).$$
Nhưng bởi vì $u = x + 1, v = y + 1, w = z + 1$, từ điều n y hệ thức cuối cùng tương đương với bất đẳng thức $xy + yz + zx \geq 3$, đây l bất đẳng thức mong muốn.

95. [Gabriel Dospinescu] Cho n l số nguyên lớn hơn 2. Tìm số thực m_n lớn nhất v số thực M_n nhỏ nhất sao cho với mọi số thực dương $x_1, x_2, ..., x_n$ (với $x_n = x_0, x_{n+1} = x_1$), ta có
$$m_n \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \leq M_n.$$

Lời giải:
Ta sẽ chứng minh rằng $m_n = \dfrac{1}{2(n-1)}, M_n = \dfrac{1}{2}$. Đầu tiên, chúng ta thấy bất đẳng thức
$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \geq \frac{1}{2(n-1)}$$

là tầm thường, bởi vì $x_{i-1}+2(n-1)x_i+x_{i+1} \leq 2(n-1).\sum\limits_{k=1}^{n} x_k$ với mọi i. Điều này chỉ ra rằng $m_n \geq \dfrac{1}{2(n-1)}$. Đặt $x_i = x^i$, biểu thức trở thành

$$\frac{1}{x+x^{n-1}+2(n-1)}+\frac{(n-2)x}{1+2(n-1)x+x^2}+\frac{x^{n-1}}{1+2(n-1)x^{n-1}+x^{n-2}}$$

và lấy giới hạn khi x tiến đến 0, ta được $m_n \leq \dfrac{1}{2(n-1)}$ và suy ra $m_n = \dfrac{1}{2(n-1)}$.

Bây giờ, ta sẽ chứng minh rằng $M_n \geq \dfrac{1}{2}$. Rõ ràng, ta chỉ cần chứng minh rằng với mọi $x_1, x_2, ..., x_n > 0$ ta có

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}+2(n-1)x_i+x_{i+1}} \leq \frac{1}{2}.$$

Nhưng ta có

$$\begin{aligned}\sum_{i=1}^{n} \frac{2x_i}{x_{i-1}+2(n-1)x_i+x_{i+1}} &\leq \sum_{i=1}^{n} \frac{2x_i}{2\sqrt{x_{i-1}x_{i+1}}+2(n-1)x_i}\\ &= \sum_{i=1}^{n} \frac{1}{n-1+\dfrac{\sqrt{x_{i-1}x_{i+1}}}{x_i}}.\end{aligned}$$

Đặt $\dfrac{\sqrt{x_{i-1}x_{i+1}}}{x_i} = a_i$, ta phải chứng minh rằng nếu $\prod\limits_{i=1}^{n} a_i = 1$ thì $\sum\limits_{i=1}^{n} \dfrac{1}{n-1+a_i} \leq 1$. Nhưng bất đẳng thức này đã được chứng minh trong bài toán 84. Vậy $M_n \geq \dfrac{1}{2}$ và bởi vì cho $x_1 = x_2 = ... = x_n$ ta có dấu bằng, ta suy ra $M_n = \dfrac{1}{2}$, điều này giải quyết bài toán.

96. [Vasile Cirtoaje] Nếu x, y, z là các số thực dương, thì

$$\frac{1}{x^2+xy+y^2}+\frac{1}{y^2+yz+z^2}+\frac{1}{z^2+zx+x^2} \geq \frac{9}{(x+y+z)^2}.$$

Gazeta Matematiă

Lời giải:
Xét đẳng thức

$$x^2+xy+y^2 = (x+y+z)^2-(xy+yz+zx)-(x+y+z)z,$$

ta có

$$\frac{(x+y+z)^2}{x^2+xy+y^2} = \frac{1}{1 - \dfrac{xy+yz+zx}{(x+y+z)^2} - \dfrac{z}{x+y+z}},$$

hoặc

$$\frac{(x+y+z)^2}{x^2+xy+y^2} = \frac{1}{1-(ab+bc+ca)-c},$$

với $a = \dfrac{x}{x+y+z}, b = \dfrac{y}{x+y+z}, c = \dfrac{x}{x+y+z}$. Bất đẳng thức có thể được viết lại như sau

$$\frac{1}{1-d-c} + \frac{1}{1-d-b} + \frac{1}{1-d-a} \geq 9,$$

với a, b, c l các số thực dương thoả mãn $a+b+c=1$ v $d = ab+bc+ca$. Sau một v i bước tính toán bất đẳng thức trở th nh

$$9d^3 - 6d^2 - 3d + 1 + 9abc \geq 0$$

hoặc

$$d(3d-1)^2 + (1-4d+9abc) \geq 0.$$

đây l bất đẳng thức **Schur**.

97. [Vasile Cirtoaje] Với mọi $a, b, c, d > 0$, chứng minh rằng

$$2(a^3+1)(b^3+1)(c^3+1)(d^3+1) \geq (1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

Gazeta Matematiă

Lời giải:
Sử dụng bất đẳng thức **Huygens**

$$\prod(1+a^4) \geq (1+abcd)^4,$$

chú ý rằng ta chỉ cần chỉ ra

$$2^4 \prod(a^3+1)^4 \geq \prod(1+a^4)(1+a^2)^4.$$

Hiển nhiên, ta chỉ cần chứng minh

$$2(a^3+1)^4 \geq (a^4+1)(a^2+1)^4$$

với mọi số thực a. Nhưng

$$(a^2+1)^4 \leq (a+1)^2(a^3+1)^2$$

và ta chỉ còn phải chứng minh bất đẳng thức

$$\begin{aligned}
&2(a^3+1)^2 \geq (a+1)^2(a^4+1)\\
\Longleftrightarrow &2(a^2-a+1)^2 \geq a^4+1\\
\Longleftrightarrow &(a-1)^4 \geq 0,
\end{aligned}$$

điều này là hiển nhiên.

98. Chứng minh rằng với mọi số thực a, b, c, ta có

$$(a+b)^4 + (b+c)^4 + (c+a)^4 \geq \frac{4}{7}(a^4+b^4+c^4).$$

Vietnam TST, 1996

Lời giải:
Ta thay thế $a+b=2z, b+c=2x, c+a=2y$. bất đẳng thức trở thành $\sum(y+z-x)^4 \leq 28\sum x^4$. Bây giờ ta có một chuỗi đồng nhất thức sau

$$\begin{aligned}
\sum(y+z-x)^4 =&\sum\left(\sum x^2+2yz-2xy-2xz\right)^2\\
=&3\left(\sum x^2\right)^2+4\left(\sum x^2\right)\left(\sum(yz-xy-xz)\right)+4\sum(xy+xz-yz)^2\\
=&3\left(\sum x^2\right)^2-4\left(\sum xy\right)\left(\sum x^2\right)+16\sum x^2y^2-4\left(\sum xy\right)^2\\
=&4\left(\sum x^2\right)^2+16\sum x^2y^2-\left(\sum x\right)^4\\
\leq&28\sum x^4,
\end{aligned}$$

bởi vì $\left(\sum x^2\right)^2 \geq 3\sum x^4, \sum x^2y^2 \geq x^4$.

99. Chứng minh rằng nếu a, b, c là các số thực dương thỏa mãn $abc = 1$, thì

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}.$$

Bulgaria, 1997

Lời giải:
Đặt $x=a+b+c$ và $y=ab+bc+ca$. Dễ thấy rằng vế trái của bất đẳng thức là $\dfrac{x^2+4x+y+3}{x^2+2x+y+xy}$, trong khi vế phải là $\dfrac{12+4x+y}{9+4x+2y}$. Bây giờ bất đẳng thức trở thành

$$\frac{x^2+4x+y+3}{x^2+2x+y+xy}-1 \leq \frac{12+4x+y}{9+4x+2y}-1 \Longleftrightarrow \frac{2x+3-xy}{x^2+2x+y+xy} \leq \frac{3-y}{9+4x+2y}.$$

Với bất đẳng thức cuối cùng, chúng ta quy đồng mẫu số. Sau đó sử dụng các bất đẳng thức $x \geq 3, y \geq 3, x^2 \geq 3y$, ta có

$$\frac{5}{3}x^2y \geq 5x^2, \frac{x^2y}{3} \geq y^2, xy^2 \geq 9x, 5xy \geq 15x, xy \geq 3y \text{ v } \; x^2y \geq 27.$$

Cộng các bất đẳng thức n y lại, ta được bất đẳng thức cần chứng minh.

100. [Tran Nam Dung] Tìm giá trị nhỏ nhất của biểu thức

$$\frac{1}{a} + \frac{2}{b} + \frac{3}{c},$$

trong đó a, b, c l các số thực dương thỏa mãn $21ab + 2bc + 8ca \leq 12$.

Vietnam, 2001

Lời giải 1: (bởi Trần Nam Dũng):
Đặt $\frac{1}{a} = x, \frac{2}{b} = y, \frac{3}{c} = z$. Khi đó dễ kiểm tra điều kiện của b i toán trở th nh $2xyz \geq 2x + 4y + 7z$. V ta cần tìm giá trị nhỏ nhất của $x + y + z$. Nhưng

$$z(2xy - 7) \geq 2x + 4y \Longrightarrow \begin{cases} 2xy > 7 \\ z \geq \dfrac{2x + 4y}{2xy - 7} \end{cases}$$

Bây giờ, ta biến đổi biểu thức để sau khi áp dụng bất đẳng thức **AM-GM** tử số $2xy - 7$ cần triệt tiêu

$$\begin{aligned} x + y + z \geq & x + y + \frac{2x + 4y}{2xy - 7} \\ = & x + \frac{11}{2x} + y - \frac{7}{2x} + \frac{2x + \frac{14}{x}}{2xy - 7} \\ \geq & x + \frac{11}{2x} + 2\sqrt{1 + \frac{7}{x^2}}. \end{aligned}$$

Nhưng, chứng minh trực tiếp được $2\sqrt{1 + \frac{7}{x^2}} \geq \dfrac{3 + \frac{7}{x}}{2}$ v do đó $x + y + z \geq \frac{3}{2} + x + \frac{9}{x} \geq \frac{15}{2}$. Ta có dấu bằng với $x = 3, y = \frac{5}{2}, z = 2$. Vậy, câu trả lời cho b i toán ban đầu l $\frac{15}{2}$, đạt được với $a = \frac{1}{3}, b = \frac{4}{5}, c = \frac{3}{2}$.

Lời giải 2:
Ta sử dụng sự thay thế tương tự và quy bài toán về tìm giá trị nhỏ nhất của $x+y+z$ khi $2xyz \geq 2x+4y+7z$. Áp dụng bất đẳng thức **AM-GM** ta thấy rằng

$$x+y+z = \frac{1}{15}\left(\underbrace{\frac{5}{2}x+...+\frac{5}{2}x}_{6 \text{ số}}+\underbrace{3y+...+3y}_{5 \text{ số}}+\underbrace{\frac{15}{4}x+...+\frac{15}{4}z}_{4 \text{ số}}\right) \geq (\frac{5x}{2})^{\frac{2}{5}}(3y)^{\frac{1}{3}}(\frac{15z}{4})^{\frac{4}{15}}.$$

Và cũng có

$$\begin{aligned} 2x+4y+7z =& \frac{1}{15}\left(\underbrace{10x+...+10x}_{3 \text{ số}}+\underbrace{12y+...+12y}_{5 \text{ số}}+\underbrace{15x+...+15z}_{7 \text{ số}}\right) \\ \geq& 10^{\frac{1}{5}}.12^{\frac{1}{3}}.15^{\frac{7}{15}}.x^{\frac{1}{5}}.y^{\frac{1}{3}}.z^{\frac{7}{15}}. \end{aligned}$$

Điều này có nghĩa là

$$(x+y+z)^2(2x+4y+7z) \geq \frac{225}{2}xyz.$$

Vì $2xyz \geq 2x+4y+7z$, ta sẽ có

$$(x+y+z)^2 \geq \frac{225}{4} \Longrightarrow x+y+z \geq \frac{15}{2},$$

dấu bằng xảy ra với $x=3, y=\frac{5}{2}, z=2$.

101. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng với mọi $x,y,z,a,b,c>0$ thỏa mãn $xy+yz+zx=3$, ta có

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq 3.$$

Lời giải:
Ta sẽ chứng minh bất đẳng thức

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq \sqrt{3(xy+yz+zx)}$$

với mọi a,b,c,x,y,z. Bởi vì bất đẳng thức thuần nhất với x,y,z ta có thể giả sử rằng

$x+y+z=1$. Áp dụng bất đẳng thức **Cauchy-Schwarz** ta có

$$\begin{aligned}
\frac{ax}{b+c}+\frac{by}{c+a}+\frac{cz}{a+b}+\sqrt{3(xy+yz+zx)} &\leq \\
&\leq\sqrt{\sum\left(\frac{a}{b+c}\right)^2}\sqrt{\sum x^2}+\sqrt{\frac{3}{4}}\sqrt{\sum xy}+\sqrt{\frac{3}{4}}\sqrt{\sum xy} \\
&\leq\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}\cdot\sqrt{\sum x^2+2\sum xy} \\
&=\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}.
\end{aligned}$$

Vậy, ta còn phải chứng minh bất đẳng thức $\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}}\leq\sum\frac{a}{b+c}$. Nhưng bất đẳng thức n y tương đương với $\sum\frac{ab}{(c+a)(c+b)}\geq\frac{3}{4}$, đây l bất đẳng thức tầm thường.

Chú ý
Một bất đẳng thức mạnh hơn sau đây:

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y)\geq\sum\sqrt{(x+y)(x+z)}-(x+y+z),$$

có thể thu được bởi áp dụng bất đẳng thức **Cauchy-Schwarz** , như sau

$$\begin{aligned}
\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) &= \\
&=(a+b+c)\left(\frac{y+z}{b+c}+\frac{z+x}{c+a}+\frac{x+y}{a+b}\right)-2(x+y+z) \\
&\geq\frac{1}{2}\left(\sqrt{y+z}+\sqrt{z+x}+\sqrt{x+y}\right)^2-2\left(x+y+z\right) \\
&=\sum\sqrt{(x+y)(x+z)}-(x+y+z).
\end{aligned}$$

Một b i tập hay cho người đọc l : chứng minh rằng

$$\sum\sqrt{(x+y)(x+z)}\geq x+y+z+\sqrt{3(xy+yz+zx)}.$$

102. Cho a,b,c l các số thực dương. Chứng minh rằng

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(a+c-b)^2}{(a+c)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Japan, 1997

Lời giải 1:
Đặt $x = \dfrac{b+c}{a}, y = \dfrac{c+a}{b}, z = \dfrac{a+b}{c}$. Bất đẳng thức có thể được viết thành

$$\sum \frac{(x-1)^2}{x^2+1} \geq \frac{3}{5}.$$

Sử dụng bất đẳng thức **Cauchy-Schwarz** , ta có

$$\sum \frac{(x-1)^2}{x^2+1} \geq \frac{(x+y+z-3)^2}{x^2+y^2+z^2+3}$$

và chỉ cần chứng minh rằng

$$\frac{(x+y+z-3)^2}{x^2+y^2+z^2+3} \geq \frac{3}{5} \Longleftrightarrow \left(\sum x\right)^2 - 15\sum x + 3\sum xy + 18 \geq 0.$$

Từ bất đẳng thức **Schur**, sau một vài bước tính toán, ta suy ra $\sum xy \geq 2\sum x$.
Vậy ta có

$$\left(\sum x\right)^2 - 15\sum x + 3\sum xy + 18 \geq \left(\sum x\right)^2 - 9\sum x + 18 \geq 0,$$

bất đẳng thức cuối cùng hiển nhiên đúng vì $\sum x \geq 6$.

Lời giải 2:
Hiển nhiên, ta có thể đặt $a+b+c=2$. Bất đẳng thức trở thành

$$\sum \frac{4(1-a)^2}{2+2(1-a)^2} \geq \frac{3}{5} \Longleftrightarrow \sum \frac{1}{1+(1-a)^2} \leq \frac{27}{10}.$$

Nhưng với sự thay thế $1-a=x, 1-b=y, 1-c=z$, bất đẳng thức được suy ra từ bài toán 47.

103. [Vasile Cirtoaje, Gabriel Dospinescu] Chứng minh rằng nếu $a_1, a_2, ..., a_n \geq 0$ thì

$$a_1^n + a_2^n + ... + a_n^n - na_1a_2...a_n \geq (n-1)\left(\frac{a_1+a_2+...+a_{n-1}}{n-1} - a_n\right)^n,$$

trong đó a_n là số nhỏ nhất trong các số $a_1, a_2, ..., a_n$.

Lời giải:
Đặt $a_i - a_n = x_i \geq 0$ với $i \in \{1, 2, 3, ..., n-1\}$. Bây giờ, ta xem

$$\sum_{i=1}^{n} a_i^n - n\prod_{i=1}^{n} a_i - (n-1)\left(\frac{\sum\limits_{i=1}^{n} a_i}{n-1} - a_n\right)^n$$

như l một đa thức của $a = a_n$. Thực tế l

$$a^n + \sum_{i=1}^{n-1}(a+x_i)^n - na\prod_{i=1}^{n}(a+x_i) - (n-1)\left(\frac{x_1 + x_2 + ... + x_{n-1}}{n-1}\right)^n.$$

Ta sẽ chứng minh rằng hệ số của a^k l không âm với mọi $k \in \{0, 1, ..., n-1\}$, bởi vì hiển nhiên l bậc của đa thức n y nhiều nhất l $n-1$. Với $k = 0$, điều n y suy ra từ tính lồi của h m $f(x) = x^n$

$$\sum_{i=1}^{n-1} x_i^n \geq (n-1)\left(\frac{\sum\limits_{i=1}^{n-1} x_i}{n-1}\right)^n.$$

Với $k > 0$, hệ số của a^k l

$$\binom{n}{k}\sum_{i=1}^{n-1} x_i^{n-k} - n\sum_{1 \leq i_1 < i_2 < ... < i_{n-k} \leq n-1} x_{i_1}x_{i_2}...x_{i_{n-k}}.$$

Ta sẽ chứng minh đây l số không âm. Từ bất đẳng thức **AM-GM** ta có

$$\begin{aligned} n\sum_{1 \leq i_1 < i_2 < ... < i_{n-k} \leq n-1} x_{i_1}x_{i_2}...x_{i_{n-k}} &\leq \\ &\leq \frac{n}{n-k}\sum_{1 \leq i_1 < i_2 < ... < i_{n-k} \leq n-1} x_{i_1}^{n-k}x_{i_2}^{n-k}...x_{i_{n-k}}^{n-k} \\ &= \frac{k}{n-1}\binom{n}{k}\sum_{i=1}^{n-1} x_i^{n-k}. \end{aligned}$$

Rõ r ng $\frac{k}{n-1}\binom{n}{k}\sum\limits_{i=1}^{n-1} x_i^{n-k} < \binom{n}{k}\sum\limits_{i=1}^{n-1} x_i^{n-k}$. Điều n y chỉ ra rằng mỗi hệ số của đa thức l không âm, vậy đa thức nhận giá trị không âm khi hạn chế trên tập những số không âm.

104. [Turkevici] Chứng minh rằng với mọi số thực dương x, y, z, t, ta có

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$$

Kvant

Lời giải:
Hiển nhiên, ta chỉ cần chứng minh bất đẳng thức với $xyzt = 1$, do đó bài toán được xét với giả thiết này.

Nếu a, b, c, d có tích bằng 1, thì

$$a^2 + b^2 + c^2 + d^2 + 2 \geq ab + bc + cd + da + ac + bd.$$

Giả sử d là só nhỏ nhất trong các số a, b, c, d và đặt $m = \sqrt[3]{abc}$. Ta sẽ chứng minh

$$a^2 + b^2 + c^2 + d^2 + 2 - (ab + bc + cd + da + ac + bd) \geq d^2 + 3m^2 + 2 - (3m^2 + 3md),$$

Bất đẳng thức này thực tế là

$$a^2 + b^2 + c^2 - ab - bc - ca \geq d(a + b + c - 3\sqrt[3]{abc}).$$

Bởi vì $d \leq \sqrt[3]{abc}$, nên để chứng minh bất đẳng thức đầu tiên ta cần chứng minh

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \sqrt[3]{abc}(a + b + c - 3\sqrt[3]{abc}).$$

Đặt $u = \dfrac{a}{\sqrt[3]{abc}}, v = \dfrac{b}{\sqrt[3]{abc}}, w = \dfrac{c}{\sqrt[3]{abc}}$. Sử dụng bài toán 74, ta có

$$u^2 + v^2 + w^2 + 3 \geq u + v + w + uv + vw + wu,$$

bất đẳng thức này chính xác là

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \sqrt[3]{abc}(a + b + c - 3\sqrt[3]{abc}).$$

Vậy, ta chỉ còn phải chứng minh rằng $d^2 + 2 \geq 3md \iff d^2 + 2 \geq 3\sqrt[3]{d^2}$, bất đẳng thức này hiển nhiên đúng.

105. Chứng minh rằng với mọi số thực $a_1, a_2, ..., a_n$, bất đẳng thức sau đúng

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ij}{i + j - 1} a_i a_j.$$

Lời giải:
Ta thấy rằng

$$\sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j = \sum_{i,j=1}^{n} ia_i.ja_j \int_0^1 t^{i+j-2} dt$$
$$= \int_0^1 \left(\sum_{i,j=1}^{n} ia_i.ja_j.t^{i-1+j-1} \right) dt$$
$$= \int_0^1 \left(\sum_{i=1}^{n} ia_i.t^{i-1} \right)^2 dt.$$

Bây giờ, sử dụng bất đẳng thức **Cauchy-Schwarz** cho tích phân, ta có

$$\int_0^1 \left(\sum_{i=1}^{n} ia_i.t^{i-1} \right)^2 dt \geq \left(\int_0^1 \left(\sum_{i=1}^{n} ia_i.t^{i-1} \right) dt \right)^2 = \left(\sum_{i=1}^{n} a_i \right)^2,$$

điều n y kết thúc chứng minh.

106. Chứng minh rằng nếu $a_1, a_2, ..., a_n, b_1, ..., b_n$ l các số thực nằm giữa 1001 v 2002, thỏa mãn

$$a_1^2 + a_2^2 + ... + a_n^2 = b_1^2 + b_2^2 + ... + b_n^2,$$

thì ta có bất đẳng thức

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + ... + \frac{a_n^3}{b_n} \leq \frac{17}{10}(a_1^2 + a_2^2 + ... + a_n^2).$$

TST Singapore

Lời giải:
Ý tưởng then chốt l $\frac{a_i}{b_i} \in [\frac{1}{2}, 2]$ với mọi i v với mọi $x \in [\frac{1}{2}, 2]$, ta có bất đẳng thức $x^2 + 1 \leq \frac{5}{2}x$. Do đó, ta có

$$\frac{5}{2}.\frac{a_i}{b_i} \geq 1 + \frac{a_i^2}{b_i^2} \Longleftrightarrow \frac{5}{2} a_i b_i \geq a_i^2 + b_i^2$$

v cũng có

$$(1) \qquad \frac{5}{2} \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} (a_i^2 + b_i^2) = 2 \sum_{i=1}^{n} a_i^2.$$

Bây giờ, ta thấy rằng $\frac{a_i^2}{b_i^2} = \frac{\frac{a_i^3}{b_i}}{a_ib_i}$ và bất đẳng thức $\frac{5}{2}.\frac{a_i}{b_i} \geq 1 + \frac{a_i^2}{b_i^2}$ cho ta $\frac{5}{2}a_i^2 \geq \frac{a_i^3}{b_i} + a_ib_i$ và cộng các bất đẳng thức lại, ta được

(2) $$\frac{5}{2}\sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} \frac{a_i^3}{b_i} + \sum_{i=1}^{n} a_ib_i.$$

Sử dụng (1) và (2) ta có

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + ... + \frac{a_n^3}{b_n} \leq \frac{17}{10}(a_1^2 + a_2^2 + ... + a_n^2),$$

đây là bất đẳng thức cần chứng minh.

107. [Titu Andreescu, Gabriel Dospinescu] Chứng minh rằng nếu a, b, c là các số thực dương có tổng bằng 1, thì

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8(a^2b^2 + b^2c^2 + c^2a^2)^2.$$

Lời giải:
Đặt $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. Ta tìm thấy dạng tương đương nếu $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ thì

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \geq 8(x^2 + y^2 + z^2)^2.$$

Ta sẽ chứng minh bất đẳng thức sau

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^2 \geq 8(x^2 + y^2 + z^2)^2,$$

với mọi số thực dương x, y, z. Viết $x^2 + y^2 = 2c, y^2 + z^2 = 2a, z^2 + x^2 = 2b$. Khi đó bất đẳng thức trở thành

$$\sum \sqrt{\frac{abc}{b + c - a}} \geq \sum a.$$

Nhắc lại bất đẳng thức **Schur**

$$\sum a^4 + abc(a + b + c) \geq \sum a^3(b + c) \Longleftrightarrow abc(a + b + c) \geq \sum a^3(b + c - a).$$

Bây giờ, sử dụng bất đẳng thức **Holder,** ta có

$$\sum a^3(b + c - a) = \sum \frac{a^3}{\left(\frac{1}{\sqrt{b + c - a}}\right)^2}.$$

Kết hợp hai bất đẳng thức trên, ta được

$$\sum \sqrt{\frac{abc}{b+c-a}} \geq \sum a$$

v bất đẳng thức được chứng minh.

108. [Vasile Cirtoaje] Nếu a, b, c, d l các số thực dương thỏa mãn $abcd = 1$, thì

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

Gazeta Matematiă

Lời giải:
Bất đẳng thức cần chứng minh được suy ra bởi tổng các bất đẳng thức

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab},$$

$$\frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{1}{1+cd}.$$

Các bất đẳng thức n y suy ra từ

$$\begin{aligned}\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} - \frac{1}{1+ab} &= \frac{ab(a^2+b^2) - a^2b^2 - 2ab + 1}{(1+a)^2(1+b)^2(1+ab)} \\ &= \frac{ab(a-b)^2 + (ab-1)^2}{(1+a)^2(1+b)^2(1+ab)} \geq 0.\end{aligned}$$

Dấu bằng xảy ra nếu $a = b = c = d = 1$.

109. [Vasile Cirtoaje] Cho a, b, c l các số thực dương. Chứng minh rằng

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^+b^2} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Gazeta Matematiă

Lời giải:
Ta có các đồng nhất thức sau

$$\frac{a^2}{b^2+c^2} - \frac{a}{b+c} = \frac{ab(a-b+ac(a-c)}{(b+c)(b^2+c^2)}$$

$$\frac{b^2}{c^2+a^2} - \frac{b}{c+a} = \frac{bc(b-c)+ab(b-a)}{(c+a)(c^2+a^2)}$$
$$\frac{c^2}{a^2+b^2} - \frac{c}{a+b} = \frac{ac(c-a)+bc(c-b)}{(b+a)(b^2+a^2)}.$$

Vậy ta có

$$\begin{aligned}\sum \frac{a^2}{b^2+c^2} - \sum \frac{a}{b+c} &\\ &= \sum \left[\frac{ab(a-b)}{(b+c)(b^2+c^2)} - \frac{ab(a-b)}{(a+c)(a^2+c^2)}\right] \\ &= (a^2+b^2+c^2+ab+bc+ca). \sum \frac{ab(a-b)^2}{(b+c)(c+a)(b^2+c^2)(c^2+a^2)} \geq 0.\end{aligned}$$

110. [Gabriel Dospinescu] Cho $a_1, a_2, ..., a_n$ là các số thực và S là một tập con khác rỗng của $\{1, 2, ..., n\}$. Chứng minh rằng

$$\left(\sum_{i\in S} a_i\right)^2 \leq \sum_{1\leq i\leq j\leq n} (a_i + ... + a_j)^2.$$

TST 2004, Romania

Lời giải 1:
Ki hiệu $s_k = a_1 + a_2 + ... + a_k$ với $k = 1, ..., n$ và $s_{n+1} = 0$. Bây giờ định nghĩa

$$b_i = \begin{cases} 1 \text{ nếu } & i \in S \\ 0, \text{ nếu } & i \notin S \end{cases}$$

Sử dụng tổng **Abel** ta có

$$\begin{aligned}\sum_{i\in S} a_i &= a_1b_1 + a_2b_2 + ... + a_nb_n \\ &= s_1(b_1-b_2) + s_2(b_2-b_3) + ... + s_{n-1}(b_{n-1}-b_n) + s_nb_n + s_{n+1}(-b_1).\end{aligned}$$

Bây giờ, đặt $b_1 - b_2 = x_1, b_2 - b_3 = x_2, ..., b_{n-1} - b_n = x_{n-1}, b_n = x_n, x_{n+1} = -b_1$. Vậy ta có

$$\sum_{i\in S} a_i = \sum_{i=1}^{n+1} x_i s_i$$

v $x_i \in \{-1, 0, 1\}$. Hiển nhiên, $\sum\limits_{i=1}^{n+1} x_i = 0$. Mặt khác, sử dụng đồng nhất thức Lagrange ta có

$$\begin{aligned}\sum_{1\le i\le j\le n} (a_i + ... + a_j)^2 =& \sum_{i=1}^{n} s_i^2 + \sum_{1\le i<j\le n} (s_j - s_i)^2 \\ =& \sum_{1\le i<j\le n+1} (s_j - s_i)^2 \\ =& (n+1)\sum_{i=1}^{n+1} s_i^2 - \left(\sum_{i=1}^{n+1} s_i\right)^2.\end{aligned}$$

Vậy ta cần phải chứng minh rằng

$$(n+1)\sum_{i=1}^{n+1} s_i^2 \ge \left(\sum_{i=1}^{n+1} s_i x_i\right)^2 + \quad \sum_{i=1}^{n+1} s_i \Bigg)^2.$$

Nhưng ta có

$$\sum_{i=1}^{n+1} s_i x_i \Bigg)^2 + \quad \sum_{i=1}^{n+1} s_i \Bigg)^2 = \sum_{i=1}^{n+1} s_i^2(1 + x_i^2) + 2\sum_{1\le i<j\le n+1} s_i s_j (x_i x_j + 1).$$

Bây giờ, sử dụng hệ thức $2s_i s_j \le s_i^2 + s_j^2, 1 + x_i x_j \ge 0$, ta có thể viết

$$\begin{aligned}&2\sum_{1\le i<j\le n+1} s_i s_j (x_i x_j + 1) \le \sum_{1\le i<j\le n+1} (s_i^2 + s_j^2)(1 + x_i x_j) = \\ =& n(s_1^2 + ... + s_{n+1}^2) + s_1^2 x_1 (x_2 + x_3 + ... + x_n) + ... + s_n^2 x_n (x_1 + x_2 + ... + x_{n-1}) \\ =& - s_1^2 x_1^2 - ... - s_n^2 x_n^2 + n(s_1^2 + ... + s_n^2).\end{aligned}$$

Vậy

$$\sum_{i=1}^{n+1} s_i x_i \Bigg)^2 + \quad \sum_{i=1}^{n+1} s_i \Bigg)^2 \le \sum_{k=1}^{n+1} s_k^2 (x_k^2 + 1) + \sum_{k=1}^{n+1} s_k^2 (n - x_k^2) = (n+1)\sum_{k=1}^{n+1} s_k^2$$

v ta có điều phải chứng minh.

Lời giải 2 (bởi Andrei Negut):
Đầu tiên, ta chứng minh bổ đề
Bổ đề
Với mọi $a_1, a_2, ..., a_{2k+1} \in \mathbb{R}$ ta có bất đẳng thức

$$\sum_{i=0}^{k} a_{2i+1} \Bigg)^2 \le \sum_{1\le i\le j\le 2k+1} (a_i + ... + a_j)^2.$$

Chứng minh của bổ đề. Đặt $s_k = a_1 + ... + a_k$. Ta có

$$\sum_{i=0}^{k} a_{2i+1} = s_1 + s_3 - s_2 + ... + s_{2k+1} - s_{2k},$$

vậy vế trái trong bổ đề là

$$\sum_{i=1}^{2k+1} s_i^2 + 2 \sum_{0\le i<j\le k} s_{2i+1}s_{2j+1} + 2 \sum_{1\le i<j\le k} s_{2i}s_{2j} - 2 \sum_{\substack{0\le i\le k \\ 1\le j\le k}} s_{2i+1}s_{2j}$$

và vế phải là

$$(2k+1)\sum_{i=1}^{2k+1} s_i^2 - 2 \sum_{1\le i<j\le 2k+1} s_i s_j.$$

Do đó, ta cần chứng minh rằng

$$2k \sum_{i=1}^{2k+1} s_i^2 \ge 4 \sum_{0\le i<j\le k} s_{2i+1}s_{2j+1} + 4 \sum_{1\le i<j\le k} s_{2i}s_{2j}$$

và bất đẳng thức có được bởi cộng các bất đẳng thức

$$2s_{2j+1}s_{2i+1} \le s_{2i+1}^2 + s_{2j+1}^2, 2s_{2j}s_{2i} \le s_{2i}^2 + s_{2j}^2.$$

Bây giờ, ta quay lại lời giải của bài toán. Ta có thể biểu diễn S dưới dạng sau

$$S = \{a_{i_1}, a_{i_1+1}, ..., a_{i_1+k_1}, a_{i_2}, a_{i_2+1}, ..., a_{i_2+k_2}, ..., a_{i_r}, ..., a_{i_r+k_r}\}$$

với $i_j + k_j < i_{j+1} - 1$. Do đó, ta viết S như là một dãy, theo sau là một khoảng trống, theo sau là một dãy, theo sau là một khoảng trống, vân vân. Bây giờ ta đặt

$$s_1 = a_{i_1} + ... + a_{i_1+k_1}, s_2 = a_{i_1+k_1+1} + ... + a_{i_2-1}, ..., s_{2r-1} = a_{i_r} + ... + a_{i_r+k_r}.$$

Khi đó

$$\begin{aligned}\left(\sum_{i\in S} a_i\right)^2 &= (s_1 + s_3 + ... + s_{2r-1})^2 \\ &\le \sum_{1\le i\le j\le 2r-1} (s_i + ... + s_j)^2 \\ &\le \sum_{1\le i\le j\le n} (a_i + ... + a_j)^2.\end{aligned}$$

Bất đẳng thức cuối cùng hiển nhiên đúng, bởi vì các số hạng ở vế trái nằm trong số các số hạng ở vế phải.

Lời giải 3:
Ta sẽ chứng minh bất đẳng thức bằng quy nạp. Với $n = 2$ v $n = 3$ bất đẳng thức dễ chứng minh. Giải sử bất đẳng thức đúng cho mọi $k < n$ v ta phải chứng minh bất đẳng thức đúng cho n. Nếu $1 \notin S$ thì chúng ta chỉ áp dụng bước quy nạp cho các số $a_2, ..., a_n$, bởi vì vế phải không giảm. Bây giờ, giả sử $1 \in S$. Nếu $2 \in S$, thì ta áp dụng bước quy nạp với các số $a_1 + a_2, a_3, ..., a_n$. Do đó, ta có thể giả thiết rằng $2 \notin S$. Dễ thấy rằng $(a+b+c)^2 + c^2 \geq \dfrac{(a+b)^2}{2} \geq 2ab$ v suy ra ta có

$$(1) \qquad (a_1 + a_2 + ... + a_k)^2 + (a_2 + a_3 + ... + a_{k-1})^2 \geq 2a_1a_k.$$

Cũng vậy, bước quy nạp cho $a_3, ..., a_n$ chỉ ra rằng

$$\left(\sum_{i \in S \setminus \{1\}} a_i\right)^2 \leq \sum_{3 \leq i \leq j \leq n} (a_i + ... + a_j)^2.$$

Vậy chỉ cần chứng minh rằng

$$a_1^2 + 2a_1 \sum_{i \in S\setminus\{1\}} a_i \leq \sum_{i=1}^{n} (a_1 + ... + a_i)^2 + \sum_{i=2}^{n} (a_2 + ... + a_i)^2.$$

Nhưng bất đẳng thức n y rõ r ng vì a_1^2 xuất hiện trong vế phải v bởi tổng các bất đẳng thức trong (1).

111. [Dung Tran Nam] Cho $x_1, x_2, ..., x_{2004}$ l các số thực trong đoạn $[-1, 1]$ thỏa mãn $x_1^3 + x_2^3 + ... + x_{2004}^3 = 0$. Tìm giá trị lớn nhất của $x_1 + x_2 ... + x_{2004}$.

Lời giải:
Ta đặt $a_i = x_i^3$ v h m số $f : [-1, 1] \to \mathbb{R}, f(x) = x^{\frac{1}{3}}$. Đầu tiên ta sẽ chứng minh những tính chất của f:
1. $f(x+y+1) + f(-1) \geq f(x) + f(y)$ nếu $-1 < x, y < 0$.
2. f l lồi trên $[-1, 0]$ v lõm trên $[0, 1]$.
3. Nếu $x > 0$ v $y < 0$ v $x + y < 0$ thì $f(x) + f(y) \leq f(-1) + f(x+y+1)$ v nếu $x + y > 0$ thì $f(x) + f(y) \leq f(x+y)$.
Chứng minh các kết quả n y dễ d ng. Thật vậy, với kết quả đầu tiên ta đặt $x = -a^3, y = -b^3$ v bất đẳng thức trở th nh

$$\begin{aligned} & 1 > a^3 + b^3 + (1-a-b)^3 \\ \Longleftrightarrow & 1 > (a+b)(a^2 - ab + b^2) + 1 - 3(a+b) + 3(a+b)^2 - (a+b)^3 \\ \Longleftrightarrow & 3(a+b)(1-a)(1-b) > 0, \end{aligned}$$

bất đẳng thức này đúng. Kết quả thứ hai là hiển nhiên và kết quả thứ ba có thể dễ dàng suy ra như trong cách chứng minh của 1.
Từ lập luận trên ta suy ra rằng nếu $(t_1, t_2, ..., t_{2004}) = t$ là điểm mà hàm số

$$g : A = \left\{x \in [-1,1]^{2004} \middle| x_1 + x_2 + ... + x_n = 0\right\} \to \mathbb{R}, \quad g(x_1, ..., x_{2004}) = \sum_{k=1}^{2004} f(x_k)$$

đạt giá trị lớn nhất (giá trị lớn nhất tồn tại bời vì hàm số xác định trên một tập compact) khi đó ta có tất cả các thành phần dương của t phải bằng nhau và tất cả các thành phần âm bằng -1. Vậy giả sử ta có k thành phần bằng -1 và $2004-k$ thành phần bằng số a. Bởi vì $t_1 + t_2 + ... + t_{2004} = 0$ ta có $a = \dfrac{k}{2004-k}$ và giá trị của g tại điểm này bằng $(2004-k)\sqrt[3]{\dfrac{k}{2004-k}} - k$. Vậy ta phải tìm giá trị lớn nhất của $(2004-k)\sqrt[3]{\dfrac{k}{2004-k}} - k$, khi k trong tập $\{0, 1, ..., 2004\}$. Bằng cách dùng đạo hàm ta thấy rằng giá trị lớn nhất đạt được khi $k = 223$ và giá trị lớn nhất bằng $\sqrt[3]{223}.\sqrt[3]{1718^2} - 223$.

112. [Gabriel Dospinescu, Călin Popa] Chứng minh rằng nếu $n \geq 2$ và $a_1, a_2, ..., a_n$ là các số thực có tích bằng 1, thì

$$a_1^2 + a_2^2 + ... + a_n^2 - n \geq \frac{2n}{n-1}.\sqrt[n]{n-1}(a_1 + a_2 + ... + a_n - n).$$

Lời giải:
Ta sẽ chứng minh bất đẳng thức bằng quy nạp. Với $n = 2$ bất đẳng thức là tầm thường. Bây giờ giả sử bất đẳng thức đúng với $n-1$ số, ta chứng minh bất đẳng thức đúng cho n số. Đâu tiên, dễ thấy rằng ta chỉ cần chứng minh cho $a_1, ..., a_2 > 0$ (ngược lại ta có thể thay thế $a_1, ..., a_n$ với $|a_1|, ..., |a_n|$, mà có tích bằng 1, khi đó vế phải tăng). Bây giờ, giả sử a_n là số lớn nhất giữa các số $a_1, a_2, ..., a_n$ và giả sử G là trung bình nhân của $a_1, a_2, ..., a_{n-1}$. Đầu tiên, ta sẽ chứng minh rằng

$$\begin{aligned}
&a_1^2 + a_2^2 + ... + a_n^2 - n - \frac{2n}{n-1}.\sqrt[n]{n-1}(a_1 + a_2 + ... + a_n - n) \\
&\geq a_n^2 + (n-1)G^2 - n - \frac{2n}{n-1}.\sqrt[n]{n-1}\left(a_n + (n-1)G - n\right),
\end{aligned}$$

bất đẳng thức này tương đương với

$$\begin{aligned}
&a_1^2 + a_2^2 + ... + a_{n-1}^2 - (n-1)\sqrt[n-1]{a_1^2.a_2^2...a_{n-1}^2} \\
\geq &\frac{2n}{n-1}\sqrt[n]{n-1}\left(a_1 + a_2 + ... + a_{n-1} - (n-1)\sqrt[n-1]{a_1.a_2...a_{n-1}}\right).
\end{aligned}$$

Vì $\sqrt[n-1]{a_1.a_2...a_{n-1}} \leq 1$ v $a_1 + a_2 + ... + a_{n-1} - (n-1)\sqrt[n-1]{a_1.a_2...a_{n-1}} \geq 0$, nên để chứng minh bất đẳng thức trên ta sẽ chứng minh bất đẳng thức

$$(1) \qquad a_1^2 + ... + a_{n-1}^2 - (n-1)G^2 \geq \frac{2n}{n-1}\sqrt[n]{n-1}.G.\left(a_1 + ... + a_{n-1} - (n-1)G\right).$$

Bây giờ áp dụng giả thiết quy nap cho các số $\frac{a_1}{G}, ..., \frac{a_{n-1}}{G}$ có tích bằng 1 v ta suy ra rằng

$$\frac{a_1^2 + ... + a_{n-1}^2}{G^2} - n + 1 \geq \frac{2(n-1)}{n-2}.\sqrt[n-1]{n-2}\left(\frac{a_1 + ... + a_{n-1}}{G} - n + 1\right),$$

vậy để chứng minh bất đẳng thức (1) chỉ cần chứng minh rằng

$$\frac{2(n-1)}{n-2}.\sqrt[n-1]{n-2}\Big(a_1+...+a_{n-1}-(n-1)G\Big) \geq \frac{2n}{n-1}\sqrt[n]{n-1}\Big(a_1+...+a_{n-1}-(n-1)G\Big),$$

bất đẳng thức n y tương đương với $1+\frac{1}{n(n-2)} \geq \frac{\sqrt[n]{n-1}}{\sqrt[n-1]{n-2}}$. Bất đẳng thức trở th nh

$$\left(1 + \frac{1}{n(n-2)}\right)^{n(n-1)} \geq \frac{(n-1)^{n-1}}{(n-2)^n},$$

bất đẳng thức trên đúng với $n > 4$ vì

$$\left(1 + \frac{1}{n(n-2)}\right)^{n(n-1)} > 2$$

v

$$\frac{(n-1)^{n-1}}{(n-2)^n} = \frac{1}{n-2}(1 + \frac{1}{n-2})(1 + \frac{1}{n-2})^{n-2} < \frac{e}{n-2}(1 + \frac{1}{n-2}) < 2.$$

Với $n = 3$ v $n = 4$ điều n y dễ kiểm tra. Vậy ta đã chứng minh được rằng

$$(2) \qquad \begin{aligned} &a_1^2 + a_2^2 + ... + a_n^2 - n - \frac{2n}{n-1}.\sqrt[n]{n-1}\left(a_1 + a_2 + ... + a_n - n\right) \\ &\geq a_n^2 + (n-1)G^2 - n - \frac{2n}{n-1}.\sqrt[n]{n-1}\left(a_n + (n-1)G - n\right). \end{aligned}$$

Đặt $x = \frac{1}{G}$, khi đó vế phải của bất đẳng thức (2) tương đương với

$$x^{2(n-1)} + \frac{n-1}{x^2} - n - \frac{2n}{n-1}\sqrt[n]{n-1}\left(x^{n-1} + \frac{n-1}{x} - n\right).$$

Để kết thúc bước quy nạp ta chỉ cần chứng minh rằng

$$x^{2(n-1)} + \frac{n-1}{x^2} - n \geq \frac{2n}{n-1}\sqrt[n]{n-1}\left(x^{n-1} + \frac{n-1}{x} - n\right)$$

với $x \geq 1$. Xét hàm số

$$f(x) = x^{2(n-1)} + \frac{n-1}{x^2} - n - \frac{2n}{n-1}\sqrt[n]{n-1}\left(x^{n-1} + \frac{n-1}{x} - n\right).$$

Ta có

$$f'(x) = 2.\frac{x^n-1}{x^2}.\left[\frac{(n-1)(x^n+1)}{x} - n\sqrt[n]{n-1}\right] \geq 0$$

bởi vì

$$x^{n-1} + \frac{1}{x} = x^{n-1} + \frac{1}{(n-1)x} + ... + \frac{1}{(n-1)x} \geq n\sqrt[n]{\frac{1}{(n-1)^{n-1}}}.$$

Vậy f là hàm tăng, do đó $f(x) \geq f(1) = 0$. Điều này chứng minh bất đẳng thức.

Lời người dịch: Cách chứng minh này chưa giải được bài toán, vì tác giả vẫn chưa chứng minh được bất đẳng thức (2). Bởi vì, vế phải của bất đẳng thức (1) không lớn hơn vế phải của bất đẳng thức (2) do $G := \sqrt[n-1]{a_1.a_2...a_{n-1}} \leq 1$.

113. [Vasile Cirtoaje] Nếu a, b, c là cá số thực dương, thì

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \leq 3.$$

Gazeta Matematiă

Lời giải 1:

Bằng cách đặt $x = \sqrt{\frac{b}{a}}, y = \sqrt{\frac{c}{b}}, z = \sqrt{\frac{a}{c}}$ bài toán quy về chứng minh rằng $xyz = 1$ suy ra

$$\sqrt{\frac{2}{1+x^2}} + \sqrt{\frac{2}{1+y^2}} + \sqrt{\frac{2}{1+z^2}} \leq 3.$$

Ta giả sử $x \le y \le z$, điều n y suy ra $xy \le 1$ v $z \ge 1$. Ta có

$$\begin{aligned}\left.\sqrt{\frac{2}{1+x^2}}+\sqrt{\frac{2}{1+y^2}}\right)^2 \le& 2\left(\frac{2}{1+x^2}+\frac{2}{1+y^2}\right)\\ =&4\left[1+\frac{1-x^2y^2}{(1+x^2)(1+y^2)}\right]\\ \le& 4\left[1+\frac{1-x^2y^2}{(1+xy)^2}\right]\\ =&\frac{8}{1+xy}=\frac{8z}{z+1}.\end{aligned}$$

Vậy

$$\sqrt{\frac{2}{1+x^2}}+\sqrt{\frac{2}{1+y^2}} \le 2\sqrt{\frac{2z}{z+1}}$$

v ta cần chứng minh

$$2\sqrt{\frac{2z}{z+1}}+\sqrt{\frac{2}{1+z^2}} \le 3.$$

Bởi vì

$$\sqrt{\frac{2z}{z+1}} \le \frac{2}{1+z}$$

nên ta chỉ cần chứng minh

$$2\sqrt{\frac{2z}{z+1}}+\frac{2}{1+z} \le 3.$$

Bất đẳng thức n y tương đương với

$$1+3z-2\sqrt{2z(1+z)} \ge 0, (\sqrt{2z}-\sqrt{z+1})^2 \ge 0,$$

v ta có điều phải chứng minh.

Lời giải 2:
Hiển nhiên, b i toán yêu cầu chứng minh rằng nếu $xyz = 1$ thì

$$\sum \sqrt{\frac{2}{x+1}} \le 3.$$

Ta có hai trường hợp. Trường hợp đầu l dễ khi $xy+yz+zx \ge x+y+z$. Trong trường hợp n y chúng ta có thể áp dụng bất đẳng thức **Cauchy-Schwarz** để có

$$\sum \sqrt{\frac{2}{x+1}} \le \sqrt{3\sum \frac{2}{x+1}}.$$

Nhưng

$$\begin{aligned}
&\sum \frac{1}{x+1} \le \frac{3}{2}\\
\Longleftrightarrow &\sum 2(xy+x+y+1) \le 3(2+x+y+z+xy+yz+zx)\\
\Longleftrightarrow &x+y+z \le xy+yz+zx
\end{aligned}$$

và trong trường hợp này bất đẳng thức được chứng minh.
Trường hợp thứ hai khi $xy+yz+zx < x+y+z$. Suy ra

$$(x-1)(y-1)(z-1) = x+y+z-xy-yz-zx > 0,$$

do đó có chính xác hai trong các số x, y, z nhỏ hơn 1, giả sử là x và y. Vậy ta phải chứng minh rằng nếu x và y nhỏ hơn 1, thì

$$\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2xy}{xy+1}} \le 3.$$

Sử dụng bất đẳng thức **Cauchy-Schwarz**, ta có

$$\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2xy}{xy+1}} \ge 2\sqrt{\frac{1}{x+1}+\frac{1}{y+1}} + \sqrt{\frac{2xy}{xy+1}}$$

và ta chỉ cần chứng minh rằng số hạng cuối cùng nhiều nhất là 3. Điều này tương đương với

$$2.\frac{\frac{1}{x+1}+\frac{1}{y+1}-1}{1+\sqrt{\frac{1}{x+1}+\frac{1}{y+1}}} \le \frac{1-\frac{2xy}{xy+1}}{1+\sqrt{\frac{2xy}{xy+1}}}.$$

Bởi vì ta có $\frac{1}{x+1}+\frac{1}{y+1} \ge 1$, vế trái bất đẳng thức trên nhiều nhất là

$$\frac{1}{x+1}+\frac{1}{y+1}-1 = \frac{1-xy}{(x+1)(y+1)}.$$

Ta còn phải chứng minh bất đẳng thức

$$\begin{aligned}
&\frac{1-xy}{(x+1)(y+1)} \le \frac{1-xy}{(xy+1)\left(1+\sqrt{\frac{2xy}{xy+1}}\right)}\\
\Longleftrightarrow &xy+1+(xy+1)\sqrt{\frac{2xy}{xy+1}} \le xy+1+x+y\\
\Longleftrightarrow &x+y \ge \sqrt{2xy(xy+1)},
\end{aligned}$$

bất đẳng thức này suy ra từ $\sqrt{2xy(xy+1)} \le 2\sqrt{xy} \le x+y$. Bài toán đã giải xong.

114. Chứng minh rằng bất đẳng thức sau đúng với mọi số thực dương x, y, z

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}\right)\geq\frac{9}{4}.$$

Iran, 1996

Lời giải 1 (bởi Iurie Boreico)
Với sự thay thế $x+y=c, y+z=a, z+x=b$, sau một v i bước tính toán bất đẳng thức trở th nh

$$\sum\left(\frac{2}{ab}-\frac{1}{c^2}\right)(a-b)^2\geq 0.$$

Giả sử $a\geq b\geq c$. Nếu $2c^2\geq ab$, mỗi số hạng trong biểu thức ở trên l dương v ta có điều phải chứng minh. Vậy giả sử $2c^2<ab$. Đầu tiên, chúng ta chứng minh rằng $2b^2\geq ac, 2a^2\geq bc$. Giả sử $2b^2<ac$. Khi đó $(b+c)^2\leq 2(b^2+c^2)<a(b+c)$, do đó $b+c<a$, mâu thuẫn. Hiển nhiên, ta có thể viết bất đẳng thức như l

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)(a-c)^2+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2\geq\left(\frac{1}{c^2}-\frac{1}{ab}\right)(a-b)^2.$$

Chúng ta có thể nhìn thấy ngay rằng $(a-c)^2\geq(a-b)^2+(b-c)^2$ l đúng v ta chỉ cần chứng minh rằng

$$\left(\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\right)(b-c)^2\geq\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}\right)(a-b)^2.$$

Nhưng ta có $\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}<\left(\frac{1}{b}-\frac{1}{c}\right)^2$ v vế phải của bất đẳng thức nhiều nhất l $\frac{(a-b)^2(b-c)^2}{b^2c^2}$. Cũng dễ thấy rằng

$$\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\geq\frac{1}{ac}+\frac{1}{bc}>\frac{(a-b)^2}{b^2c^2},$$

điều n y chỉ ra rằng về trái ít nhất bằng $\frac{(a-b)^2(b-c)^2}{b^2c^2}$ v chứng minh được ho n th nh.

Lời giải 2:
Vì bất đẳng thức l thuần nhất, ta có thể giả sử rằng $xy+yz+zx=3$. Ta cũng có thể đặt $x+y+z=3a$. Từ $(x+y+z)^2\geq 3(xy+yz+zx)$, ta có $a\geq 1$. Bây giờ ta viết

bất đẳng thức như sau

$$\begin{aligned}
&\frac{1}{(3a-z)^2}+\frac{1}{(3a-y)^2}+\frac{1}{(3a-x)^2}\geq\frac{3}{4}\\
\Longleftrightarrow & 4\left[(xy+3az)^2+(yz+3ax)^2+(zx+3ay)^2\right]\geq 3(9a-xyz)^2,\\
\Longleftrightarrow & 4(27a^4-18a^2+3+4axyz)\geq(9a-xyz)^2,\\
\Longleftrightarrow & 3(12a^2-1)(3a^2-4)+xyz(34a-xyz)\geq 0, \qquad (1)\\
\Longleftrightarrow & 12(3a^2-1)^2+208a^2\geq(17a-xyz)^2. \qquad (2)
\end{aligned}$$

Ta có hai trường hợp.
i) Trường hợp $3a^2-4\geq 0$. Từ

$$34a-xyz=\frac{1}{9}[34(x+y+z)(xy+yz+zx)-9xyz]>0,$$

suy ra bất đẳng thức (1) đúng.
ii) Trường hợp $3a^2-4<0$. Từ bất đẳng thức **Schur**

$$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz\geq 0,$$

suy ra $3a^3-4a+xyz\geq 0$. Vậy

$$\begin{aligned}
12(3a^2-1)^2+208a^2-(17a-xyz)^2&\geq 12(3a^2-1)^2+208a^2-a^2(3a^2+13)^2\\
&=3(4-11a^2+10a^4-3a^6)\\
&=3(1-a^2)^2(4-3a^2)^2\geq 0.
\end{aligned}$$

115. Chứng minh rằng với mọi x, y trong đoạn $[0,1]$,

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2}\geq(1+\sqrt{5})(1-xy).$$

Lời giải (bởi Faruk F. Abi-Khuzam và Roy Barbara - " A sharp inequality and the iradius conjecture"):
Cho hàm $F:[0,1]^2\to\mathbb{R}, F(x,y)=\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2}-(1+\sqrt{5})(1-xy)$. F là hàm đối xứng với x và y và từ tính lồi của hàm $x\mapsto\sqrt{1+x^2}$ suy ra $F(x,0)\geq 0$ với mọi x. Bây giờ ta cố định y và xem F như là một hàm của x. Đạo hàm là

$$f'(x)=(1+\sqrt{5})y+\frac{x}{\sqrt{x^2+1}}-\frac{1-x}{\sqrt{(1-x)^2+(1-y)^2}}$$

và

$$f''(x)=\frac{1}{\sqrt{(1+x^2)^3}}+\frac{(1-y)^2}{\sqrt{((1-x)^2+(1-y)^2)^3}}.$$

Vậy, f l h m lồi v f' l h m tăng. Bây giờ đặt

$$r = \sqrt{1 - \frac{6\sqrt{3}}{(1+\sqrt{5})^2}}, c = 1 + \sqrt{5}.$$

Trường hợp đầu tiên ta xét l $y \geq \frac{1}{4}$. Dễ thấy rằng trong trường hợp n y $cy > \frac{1}{\sqrt{y^2 - 2y + 2}}$ (đạo h m của h m $y \mapsto y^2c^2(y^2 - 2y + 2) - 1$ l dương) v do đó $f'(0) > 0$. Bởi vì f' l h m tăng, ta có $f'(x) > 0$ nên f l h m tăng với $f(0) = F(0, y) = F(y, 0) \geq 0$. Vậy, trong trường hợp n y bất đẳng thức được chứng minh.
Từ trường hợp 1 v tính đối xứng, ta chỉ còn phải chứng minh bất đẳng thức đúng trong các trường hợp $x \in \left[0, \frac{1}{4}\right], y \in [0, r]$ v $x \in \left[r, \frac{1}{4}\right], y \in \left[r, \frac{1}{4}\right]$.
Trong trường hợp đầu tiên ta có ngay

$$f'(x) \leq r(1+\sqrt{5}) + \frac{x}{\sqrt{1+x^2}} - \frac{1-x}{\sqrt{1+(x-1)^2}}.$$

Vậy ta có $f'(\frac{1}{4}) < 0$, suy ra f l h m giảm trên $\left[0, \frac{1}{4}\right]$. Bởi vì từ trường hợp đầu tiên đã xét, ta có $f(\frac{1}{4}) \geq 0$, suy ra $F(x, y) = f(x) \geq 0$ với mọi điểm (x, y) trong đó $x \in \left[0, \frac{1}{4}\right], y \in [0, r]$.

Bây giờ ta sẽ xét trường hợp quan trọng nhất, khi $x \in \left[r, \frac{1}{4}\right], y \in \left[r, \frac{1}{4}\right]$. Cho các điểm $O(0,0), A(1,0), B(1,1), C(0,1), M(1,y), N(x,1)$. Tam giác OMN có chu vi

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2 + (1-y)^2} \text{ v diện tích } \frac{1-xy}{2}.$$

Nhưng dễ chỉ ra rằng trong bất kì tam giác với chu vi P v diện tích S, ta có bất đẳng thức $S \leq \frac{P^2}{12\sqrt{3}}$. Vậy ta có

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2 + (1-y)^2} \geq \sqrt{6\sqrt{3}}\sqrt{1-xy} \geq (1+\sqrt{5})(1-xy),$$

bất đẳng thức n y suy ra từ hệ thức $xy \geq r^2$. Chứng minh được ho n th nh.

116. [Suranyi] Chứng minh rằng với mọi số thực dương $a_1, a_2, ..., a_n$, bất đẳng thức sau đúng

$$(n-1)(a_1^n + a_2^n + ... + a_n^n) + na_1a_2...a_n \geq (a_1 + a_2 + ... + a_n)(a_1^{n-1} + a_2^{n-1} + ... + a_n^{n-1}).$$

Lời giải:
Một lần nữa, ta sẽ sử dụng quy nạp để chứng minh bất đẳng thức này, nhưng chứng minh của bước quy nạp thực sự không tầm thường. Thật vậy, giả sử bất đẳng thức đúng cho n số và chúng ta sẽ chứng minh bất đẳng thức đúng cho $n+1$ số.
Nhờ tính đối xứng và thuần nhất của bất đẳng thức, nên ta chỉ cần chứng minh bất đẳng thức dưới điều kiện $a_1 \geq a_2 \geq ... \geq a_{n+1}$ và $a_1 + a_2 + ... + a_n = 1$. Ta phải chứng minh

$$n\sum_{i=1}^{n} a_i^{n+1} + na_{n+1}^{n+1} + na_{n+1}.\prod_{i=1}^{n} a_i + a_{n+1}\prod_{i=1}^{n} a_i - (1+a_{n+1})\left(\sum_{i=1}^{n} a_i^n + a_{n+1}^n\right) \geq 0.$$

Nhưng từ giả thiết quy nạp ta có

$$(n-1)(a_1^n + a_2^n + ... + a_n^n) + na_1a_2...a_n \geq a_1^{n-1} + a_2^{n-1} + ... + a_n^{n-1}$$

và từ đó

$$na_{n+1}\prod_{i=1}^{n} a_i \geq a_{n+1}\sum_{i=1}^{n} a_i^{n-1} - (n-1)a_{n+1}\sum_{i=1}^{n} a_i^n.$$

Sử dụng bất đẳng thức cuối cùng, ta chỉ còn phải chứng minh

$$\left(n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^n\right) - a_{n+1}\left(n\sum_{i=1}^{n} a_i^n - \sum_{i=1}^{n} a_i^{n-1}\right) +$$
$$+ a_{n+1}\left(\prod_{i=1}^{n} a_i + (n-1)a_{n+1}^n - a_{n+1}^{n-1}\right) \geq 0.$$

Bây giờ chúng ta sẽ chia bất đẳng thức này thành

$$a_{n+1}\left(\prod_{i=1}^{n} a_i + (n-1)a_{n+1}^n - a_{n+1}^{n-1}\right) \geq 0$$

và

$$\left(n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^n\right) - a_{n+1}\left(n\sum_{i=1}^{n} a_i^n - \sum_{i=1}^{n} a_i^{n-1}\right).$$

Chúng ta chứng minh hai bất đẳng thức này. Bất đẳng thức đầu tiên là rất rõ ràng

$$\begin{aligned}\prod_{i=1}^{n} a_i + (n-1)a_{n+1}^n - a_{n+1}^{n-1} =& \prod_{i=1}^{n}(a_i - a_{n+1} + a_{n+1}) + (n-1)a_{n+1}^n - a_{n+1}^{n-1}\\ \geq& a_{n+1}^n + a_{n+1}^{n-1}.\sum_{i=1}^{n}(a_i - a_{n+1}) + (n-1)a_{n+1}^n - a_{n+1}^{n-1} = 0\end{aligned}$$

Bây giờ ta chứng minh bất đẳng thức thứ hai. Bất đẳng thức n y có thể được viết lại l

$$n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^{n} \geq a_{n+1}\left(n\sum_{i=1}^{n} a_i^{n} - \sum_{i=1}^{n} a_i^{n-1}\right).$$

Bởi vì $n\sum\limits_{i=1}^{n} a_i^{n} - \sum\limits_{i=1}^{n} a_i^{n-1} \geq 0$ (sử dụng bất đẳng thức **Chebyshev**) v $a_{n+1} \leq \frac{1}{n}$, ta chỉ cần chứng minh rằng

$$n\sum_{i=1}^{n} a_i^{n+1} - \sum_{i=1}^{n} a_i^{n} \geq \frac{1}{n}\quad \left. n\sum_{i=1}^{n} a_i^{n} - \sum_{i=1}^{n} a_i^{n-1}\right).$$

Nhưng bất đẳng thứ n y đúng, vì $na_i^{n+1} + \frac{1}{n}a_i^{n-1} \geq a_i^{n}$ với mọi i. Vậy, bước quy nạp được chứng minh.

117. Chứng minh rằng với mọi $x_1, x_2, ..., x_n$ có tích bằng 1,

$$\sum_{1\leq i<j\leq n} (x_i - x_j)^2 \geq \sum_{i=1}^{n} x_i^2 - n.$$

Một dạng tổng quát của bất đẳng thức Turkevici

Lời giải:
Hiển nhiên, bất đẳng thức có thể được viết dưới dạng sau

$$n \geq -(n-1)\quad \left. \sum_{k=1}^{n} x_k^2\right) + \quad \left. \sum_{k=1}^{n} x_k\right)^2.$$

Ta sẽ chứng minh bất đẳng thức bởi quy nạp. Trường hợp $n = 2$ l tầm thường. Giả sử bất đẳng thức đúng cho $n-1$ v ta sẽ chứng minh bất đẳng thức đúng với n. Đặt

$$f(x_1, x_2, ..., x_n) = -(n-1)\quad \left. \sum_{k=1}^{n} x_k^2\right) + \quad \left. \sum_{k=1}^{n} x_k\right)^2$$

v đặt $G = \sqrt[n-1]{x_2x_3...x_n}$, khi ta đã chọn $x_1 = \min\{x_1, x_2, ..., x_n\}$. Dễ thấy rằng bất đẳng thức $f(x_1, x_2, ..., x_n) \leq f(x_1, G, G, ..., G)$ tương đương với

$$(n-1)\sum_{k=2}^{n} x_k^2 - \left(\sum_{k=2}^{n} x_k\right)^2 \geq 2x_1\quad \left. \sum_{k=2}^{n} x_k - (n-1)G\right).$$

Hiển nhiên, ta có $x_1 \leq G$ và $\sum\limits_{k=2}^{n} x_k \geq (n-1)G$, vậy ta chỉ cần chứng minh rằng

$$(n-1)\sum_{k=2}^{n} x_k^2 - \left(\sum_{k=2}^{n} x_k\right)^2 \geq 2G\left(\sum_{k=2}^{n} x_k - (n-1)G\right).$$

Ta sẽ chứng minh bất đẳng thức này đúng cho mọi $x_2, ..., x_n > 0$. Bởi vì bất đẳng thức là thuần nhất, nên ta chỉ cần chứng minh bất đẳng thức khi $G = 1$. Trong trường hợp này, từ bước quy nạp ta có

$$(n-2)\sum_{k=2}^{n} x_k^2 + n - 1 \geq \left(\sum_{k=2}^{n} x_k\right)^2$$

và ta chỉ cần chứng minh rằng

$$\sum_{k=2}^{n} x_k^2 + n - 1 \geq \sum_{k=2}^{n} 2x_k \Leftrightarrow \sum_{k=2}^{n} (x_k - 1)^2 \geq 0,$$

điều này hiển nhiên đúng.

Vậy, ta đã chứng minh được rằng $f(x_1, x_2, ..., x_n) \leq f(x_1, G, G, ..., G)$. Bây giờ, để hoàn thành bước quy nạp, ta sẽ chứng minh rằng $f(x_1, G, G, ..., G) \leq n$. Bởi vì, khẳng định cuối cùng quy về chứng minh rằng

$$(n-1)\left((n-1)G^2 + \frac{1}{G^{2(n-1)}}\right) + n \geq \left((n-1)G + \frac{1}{G^{n-1}}\right)^2$$

bất đẳng thức này dẫn đến

$$\frac{n-2}{G^{2n-2}} + n \geq \frac{2n-2}{G^{n-2}}$$

và đây là một hệ quả trực tiếp của bât đẳng thức **AM-GM** .

118. [Gabriel Dospinescu] Tìm giá trị nhỏ nhất của biểu thức

$$\sum_{i=1}^{n} \sqrt{\frac{a_1 a_2 ... a_2}{1 - (n-1)a_i}}$$

với $a_1, a_2, ..., a_n < \dfrac{1}{n-1}$ có tổng bằng 1 và $n > 2$ là một số nguyên.

Lời giải:

Ta sẽ chứng minh giá trị nhỏ nhất l $\dfrac{1}{\sqrt{n^{n-3}}}$. Thật vậy, sử dụng bất đẳng thức **Suranyi**, ta thấy rằng

$$(n-1)\sum_{i=1}^{n} a_i^n + na_1a_2...a_n \ge \sum_{i=1}^{n} a_i^{n-1} \Rightarrow na_1a_2...a_n \ge \sum_{i=1}^{n} a_i^{n-1}\left(1-(n-1)a_i\right).$$

Bây giờ, ta nhận thấy

$$\sum_{k=1}^{n} a_k^{n-1}\left(1-(n-1)a_k\right) = \sum_{k=1}^{n} \frac{\left(\sqrt[3]{a_k^{n-1}}\right)^3}{\left(\dfrac{1}{1-(n-1)a_k}\right)^2}$$

v do đó, bởi áp dụng trực tiếp bất đẳng thức **Holder**, ta có

$$\sum_{k=1}^{n} a_k^{n-1}\left(1-(n-1)a_k\right) \ge \frac{\left(\sum\limits_{k=1}^{n} a_k^{\frac{n-1}{3}}\right)^3}{\left(\sum\limits_{k=1}^{n}\sqrt{\dfrac{1}{1-(n-1)a_k}}\right)^2}.$$

Nhưng với $n > 3$, ta có thể áp dụng bất đẳng thức **Jensen** để suy ra rằng

$$\frac{\sum\limits_{k=1}^{n} a_k^{\frac{n-1}{3}}}{n} \ge \left(\frac{\sum\limits_{k=1}^{n} a_k}{n}\right)^{\frac{n-1}{3}} = \frac{1}{n^{\frac{n-1}{3}}}.$$

Vậy, kết hợp các bất đẳng thức, ta chứng minh được bất đẳng thức với $n > 3$. Với $n = 3$ bất đẳng thức quy về chứng minh rằng

$$\sum \sqrt{\frac{abc}{b+c-a}} \ge \sqrt{a},$$

bất đẳng thức n y đã được chứng minh trong lời giải của b i toán 107.

119. [Vasile Cirtoaje] Cho $a_1, a_2, ..., a_n < 1$ l các số thực không âm thỏa mãn

$$a = \sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}} \ge \frac{\sqrt{3}}{3}.$$

Chứng minh rằng

$$\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + ... + \frac{a_n}{1-a_n^2} \ge \frac{na}{1-a^2}.$$

Lời giải:
Ta chứng minh bằng quy nạp. Rõ ràng, bất đẳng thức là tầm thường với $n = 1$. Bây giờ ta giả sử rằng bất đẳng thức đúng cho $n = k-1, k \geq 2$ và sẽ chứng minh rằng

$$\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + ... + \frac{a_k}{1-a_k^2} \geq \frac{ka}{1-a^2},$$

với

$$a = \sqrt{\frac{a_1^2 + a_2^2 + ... + a_k^2}{k}} \geq \frac{\sqrt{3}}{3}.$$

Không mất tổng quát ta giả sử rằng $a_1 \geq a_2 \geq ... \geq a_k$, do đó $a \geq a_k$. Đặt

$$x = \sqrt{\frac{a_1^2 + a_2^2 + ... + a_{k-1}^2}{k-1}} = \sqrt{\frac{ka^2 - a_k^2}{k-1}},$$

từ $a \geq a_k$ suy ra rằng $x \geq \dfrac{\sqrt{3}}{3}$. Vậy, bởi giả thiết quy nạp, ta có

$$\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + ... + \frac{a_{k-1}}{1-a_{k-1}^2} \geq \frac{(k-1)x}{1-x^2},$$

và ta còn phải chỉ ra rằng

$$\frac{a_k}{1-a_k^2} + \frac{(k-1)x}{1-x^2} \geq \frac{ka}{1-a^2}.$$

Từ

$$x^2 - a^2 = \frac{ka^2 - a_k^2}{k-1} - a^2 = \frac{a^2 - a_k^2}{k-1},$$

ta được

$$(k-1)(x-a) = \frac{(a-a_k)(a+a_k)}{x+a}$$

và

$$\begin{aligned}
&\frac{a_k}{1-a_k^2} + \frac{(k-1)x}{1-x^2} - \frac{ka}{1-a^2} \\
=&\left(\frac{a_k}{1-a_k^2} - \frac{a}{1-a^2}\right) + (k-1)\left(\frac{x}{1-x^2} - \frac{a}{1-a^2}\right) \\
=&\frac{-(a-a_k)(1+aa_k)}{(1-a_k^2)(1-a^2)} + \frac{(k-1)(x-a)(1+ax)}{(1-x^2)(1-a^2)} \\
=&\frac{a-a_k}{1-a^2}\left[\frac{-(1+aa_k)}{1-a_k^2} + \frac{(a+a_k)(1+ax)}{(1-x^2)(x+a)}\right] \\
=&\frac{(a-a_k)(x-a_k)[-1+x^2+a_k^2+xa_k+a(x+a_k)+a^2+axa_k(x+a_k)+a^2xa_k]}{(1-a^2)(1-a_k^2)(1-x^2)(x+a)}.
\end{aligned}$$

Từ $x^2 - a_k^2 = \dfrac{ka^2 - a_k^2}{k-1} - a_k^2 = \dfrac{k(a-a_k)(a+a_k)}{(k-1)}$, suy ra

$$(a - a_k)(x - a_k) = \frac{k(a-a_k)^2(a+a_k)}{(k-1)(x+a_k)} \geq 0,$$

v do đó ta phải chỉ ra được rằng

$$x^2 + a_k^2 + xa_k + a(x + a_k) + a^2 + axa_k(x + a_k) + a^2xa_k \geq 1.$$

Để chỉ ra bất đẳng thức n y, ta cần chú ý rằng

$$x^2 + a_k^2 = \frac{ka^2 + (k-2)a_k^2}{k-1} \geq \frac{ka^2}{k-1}$$

v

$$x + a_k \geq \sqrt{x^2 + a_k^2} \geq a\sqrt{\frac{k}{k-1}}.$$

Do đó ta có

$$\begin{aligned}
&x^2 + a_k^2 + xa_k + a(x + a_k) + a^2 + axa_k(x + a_k) + a^2xa_k \\
\geq &(x^2 + a_k^2) + a(x + a_k) + a^2 \\
\geq &\left. \frac{k}{k-1} + \sqrt{\frac{k}{k-1}} + 1 \right) a^2 > 3a^2 = 1,
\end{aligned}$$

v chứng minh được ho n th nh. Dấu bằng xảy ra khi $a_1 = a_2 = ... = a_n$.

Chú ý

1. Từ lời giải cuối cùng, ta có thể dễ nhìn thấy rằng bất đẳng thức vẫn đúng cho điều kiện rộng hơn

$$a \geq \sqrt{1 + \sqrt{\frac{n}{n-1}} + \frac{n}{n-1}}.$$

Đây l miền rộng nhất cho a, bởi vì cho $a_1 = a_2 = ... = a_{n-1} = x$ v $a_n = 0$ (khi đó $a = x\sqrt{\dfrac{n-1}{n}}$), từ bất đẳng thức đã cho ta được $a \geq \dfrac{1}{\sqrt{1 + \sqrt{\dfrac{n}{n-1}} + \dfrac{n}{n-1}}}$.

2. Trường hợp đặc biệt $n = 3$ v $a = \dfrac{\sqrt{3}}{3}$ l một b i toán từ Crux 2003.

120. [Vasile Cirtoaje, Mircea Lascu] Cho a, b, c, x, y, z l các số thực dương thỏa mãn

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Chứng minh rằng

$$abcxyz < \frac{1}{36}.$$

Lời giải:
Sử dụng bất đẳng thức **AM-GM**, ta có

$$\begin{aligned}
&4(ab+bc+ca)(xy+yz+zx)\\
=&[(a+b+c)^2-(a^2+b^2+c^2)][(x+y+z)^2-(x^2+y^2+z^2)]\\
=&20-(a+b+c)^2(x^2+y^2+z^2)-(a^2+b^2+c^2)(x+y+z)^2\\
\leq&20-2\sqrt{(a+b+c)^2(x^2+y^2+z^2)(a^2+b^2+c^2)(x+y+z)^2}=4,
\end{aligned}$$

do đó

$$(ab+bc+ca)(xy+yz+zx) \leq 1. \tag{1}$$

Bởi nhân các bất đẳng thức đã biết

$$(ab+bc+ca)^2 \geq 3abc(a+b+c), \quad (xy+yz+zx)^2 \geq 3xyz(x+y+z),$$

ta được

$$(ab+bc+ca)^2(xy+yz+zx)^2 \geq 9abcxyz(a+b+c)(x+y+z),$$

hoặc

$$(ab+bc+ca)(xy+yz+zx) \geq 36abcxyz. \tag{2}$$

Từ (1) và (2), ta kết luận rằng

$$1 \geq (ab+bc+ca)(xy+yz+zx) \geq 36abcxyz,$$

do đó $1 \geq 36abcxyz$.
Để có $1 = 36abcxyz$, các đẳng thức $(ab+bc+ca)^2 = 36abc(a+b+c)$ và $(xy+yz+zx)^2 = 36xyz(x+y+z)$ là cần. Nhưng từ các điều kiện này suy ra $a=b=c$ và $x=y=z$, điều này mâu thuẫn với

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Vậy, suy ra $1 > 36abcxyz$.

121. [Gabriel Dospinescu] Cho $n > 2$, tìm giá nhỏ nhất của hằng số k_n, sao cho nếu $x_1, x_2, ..., x_2 > 0$ có tích bằng 1, thì

$$\frac{1}{\sqrt{1+k_nx_1}} + \frac{1}{\sqrt{1+k_nx_2}} + ... + \frac{1}{\sqrt{1+k_nx_n}} \leq n-1.$$

Lời giải:

Ta sẽ chứng minh rằng $k_n = \dfrac{2n-1}{(n-1)^2}$. Lấy $x_1 = x_2 = ... = x_n = 1$, ta được $k_n \geq \dfrac{2n-1}{(n-1)^2}$. Vậy, ta còn phải chỉ ra rằng

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + \dfrac{2n-1}{(n-1)^2} x_k}} \leq n - 1$$

nếu $x_1.x_2...x_n = 1$. Giả sử bất đẳng thức n y không đúng cho hệ n số với tích bằng 1, $x_1, x_2, ..., x_n > 0$ n o đó. Vậy, ta có thể tìm được số $M > n-1$ v các số $a_i > 0$ m có tổng bằng 1, sao cho

$$\frac{1}{\sqrt{1 + \dfrac{2n-1}{(n-1)^2} x_k}} = M a_k.$$

Vậy $a_k < \dfrac{1}{n-1}$ v ta có

$$1 = \prod_{k=1}^{n} \left[\frac{(n-1)^2}{2n-1} \left(\frac{1}{M^2 a_k^2} - 1 \right) \right] \Longrightarrow \left(\frac{2n-1}{(n-1)^2} \right)^n < \prod_{k=1}^{n} \left(\frac{1}{(n-1)^2 a_k^2} - 1 \right).$$

Bây giờ, kí hiệu $1 - (n-1)a_k = b_k > 0$ v để ý rằng $\sum\limits_{k=1}^{n} b_k = 1$. V bất đẳng thức trên trở th nh

$$(n-1)^{2n} \prod_{k=1}^{n} b_k . \prod_{k=1}^{n} (2 - b_k) > (2n-1)^n \left(\prod_{k=1}^{n} (1 - b_k) \right)^2.$$

Bởi vì từ bất đẳng thức **AM-GM** ta có

$$\prod_{k=1}^{n} (2 - b_k) \leq \left(\frac{2n-1}{n} \right)^n,$$

từ giả thiết dẫn tới

$$\prod_{k=1}^{n} (1 - b_k)^2 < \frac{(n-1)^{2n}}{n^n} b_1 b_2 ... b_n.$$

Vậy, ta chỉ cần chứng minh rằng với bất kỳ các số dương $a_1, a_2, ..., a_n$, bất đẳng thức

$$\prod_{k=1}^{n} (a_1 + a_2 + ... + a_{k-1} + a_{k+1} + ... + a_n)^2 \geq \frac{(n-1)^{2n}}{n^n} a_1 a_2 ... a_n (a_1 + a_2 + ... + a_n)^n$$

đúng.
Bất đẳng thức mạnh hơn này sẽ được chứng minh bằng quy nạp. Với $n = 3$, bất đẳng thức suy ra từ sự kiện là

$$\frac{\left(\prod(a+b)\right)^2}{abc} \geq \frac{\left(\frac{8}{9}.\sum a\right).\left(\sum ab\right)^2}{abc} \geq \frac{64}{27}\left(\sum a\right)^3.$$

Giả sử bất đẳng thức đúng cho hệ gồm n số. Cho $a_1, a_2, ..., a_{n+1}$ là các số thực dương. Bởi vì bất dẳng thức là đối xứng và thuần nhất, ta có thể giả thiết rằng $a_1 \leq a_2 \leq ... \leq a_{n+1}$ và $a_1 + a_2 + ... + a_n = 1$. Áp dụng giả thiết quy nạp ta có bất đẳng thức

$$\prod_{i=1}^{n}(1-a_i)^2 \geq \frac{(n-1)^{2n}}{n^n}a_1.a_2...a_n.$$

Để chứng minh bước quy nạp, ta phải chứng minh

$$\prod_{i=1}^{n}(a_{n+1}+1-a_i)^2 \geq \frac{n^{2n+2}}{(n+1)^{n+1}}a_1a_2...a_na_{n+1}(1+a_{n+1})^{n+1}.$$

Vậy, ta sẽ chứng minh bất đẳng thức mạnh hơn

$$\prod_{i=1}^{n}\left(1+\frac{a_{n+1}}{1-a_i}\right)^2 \geq \frac{n^{3n+2}}{(n-1)^{2n}.(n+1)^{n+1}}a_{n+1}(1+a_{n+1})^{n+1}.$$

Bây giờ, sử dụng bất đẳng thức **Huygens** và bất đẳng thức **AM-GM**, ta được

$$\prod_{i=1}^{n}\left(1+\frac{a_{n+1}}{1-a_i}\right)^2 \geq \left(1+\frac{a_{n+1}}{\sqrt[n]{\prod\limits_{i=1}^{n}(1-a_i)}}\right)^{2n} \geq \left(1+\frac{na_{n+1}}{n-1}\right)^{2n}$$

và do đó còn lại bất đẳng thức

$$\left(1+\frac{na_{n+1}}{n-1}\right)^{2n} \geq \frac{n^{3n+2}}{(n-1)^{2n}.(n+1)^{n+1}}a_{n+1}(1+a_{n+1})^{n+1}$$

nếu $a_{n+1} \geq \max\{a_1, a_2, ..., a_n\} \geq \frac{1}{n}$. Vậy ta có thể đặt $\frac{n(1+a_{n+1})}{n+1} = 1 + x$, với $x \geq 0$. Bất đẳng thức trở thành

$$\left(1+\frac{x}{n(x+1)}\right)^{2n} \geq \frac{1+(n+1)x}{(x+1)^{n-1}}.$$

Sử dụng bất đẳng thức **Bernoulli**, ta được

$$\left(1+\frac{x}{n(x+1)}\right)^{2n} \geq \frac{3x+1}{x+1}.$$

ngo i ra $(1+x)^{n-1} \geq 1+(n-1)x$ v vì vậy chỉ cần chứng minh

$$\frac{3x+1}{x+1} \geq \frac{1+(n+1)x}{1+(n-1)x},$$

bất đẳng thức n y l tầm thường.
Vậy, ta gặp mâu thuẫn khi giả sử bất đẳng thức không đúng cho hệ n số n o đó có tích bằng 1, điều n y chỉ ra rằng thực tế bất đẳng thức đúng cho $k_n = \dfrac{2n-1}{(n-1)^2}$ v đây l giá trị đã yêu cầu của b i toán.
Chú ý.
Với $n = 3$, ta tìm thấy một bât đẳng thức mạnh hơn một b i toán được đưa ra trong Olympic Toán học Trung Quốc năm 2003. Cũng vậy, trường hợp $n = 3$ l b i toán được đề nghị bởi Vasile Cărtoaje trong Gazeta Matematiă, Seria A.

122. [Vasile Cirtoaje, Gabriel Dospinescu] Cho $n > 2$, tìm giá trị lớn nhất của hằng số k_n, sao cho với bất kỳ $x_1, x_2, ..., x_n > 0$ thỏa mãn $x_1^2 + x_2^2 + ... + x_n^2 = 1$, chúng ta có bất đẳng thức

$$(1-x_1)(1-x_2)...(1-x_n) \geq k_n x_1 x_2 ... x_n.$$

Lời giải:
Chúng ta sẽ chứng minh rằng hằng số ở đây l $(\sqrt{n}-1)^n$. Thật vậy, đặt $a_i = x_i^2$. Ta phải tìm giá trị nhỏ nhất của biểu thức

$$\frac{\prod\limits_{i=1}^{n}(1-\sqrt{a_i})}{\prod\limits_{i=1}^{n}\sqrt{a_i}}$$

khi $a_1+a_2+...+a_n = 1$. Ta nhận thấy rằng chứng minh giá trị nhỏ nhất bằng $(\sqrt{n}-1)^n$ quy về chứng minh rằng

$$\prod_{i=1}^{n}(1-a_i) \geq (\sqrt{n}-1)^n.\prod_{i=1}^{n}\sqrt{a_i}.\prod_{i=1}^{n}(1+\sqrt{a_i}).$$

Nhưng từ kết quả chứng minh trong lời giải của bải toán 121, ta tìm thấy

$$\prod_{i=1}^{n}(1-a_i)^2 \geq \left(\frac{(n-1)^2}{n}\right)^n \prod_{i=1}^{n} a_i.$$

Vậy, chỉ cần chứng minh

$$\prod_{i=1}^{n}(1+\sqrt{a_i}) \geq \left(1+\frac{1}{\sqrt{n}}\right)^n.$$

Nhưng đây là một bất đẳng thức dễ, bởi vì từ bất đẳng thức **AM-GM** ta có

$$\prod_{i=1}^{n}(1+\sqrt{a_i}) \geq \left(1+\frac{\sum\limits_{i=1}^{n}\sqrt{a_i}}{n}\right)^n \leq \left(1+\frac{1}{\sqrt{n}}\right)^n,$$

bất đẳng thức cuối cùng là một hệ quả đơn giản của bất đẳng thức **Cauchy-Schwarz.**
Chú ý.
Trường hợp $n = 4$ đã được đề nghị bởi Vasile Cârtoaje trong Gazeta Matematiă Annual Contest, 2001.

Từ điển thuật ngữ

1. **Công thức tổng Abel**
Nếu $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ l các số thực hoặc số phức, v

$$S_i = a_1 + a_2 + ... + a_i, i = 1, 2, ..., n,$$

thì

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_n b_n.$$

2. **Bất đẳng thức AM-GM (Trung bình cộng-Trung bình nhân)**
Nếu $a_1, a_2, ..., a_n$ l các số thực không âm, thì

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq (a_1 a_2 ... a_n)^{\frac{1}{n}},$$

dấu bằng xảy ra nếu v chỉ nếu $a_1 = a_2 = ... = a_n$. Bất đẳng thức n y l trường hợp đặc biệt của bất đẳng thức trung bình luỹ thừa.

3. **Bất đẳng thức trung bình cộng v trung bình điều ho (AM-HM)**
Nếu $a_1, a_2, ..., a_n$ l các số thực dương, thì

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}}$$

dấu bằng xảy ra nếu v chỉ nếu $a_1 = a_2 = ... = a_n$. Bất đẳng thức n y l trường hợp đặc biệt của bất đẳng thức trung bình luỹ thừa.

4. **Bất đẳng thức Bernoulli**
Với mọi số thực $x > -1$ v $a > 1$ ta có $(1 + x)^a \geq 1 + ax$.

5. **Bất đẳng thức Cauchy-Schwarz**
Với mọi số thực $a_1, a_2, ..., a_n$ v $b_1, b_2, ..., b_n$

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2,$$

dấu bằng xảy ra nếu v chỉ nếu a_i v b_i cùng tỉ lệ, $i = 1, 2, ..., n$.

6. **Bất đẳng thức Cauchy-Schwarz cho tích phân**
Nếu a, b là các số thực, $a < b$, và $f, g : [a, b] \to \mathbb{R}$ là các hàm khả tích thì

$$\left(\int\limits_a^b f(x)g(x)\right)^2 \leq \left(\int\limits_a^b f^2(x)dx\right).\left(\int\limits_a^b g^2(x)dx\right).$$

7. **Bất đẳng thức Chebyshev**
Nếu $a_1 \leq a_2 \leq ... \leq a_n$ và $b_1, b_2, ..., b_n$ là các số thực, thì
1) Nếu $b_1 \leq b_2 \leq ... \leq b_n$ thì $\sum\limits_{i=1}^n a_ib_i \geq \frac{1}{n}\left(\sum\limits_{i=1}^n a_i\right).\left(\sum\limits_{i=1}^n b_i\right)$;
2) Nếu $b_1 \geq b_2 \geq ... \geq b_n$ thì $\sum\limits_{i=1}^n a_ib_i \leq \frac{1}{n}\left(\sum\limits_{i=1}^n a_i\right).\left(\sum\limits_{i=1}^n b_i\right)$;

8. **Bất đẳng thức Chebyshev cho tích phân**
Nếu a, b là các số thực, $a < b$, và $f, g : [a, b] \to \mathbb{R}$ là các hàm khả tích và cùng đơn điệu, thì

$$(b-a)\int\limits_a^b f(x)g(x)dx \geq \int\limits_a^b f(x)dx.\int\limits_a^b g(x)dx$$

và nếu một hàm là tăng, trong khi đó hàm khác giảm thì bất đẳng thức ngược lại đúng.

9. **Hàm lồi**
Một hàm số nhận giá trị thực trên một đoạn số thực I là lồi nếu, với mọi $x, y \in I$ và với mọi số thực không âm α, β có tổng bằng 1,

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y).$$

10. **Tính lồi**
Hàm $f(x)$ là lõm lên (xuống) trên $[a, b] \subset \mathbb{R}$ nếu $f(x)$ nằm dưới (trên) đường nối $(a_1, f(a_1))$ và $(b_1, f(b_1))$ với mọi

$$a \leq a_1 < x < b_1 \leq b.$$

Hàm $g(x)$ là lõm lên (xuống) trên mặt phẳng Eculid nếu nó lõm lên (xuống) trên mỗi đường thẳng trong mặt phẳng, ở đây, một cách tự nhiên, ta đồng nhất đường thẳng với $\mathbb{R}$.
Hàm số lõm lên và lõm xuống lần lượt được gọi là lồi và lõm.
Nếu hàm f lõm lên trên môt đoạn $[a, b]$ và $\lambda_1, \lambda_2, ..., \lambda_n$ là các số không âm có tổng bằng 1, thì

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n)$$

với mọi $x_1, x_2, ..., x_n$ trong đoạn $[a, b]$. Nếu hàm lõm xuống thì bất đẳng thức bị đảo ngược lại. Đây là bất đẳng thức **Jensen**.

11. **Tổng tuần ho n**
Giải sử n l một số nguyên dương. Cho f l một h m n biến, định nghĩa tổng tuần ho n của các biến $(x_1, x_2, ..., x_n)$ l

$$\sum_{cyc} f(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \\ + f(x_2, x_3, ..., x_n, x_1) + \\ + f(x_3, x_4, ..., x_n, x_1, x_2) + \\ + ... + f(x_n, x_1, x_2, ..., x_{n-1}).$$

12. **Bất đẳng thức Holder**
Nếu r, s l các số thực dương sao cho $\frac{1}{r} + \frac{1}{s} = 1$, thì với mọi số thực $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$,

$$\frac{\sum\limits_{i=1}^{n} a_i b_i}{n} \leq \left(\frac{\sum\limits_{i=1}^{n} a_i^r}{n} \right)^{\frac{1}{r}} . \left(\frac{\sum\limits_{i=1}^{n} b_i^s}{n} \right)^{\frac{1}{s}} .$$

13. **Bất đẳng thức Huygens**
Nếu $p_1, p_2, ..., p_n, a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ l các số thực dương với $p_1 + p_2 + ... + p_n = 1$, thì

$$\prod_{i=1}^{n} (a_i + b_i)^{p_i} \geq \prod_{i=1}^{n} a_i^{p_i} + \prod_{i=1}^{n} b_i^{p_i}.$$

14. **Bất đẳng thức Mac Laurin**
Với mọi số thực dương $x_1, x_2, ..., x_n$, thì

$$S_1 \geq S_2 \geq ... \geq S_n,$$

với

$$S_k = \sqrt[k]{\frac{\sum\limits_{1 \leq i_1 < i_2 < ... < i_k \leq n} x_{i_1} . x_{i_2} . \quad x_{i_k}}{\binom{n}{k}}}.$$

15. **Bất đẳng thức Minkowski**
Với mọi số thực $r \geq 1$ v mọi số thực dương $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$,

$$\left(\sum_{i=1}^{n} (a_i + b_i)^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} b_i^r \right)^{\frac{1}{r}} .$$

16. **Bất đẳng thức trung bình luỹ thừa**
Với mọi số thực $a_1, a_2, ..., a_n$ có tổng bằng 1, và với mọi số thực dương $x_1, x_2, ..., x_n$, ta định nghĩa $M_r = (a_1x_1^r + a_2x_2^r + ... + a_nx_n^r)^{\frac{1}{r}}$ nếu r là một số thực khác không và $M_0 = x_1^{a_1}x_2^{a_2}...x_n^{a_n}, M_\infty = \max\{x_1, x_2, ..., x_n\}, M_{-\infty} = \min\{x_1, x_2, ..., x_n\}$. Khi đó với mọi số thực $s \leq t$ ta có $M_{-\infty} \leq M_s \leq M_t \leq M_\infty$.

17. **Bất đẳng thức căn trung bình bình phương**
Nếu $a_1, a_2, ..., a_n$ là các số thực không âm, thì

$$\sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}} \geq \frac{a_1 + a_2 + ... + a_n}{n},$$

dấu bằng xảy ra nếu và chỉ nếu $a_1 = a_2 = ... = a_n$.

18. **Bất đẳng thức Schur** Với mọi số thực dương x, y, z và với mọi $r > 0$

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Trường hợp phổ biến nhất là $r = 1$, khi đó bất đẳng thức có các dạng tương đương sau:
1) $x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$;
2) $xyz \geq (x+y-z)(y+z-x)(z+x-y)$;
3) nếu $x + y + z = 1$ thì $xy + yz + zx \leq \dfrac{1 + 9xyz}{4}$.

19. **Bất đẳng thức Suranyi**
Với mọi số thực không âm $a_1, a_2, ..., a_n$,

$$(n-1)\sum_{k=1}^{n} a_k^n + n\prod_{k=1}^{n} a_k \geq \left(\sum_{k=1}^{n} a_k\right) \cdot \left(\sum_{k=1}^{n} a_k^{n-1}\right).$$

20. **Bất đẳng thức Turkevici**
Với mọi số thực dương x, y, z, t,

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$$

21. **Bất đẳng thức AM-GM có trọng số**
Với mọi số thực không âm $a_1, a_2, ..., a_n$, nếu $w_1, w_2, ..., w_n$ là những số thực không âm (trọng số) có tổng bằng 1, thì

$$w_1a_1 + w_2a_2 + ... + w_na_n \geq a_1^{w_1}a_2^{w_2}...a_n^{w_n},$$

dấu bằng xảy ra nếu và chỉ nếu $a_1 = a_2 = ... = a_n$.

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Inequalities Marathon version 1.00

http://www.mathlinks.ro/viewtopic.php?t=299899

Author:
MathLink ers

Editor:
Hassan Al-Sibyani

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Introduction

Hello everyone, this file contain 100 problem in inequalities, generally at Pre-Olympiad level. Those problems has been collected from MathLinks.ro at the topic called "'Inequalities Marathon"'. I decide to make this work to be a good reference for young students who are interested in inequalities. This work provides some of the nicest problems among international Olympiads, as well as some amazing problems by the participants. Some of the problems contains more than one solution from the marathon and even outside the marathon from other sources.

THE FILE DESCRIPTION:

I- The next page contain all the members that participate with URLs to their MathLinks.ro user-name information and also that page contain some of the real names of the participants that like to have their real names on this file.

II- Then there are some pages under the title of "'Important Definitions"', this page contain some important definition that you may see them in the problems so if you don't know one of those definitions you can look at it there.

III- Then the problems section comes each problem with its solution(s).

A- The problem format: "'**Problem No.** *'Author'* (Proposer):"'

B- The solution format: "'*Solution No. (Propser):*"' (Note that the proposer of the solution is not necessary the one who make this solution)

I would like to all the participants of this marathon for making this work, also I would like to give a special thank to Endrit Fejzullahu and Popa Alexandru for their contributing making this file.

If you have another solution to any of those problem or a suggestion or even if you find any mistake of any type on this file please PM me at my MathLinks.ro user or e-mail me at *hasan4444@gawab.com.*

Editor

Participant sorted alphabetically by thier Mathlink username:
Agr 94 Math
alex2008
apratimdefermat
beautifulliar
bokagadha
Brut3Forc3
can hang2007
dgreenb801
Dimitris X
enndb0x
Evariste-Galois
FantasyLover
geniusbliss
great math
hasan4444
keyree10
Mateescu Constantin
mathinequs
Maths Mechanic
Mircea Lascu
Pain rinnegan
peine
Potla
R.Maths
Rofler
saif
socrates
Virgil Nicula
Zhero

List of names:

Name	**Mathlink User**
Apratim De	apratimdefermat
Aravind Srinivas	Agr 94 Math
Dimitris Charisis	Dimitris X
Endrit Fejzullahu	enndb0x
Hassan Al-Sibyani	hasan4444
Hoang Quoc Viet	great math
Mateescu Constantin	Mateescu Constantin
Mharchi Abdelmalek	Evariste-Galois
Mircea Lascu	Mircea Lascu
Mohamed El-Alami	peine
Raghav Grover	Maths Mechanic
Toang Huc Khein	Pain Rinnegan
Popa Alexandru	alex2008
Redwane Khyaoui	R.Maths
Sayan Mukherjee	Potla
Virgil Nicula	Virgil Nicula
Vo Quoc Ba Can	can hang2007

Important Definitions

Abel Summation Formula
Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two finite sequances of number. Then

$$a_1b_1+a_2b_2+...+a_nb_n = (a_1-a_2)b_1+(a_2-a_3)(b_1+b_2)+\cdots+(a_{n-1}-a_n)(b_1+b_2+\cdots+b_{n-1})+a_n(b_1+b_2+\cdots+b_n)$$

AM-GM (Arithmetic Mean-Geometric Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\frac{a_1, a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Cauchy-Schwarz Inequality
For any real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \cdots, n$.

Chebyshev's Inequality
If $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are two sequences of real numbers arranged in the same order then :

$$n(a_1b_1 + a_2b_2 + ... + a_nb_n) \geq (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n)$$

Cyclic Sum
Let n be a positive integer. Given a function f of n variables, define the cyclic sum of variables $(a_1, a_2, \cdots, a_n)$ as

$$\sum_{cyc} f(a_1, a_2, a_3, \ldots a_n) = f(a_1, a_2, a_3, \ldots a_n)+f(a_2, a_3, a_4, \ldots a_n, a_1)+f(a_3, a_4, \ldots a_n, a_1, a_2)+\ldots+f(a_n, a_1, a_2, \ldots a_{n-1})$$

QM-AM-GM-HM (General Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1, a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

GM-HM (Geometric Mean-Harmonic Mean) Inequality
If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Hlawka's Inequality
Let z_1, z_2, z_3 be three complex numbers. Then:

$$|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \leq |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$$

Holder's Inequality
Given mn $(m, n \in \mathbb{N})$ positive real numbers $(x_{ij})(i\overline{1,m}\ ,\ j\overline{1,n})$, then :

$$\prod_{i1}^{m} \left(\sum_{j1}^{m} x_{ij} \right)^{\frac{1}{m}} \geq \sum_{j1}^{n} \left(\prod_{i1}^{m} x_{ij}^{\frac{1}{m}} \right)$$

Jensen's Inequality
Let F be a convex function of one real variable. Let $x_1, \ldots, x_n \in \mathbb{R}$ and let $a_1, \ldots, a_n \geq 0$ satisfy $a_1 + \cdots + a_n = 1$. Then:

$$F(a_1 x_1 + \cdots + a_n x_n) \leq a_1 F(x_1) + \cdots + a_n F(x_n)$$

Karamata's Inequality
Given that $F(x)$ is a convex function in x, and that the sequence $\{x_1, \ldots, x_n\}$ majorizes the sequence $\{y_1, \ldots, y_n\}$, the theorem states that

$$F(x_1) + \cdots + F(x_n) \geq F(y_1) + \cdots + F(y_n)$$

The inequality is reversed if $F(x)$ is concave, since in this case the function $-F(x)$ is convex.

Lagrange's Identity
The identity:

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \leq k \leq j \leq n} (a_k b_j - a_j b_k)^2$$

applies to any two sets $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ of real or complex numbers.

Minkowski's Inequality
For any real number $r \geq 1$ and any positive real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$. Then

$$((a_1+b_1)^r + (a_2+b_2)^r + \cdots + (a_n+b_n)^r)^{\frac{1}{r}} \leq (a_1^r + a_2^r + \cdots + a_n^r)^{\frac{1}{r}} + (b_1^r + b_2^r + \cdots + b_n^r)^{\frac{1}{r}}$$

Muirhead's Inequality
If a sequence A majorizes a sequence B, then given a set of positive reals $x_1, x_2, \cdots, x_n$

$$\sum_{\text{sym}} {x_1}^{a_1} {x_2}^{a_2} \cdots {x_n}^{a_n} \geq \sum_{\text{sym}} {x_1}^{b_1} {x_2}^{b_2} \cdots {x_n}^{b_n}$$

QM-AM (Quadratic Mean-Arithmetic Mean) Inequality
Also called root-mean-square and state that if $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1, a_2 + \cdots + a_n}{n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.**The Rearrangement Inequality**
Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(a_1', a_2', \cdots, a_n')$ of $(a_1, a_2, \cdots, a_n)$, we have

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &\leq a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n \\ &\leq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n \end{aligned}$$

with equality if and only if $(a_1', a_2', \cdots, a_n')$ is equal to $(a_1, a_2, \cdots, a_n)$ or $(a_n, a_{n-1}, \cdots, a_1)$ respectively.

Schur's Inequality
For all non-negative $a, b, c \in \mathbb{R}$ and $r > 0$:

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

The four equality cases occur when $a = b = c$ or when two of a, b, c are equal and the third is 0.

Common Cases
The $r = 1$ case yields the well-known inequality:$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$
When $r = 2$, an equivalent form is: $a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$

Weighted AM-GM Inequality
For any nonnegative real numbers $a_1, a_2, \cdots, a_n$ if $w_1, w_2, \cdots, w_n$ are nonnegative real numbers (weights) with sum 1, then

$$w_1a_1 + w_2a_2 + \cdots + w_na_n \geq a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Problem 1 *'India 2002'* (Hassan Al-Sibyani): For any positive real numbers a, b, c show that the following inequality holds
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{c+a}{c+b}+\frac{a+b}{a+c}+\frac{b+c}{b+a}$$

First Solution (Popa Alexandru): Ok. After not so many computations i got that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{a+b}{c+a}-\frac{b+c}{a+b}-\frac{c+a}{b+c}$$
$$=\frac{abc}{(a+b)(b+c)(c+a)}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{b}-\frac{b}{c}-\frac{c}{a}\right)$$
$$+\frac{abc}{(a+b)(b+c)(c+a)}\left(\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}-3\right)$$

So in order to prove the above inequality we need to prove $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2} \geq 3$
The second inequality is obvious by AM-GM , and the for the first we have:

$$\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2 \geq 3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$$

where i used AM-GM and the inequality $3(x^2+y^2+z^2) \geq (x+y+z)^2$ for $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$
So the inequality is proved.

Second Solution (Raghav Grover): Substitute $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$ So $xyz=1$. The inequality after substitution becomes
$x^2z+y^2x+z^2x+x^2+y^2+z^2 \geq x+y+z+3$
$x^2z+y^2x+z^2x \geq 3$ So now it is left to prove that $x^2+y^2+z^2 \geq x+y+z$ which is easy.

Third Solution (Popa Alexandru): Bashing out it gives

$$a^4c^2+b^4a^2+c^4b^2+a^3b^3+b^3c^3+a^3c^3 \geq abc(ab^2+bc^2+ca^2+3abc)$$

which is true because AM-GM gives :

$$a^3b^3+b^3c^3+a^3c^3 \geq 3a^2b^2c^2$$

and by Muirhead :
$$a^4c^2+b^4a^2+c^4b^2 \geq abc(ab^2++bc^2+ca^2)$$

Fourth Solution (Popa Alexandru): Observe that the inequality is equivalent with:

$$\sum_{cyc}\frac{a^2+bc}{a(a+b)} \geq 3$$

Now use AM-GM:

$$\sum_{cyc} \frac{a^2+bc}{a(a+b)} \geq 3\sqrt[3]{\frac{\prod(a^2+bc)}{abc\prod(a+b)}}$$

So it remains to prove:

$$\prod(a^2+bc) \geq abc\prod(a+b)$$

Now we prove

$$(a^2+bc)(b^2+ca) \geq ab(c+a)(b+c) \Leftrightarrow a^3+b^3 \geq ab^2+a^2b \Leftrightarrow (a+b)(a-b)^2 \geq 0$$

Multiplying the similars we are done.

Problem 2 *'Maxim Bogdan'* (Popa Alexandru): Let $a, b, c, d > 0$ such that $a \leq b \leq c \leq d$ and $abcd = 1$. Then show that:

$$(a+1)(d+1) \geq 3 + \frac{3}{4d^3}$$

Solution (Mateescu Constantin):From the condition $a \leq b \leq c \leq d$ we get that $a \geq \dfrac{1}{d^3}$.

$\Longrightarrow\ (a+1)(d+1) \geq \left(\dfrac{1}{d^3}+1\right)(d+1)$

Now let's prove that $\left(1+\dfrac{1}{d^3}\right)(d+1) \geq 3 + \dfrac{3}{4d^3}$

This is equivalent with: $(d^3+1)(d+1) \geq 3d^3 + \dfrac{3}{4}$

$\Longleftrightarrow [d(d-1)]^2 - [d(d-1)] + 1 \geq \dfrac{3}{4} \Longleftrightarrow \left[d(d-1)-\frac{1}{2}\right]^2 \geq 0.$

Equality holds for $a = \dfrac{1}{d^3}$ and $d(d-1) - \dfrac{1}{2} = 0 \Longleftrightarrow d = \dfrac{1+\sqrt{3}}{2}$

Problem 3 *'Darij Grinberg'* (Hassan Al-Sibyani): If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

First Solution (Dimitris Charisis):

$\sum \dfrac{a^2}{ab^2+ac^2+2abc} \geq \dfrac{(a+b+c)^2}{\sum_{sym} a^2b + 6abc}$

So we only have to prove that:

$4(a+b+c)^3 \geq 9\sum_{sym} a^2b + 54abc \Longleftrightarrow 4(a^3+b^3+c^3) + 12\sum_{sym} a^2b + 24abc \geq 9\sum_{sym} a^2b + 54abc \Longleftrightarrow$

$4(a^3+b^3+c^3) + 3\sum_{sym} a^2b \geq 30abc$

But $\sum_{sym} a^2b \geq 6abc$ and $a^3+b^3+c^3 \geq 3abc$

So $4(a^3+b^3+c^3) + 3\sum_{sym} a^2b \geq 30abc$

Second Solution (Popa Alexandru): Use Cauchy-Schwartz and Nesbitt:

$$(a+b+c)\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right) \geq \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2 \geq \frac{9}{4}$$

Problem 4 *'United Kingdom 1999'* (Dimitris Charisis): For $a, b, c \geq 0$ and $a+b+c=1$ prove that $7(ab+bc+ca) \leq 2+9abc$

First Solution (Popa Alexandru):
Homogenize to

$$2(a+b+c)^3+9abc \geq 7(ab+bc+ca)(a+b+c)$$

Expanding it becomes :

$$\sum_{sym} a^3+6\sum_{sym} a^2b+21abc \geq 7\sum_{sym} a^2b+21abc$$

So we just need to show:

$$\sum_{sym} a^3 \geq \sum_{sym} a^2b$$

which is obvious by
$a^3+a^3+b^3 \geq 3a^2b$ and similars.

Second Solution (Popa Alexandru): Schur gives $1+9abc \geq 4(ab+bc+ca)$ and use also $3(ab+bc+ca) \leq (a+b+c)^2=1$ Suming is done .

Problem 5 *'Gheorghe Szollosy, Gazeta Matematica'* (Popa Alexandru): Let $x, y, z \in \mathbb{R}_+$. Prove that:

$$\sqrt{x(y+1)}+\sqrt{y(z+1)}+\sqrt{z(x+1)} \leq \frac{3}{2}\sqrt{(x+1)(y+1)(z+1)}$$

Solution (Endrit Fejzullahu): Dividing with the square root on the RHS we have :

$$\sqrt{\frac{x}{(x+1)(z+1)}}+\sqrt{\frac{y}{(x+1)(y+1)}}+\sqrt{\frac{z}{(y+1)(z+1)}} \leq \frac{3}{2}$$

By AM-GM

$$\sqrt{\frac{x}{(x+1)(z+1)}} \leq \frac{1}{2}\left(\frac{x}{x+1}+\frac{1}{y+1}\right)$$
$$\sqrt{\frac{y}{(x+1)(y+1)}} \leq \frac{1}{2}\left(\frac{y}{y+1}+\frac{1}{x+1}\right)$$
$$\sqrt{\frac{z}{(y+1)(z+1)}} \leq \frac{1}{2}\left(\frac{z}{z+1}+\frac{1}{y+1}\right)$$

Summing we obtain

$$LHS \leq \frac{1}{2}\left(\left(\frac{x}{x+1}+\frac{1}{x+1}\right)+\left(\frac{y}{y+1}+\frac{1}{y+1}\right)+\left(\frac{z}{z+1}+\frac{1}{z+1}\right)\right)=\frac{3}{2}$$

Problem 6 *'Romanian Regional Mathematic Olympiad 2006'* (Endrit Fejzullahu): Let a, b, c be positive numbers , then prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{4a}{2a^2 + b^2 + c^2} + \frac{4b}{a^2 + 2b^2 + c^2} + \frac{4c}{a^2 + b^2 + 2c^2}$$

First Solution (Mateescu Constantin): By $AM - GM$ we have $2a^2 + b^2 + c^2 \geq 4a\sqrt{bc}$
$\Longrightarrow \dfrac{4a}{2a^2 + b^2 + c^2} \leq \dfrac{4a}{4a\sqrt{bc}} = \dfrac{1}{\sqrt{bc}}$
Addind the similar inequalities $\Longrightarrow RHS \leq \dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}}$ (1)
Using Cauchy-Schwarz we have $\left(\dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}}\right)^2 \leq \left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)^2$
So $\dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}} \leq \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$ (2)
From $(1), (2)$ we obtain the desired result .

Second Solution (Popa Alexandru): By Cauchy-Schwatz :

$$\frac{4a}{2a^2 + b^2 + c^2} \leq \frac{a}{a^2 + b^2} + \frac{a}{a^2 + c^2}$$

Then we have

$$RHS \leq \sum_{cyc} \frac{a+b}{a^2 + b^2} \leq \sum_{cyc} \frac{2}{a+b} \leq \sum_{cyc} \left(\frac{1}{2a} + \frac{1}{2b}\right) = LHS$$

Third Solution (Popa Alexandru): Since $2a^2 + b^2 + c^2 \geq a^2 + ab + bc + ca = (a+b)(c+a)$. There we have:

$$LHS \leq \sum_{cyc} \frac{4a}{(a+b)(c+a)} = \frac{8(ab + bc + ca)}{(a+b)(b+c)(c+a)} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

The last one is equivalent with $(a+b)(b+c)(c+a) \geq 8abc$.

Problem 7 *'Pham Kim Hung'* (Mateescu Constantin): Let a, b, c, d, e be non-negative real numbers such that $a + b + c + d + e = 5$. Prove that:

$$abc + bcd + cde + dea + eab \leq 5$$

Solution (Popa Alexandru): Assume $e \leq \min\{a, b, c, d\}$. Then AM-GM gives :

$$e(c+a)(b+d) + bc(a + d - e) \leq \frac{e(5-e)^2}{4} + \frac{(5-2e)^2}{27} \leq 5$$

the last one being equivalent with:

$$(e-1)^2(e+8) \geq 0$$

Problem 8 *'Popa Alexandru'* (Popa Alexandru): Let a, b, c be real numbers such that $0 \leq a \leq b \leq c$. Prove that:

$$(a+b)(c+a)^2 \geq 6abc$$

First Solution (Popa Alexandru): Let

$$b = xa\ ,\ c = yb = xya \Rightarrow x, y \geq 1$$

Then:

$$\frac{(a+b)(a+c)^2}{3} \geq 2abc$$
$$\Leftrightarrow (x+1)(xy+1)^2 \cdot a^3 \geq 6x^2ya^3$$
$$\Leftrightarrow (x+1)(xy+1)^2 \geq 6x^2y$$
$$\Leftrightarrow (x+1)(4xy+(xy-1)^2) \geq 6x^2y$$
$$\Leftrightarrow 4xy+(xy-1)^2 \cdot x+(xy-1)^2-2x^2y \geq 0$$

We have that:

$$4xy+(xy-1)^2 \cdot x+(xy-1)^2-2x^2y \geq$$
$$\geq 4xy+2(xy-1)^2-2x^2y(\text{ because } x \geq 1)$$
$$= 2x^2y^2+2-2x^2y = 2xy(y-1)+2 > 0$$

done.

Second Solution (Endrit Fejzullahu): Let $b = a+x, c = b+y = a+x+y$,sure $x, y \geq 0$
Inequality becomes

$$(2a+x)(x+y+2a)^2-6a(a+x)(a+x+y) \geq 0$$

But

$$(2a+x)(x+y+2a)^2-6a(a+x)(a+x+y) = 2a^3+2a^2y+2axy+2ay^2+x^3+2x^2y+xy^2$$

which is clearly positive.

Problem 9 *'Nesbitt'* (Raghav Grover): Prove for positive reals

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2$$

Solution (Dimitris Charisis):
From andreescu $LHS \geq \dfrac{(a+b+c+d)^2}{\sum_{sym} ab+(ac+bd)}$
So we only need to prove that:

$$(a+b+c+d)^2 \geq 2\sum_{sym} ab+2(ac+bd) \Longleftrightarrow (a-c)^2+(b-d)^2 \geq 0....$$

Problem 10 *'Vasile Cartoaje'* (Dimitris Charisis): Let a, b, c, d be REAL numbers such that $a^2+b^2+c^2+d^2 = 4$ Prove that:

$$a^3 + b^3 + c^3 + d^3 \leq 8$$

Solution (Popa Alexandru): Just observe that

$$a^3 + b^3 + c^3 + d^3 \leq 2(a^2 + b^2 + c^2 + d^2) = 8$$

because $a, b, c, d \leq 2$

Problem 11 *'Mircea Lascu'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $abc = 1$ .Prove that

$$\sum_{cyc} \frac{1}{a^2 + 2b^2 + 3} \leq \frac{1}{2}$$

Solution (Popa Alexandru): Using AM-GM we have :

$$LHS = \sum_{cyc} \frac{1}{(a^2 + b^2) + (b^2 + 1) + 2} \leq \sum_{cyc} \frac{1}{2ab + 2b + 2}$$

$$= \frac{1}{2} \sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{2}$$

because

$$\frac{1}{bc + c + 1} = \frac{1}{bc + c + abc} = \frac{1}{c} \cdot \frac{1}{ab + b + 1} = \frac{ab}{ab + b + 1}$$

and

$$\frac{1}{ca + a + 1} = \frac{1}{\frac{1}{b} + a + 1} = \frac{b}{ab + b + 1}$$

so

$$\sum_{cyc} \frac{1}{ab + b + 1} = \frac{1}{ab + b + 1} + \frac{ab}{ab + b + 1} + \frac{b}{ab + b + 1} = 1$$

Problem 12 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that:

$$\frac{1 + a + b}{2 + c} + \frac{1 + b + c}{2 + a} + \frac{1 + c + a}{2 + b} \geq \frac{15}{7}$$

First Solution (Dimitris Charisis):

$$\sum \frac{1 + a + b}{2 + c} + 1 \geq \frac{15}{7} + 3 \iff \sum \frac{3 + (a + b + c)}{2 + c} \geq \frac{36}{7} \iff \sum \frac{4}{2 + c} \geq \frac{36}{7}$$

But $\sum \frac{2^2}{2+c} \geq \frac{(2+2+2)^2}{2+2+2+a+b+c} = \frac{36}{7}$

Second Solution (Endrit Fejzullahu): Let $a \geq b \geq c$ then by Chebyshev's inequality we have
$LHS \geq \frac{1}{3}\left(1+1+1+2(a+b+c)\right)\sum_{cyc} \frac{1}{2+a} = \frac{5}{3}\sum_{cyc} \frac{1}{2+a}$
By Titu's Lemma $\sum_{cyc} \frac{1}{2+a} \geq \frac{9}{7}$, then $LHS \geq \frac{15}{7}$

Problem 13 *'Titu Andreescu, IMO 2000'* (Dimitris Charisis): Let a, b, c be positive so that $abc = 1$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

Solution (Endrit Fejzullahu):
$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$
Substitute $a = \frac{x}{y}, b = \frac{y}{z}$
Inequality is equivalent with
$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1$
$\iff (x+z-y)(y-z+x)(z-x+y) \leq xyz$
WLOG ,Let $x > y > z$, then $x+z > y, x+y > z$.If $y+z < x$, then we are done because
$(x+z-y)(y-z+x)(z-x+y) \leq 0$ and $xyz \geq 0$
Otherwise if $y+z > x$, then x, y, z are side lengths of a triangle ,and then we can make the substitution
$x = m+n, y = n+t$ and $z = t+m$
Inequality is equivalent with
$8mnt \leq (m+n)(n+t)(t+m)$, this is true by AM-GM
$m+n \geq 2\sqrt{mn}, n+t \geq 2\sqrt{nt}$ and $t+m \geq 2\sqrt{tm}$, multiply and we're done.

Problem 14 *'Korea 1998'* (Endrit Fejzullahu): Let $a, b, c > 0$ and $a+b+c = abc$. Prove that:

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}$$

First Solution (Dimitris Charisis): Setting $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$ the condition becomes $xy+yz+zx = 1$, and the inequality:
$\sum \frac{x}{\sqrt{x^2+1}} \leq \frac{3}{2}$
But $\sum \frac{x}{\sqrt{x^2+1}} = \sum \frac{x}{\sqrt{x^2+xy+xz+zy}} = \sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}}$
But $\sqrt{\frac{x}{x+y}\frac{x}{x+z}} \leq \frac{\frac{x}{x+y} + \frac{x}{x+z}}{2}$

So $\sum \sqrt{\frac{x}{x+y}\frac{x}{x+z}} \le \frac{\frac{x}{x+y}+\frac{y}{x+y}+\frac{x}{z+x}+\frac{z}{z+x}+\frac{y}{y+z}+\frac{z}{z+y}}{2} = \frac{3}{2}$

Second Solution (Raghav Grover):
Substitute $a = \tan x$,$b = \tan y$ and $c = \tan z$ where $x + y + z = \pi$
And we are left to prove
$\cos x + \cos y + \cos z \le \frac{3}{2}$
Which i think is very well known..

Third Solution (Endrit Fejzullahu): By AM-GM we have $a + b + c \ge 3\sqrt[3]{abc}$ and since $a + b + c = abc \implies (abc)^2 \ge 27$
We rewrite the given inequality as
$\frac{1}{3}\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right) \le \frac{1}{2}$
Since function $f(a) = \frac{1}{\sqrt{a^2+1}}$ is concave ,we apply Jensen's inequality
$\frac{1}{3}f(a)+\frac{1}{3}f(b)+\frac{1}{3}f(c) \le f\left(\frac{a+b+c}{3}\right) = f\left(\frac{abc}{3}\right) = \frac{1}{\sqrt{\frac{(abc)^2}{3^2}+1}} \le \frac{1}{2} \iff (abc)^2 \ge 27, \mathbb{QED}$

Problem 15 '—' (Dimitris Charisis):
If $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 3$. Find the minimum value of $A = ab + bc + ca - 3(a + b + c)$.

Solution (Endrit Fejzullahu):
$ab + bc + ca = \frac{(a+b+c)^2 - a^2 - b^2 - c^2}{2} = \frac{(a+b+c)^2 - 3}{2}$
Let $a + b + c = x$
Then $A = \frac{x^2}{2} - 3x - \frac{3}{2}$
We consider the second degree fuction $f(x) = \frac{x^2}{2} - 3x - \frac{3}{2}$
We obtain minimum for $f\left(\frac{-b}{2a}\right) = f(3) = -6$
Then $A_{min} = -6$,it is attained for $a = b = c = 1$

Problem 16 *'Cezar Lupu'* (Endrit Fejzullahu): If a, b, c are positive real numbers such that $a+b+c = 1$.Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \ge \sqrt{\frac{3}{2}}$$

First Solution (keyree10): Let $f(x) = \frac{x}{\sqrt{1-x}}$. $f''(x) > 0$
Therefore, $\sum \frac{a}{\sqrt{1-a}} \ge \frac{3s}{\sqrt{1-s}}$,where $s = \frac{a+b+c}{3} = \frac{1}{3}$, by jensen's.
$\implies \frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \ge \sqrt{\frac{3}{2}}$. Hence proved

Second Solution (geniusbliss):
By holders' inequality,
$$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)\left(\sum_{cyclic}a(b+c)\right)\geq(a+b+c)^3$$
thus,
$$\left(\sum_{cyclic}\frac{a}{(b+c)^{\frac{1}{2}}}\right)^2\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$$
or,
$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}}$$
Third Solution (Redwane Khyaoui):
$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\frac{1}{3}(a+b+c)\left(\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\right)$$
$f(x)=\frac{1}{\sqrt{x}}$ is a convex function, so jensen's inequality gives:
$$LHS\geq\frac{1}{3}\left(3\cdot\frac{1}{\sqrt{\frac{2}{3}(a+b+c)}}\right)=\sqrt{\frac{3}{2}}$$

Problem 17 *'Adapted after IMO 1987 Shortlist'* (keyree10): If a,b,c are REALS such that $a^2+b^2+c^2=1$
Prove that
$$a+b+c-2abc\leq\sqrt{2}$$

Solution (Popa Alexandru):
Use Cauchy-Schwartz:
$$LHS=a(1-2bc)+(b+c)\leq\sqrt{(a^2+(b+c)^2)((1-2bc)^2+1)}$$
So it'll be enough to prove that :
$$(a^2+(b+c)^2)((1-2bc)^2+1)\leq 2\Leftrightarrow(1+2bc)(1-bc+2b^2c^2)\leq 1\Leftrightarrow 4b^2c^2\leq 1$$
which is true because
$$1\geq b^2+c^2\geq 2bc$$
done

Problem 18 *'Russia 2000'* (Popa Alexandru): Let $x,y,z>0$ such that $xyz=1$. Show that:
$$x^2+y^2+z^2+x+y+z\geq 2(xy+yz+zx)$$

First Solution (Hoang Quoc Viet): To solve the problem of alex, we need Schur and Cauchy inequality as demonstrated as follow

$$a+b+c \geq 3\sqrt[3]{abc} = \sqrt[3]{(abc)^2} \geq \frac{9abc}{a+b+c} \geq 2(ab+bc+ca)-(a^2+b^2+c^2)$$

Note that we possess the another form of Schur such as

$$(a^2+b^2+c^2)(a+b+c)+9abc \geq 2(ab+bc+ca)(a+b+c)$$

Therefore, needless to say, we complete our proof here.

Second Solution (Popa Alexandru): Observe that :

$$2(xy+yz+zx)-(x^2+y^2+z^2) = (\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})(\sqrt{x}-\sqrt{y}+\sqrt{z})(-\sqrt{x}+\sqrt{y}+\sqrt{z})$$
$$\leq \sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z}) = \sqrt{x}+\sqrt{y}+\sqrt{z} \leq x+y+z$$

so the conclusion follows .

Third Solution (Mohamed El-Alami): Let put $p = x+y+z$ and $q = xy+yz+zx$, so then we can rewrite our inequality as $p^2+p \geq 2q$. Using Schur's inequality we have that $4q \leq \frac{p^3+9}{p}$, so it'll be enough to prove that $p^3+9 \leq p^3+p^2 \Leftrightarrow p^2 \geq 9$, which is true by AM - GM .

Problem 19 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2+b^2+c^2=3$. Prove that

$$(abc)^2(a^3+b^3+c^3) \leq 3$$

First Solution (Hoang Quoc Viet): Let

$$A = (abc)^2(a^3+b^3+c^3)$$

Therefore, we only need to maximize the following expression

$$A^3 = (abc)^6(a^3+b^3+c^3)^3$$

Using Cauchy inequality as follows, we get

$$A^3 = \frac{1}{3^6}\left(3a^2b\right)\left(3a^2c\right)\left(3b^2a\right)\left(3b^2c\right)\left(3c^2a\right)\left(3c^2b\right)\left(a^3+b^3+c^3\right)^3 \leq \left(\frac{3(a^2+b^2+c^2)(a+b+c)}{9}\right)^9$$

It is fairly straightforward that

$$a+b+c \leq \sqrt{3(a^2+b^2+c^2)} = 3$$

Therefore,

$$A^3 \leq 3^3$$

which leads to $A \leq 3$ as desired. The equality case happens $\iff a=b=c=1$

Second Solution (FantasyLover):
For the sake of convenience, let us introduce the new unknowns u, v, w as follows:

$$u = a + b + c$$
$$v = ab + bc + ca$$
$$w = abc$$

Now note that $u^2 - 2v = 3$ and $a^3 + b^3 + c^3 = u(u^2 - 3v) = u\left(\frac{9-u^2}{2}\right)$.
We are to prove that $w^2\left(u \cdot \frac{9-u^2}{2} + 3w\right) \le 3$.
By AM-GM, we have $\sqrt[3]{abc} \le \frac{a+b+c}{3} \implies w \le \frac{u^3}{3^3}$.
Hence, it suffices to prove that $u^7 \cdot \frac{9-u^2}{2} + \frac{u^9}{3^2} \le 3^7$.
However, by QM-AM we have $\sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3} \implies u \le 3$
differentiating, u achieves its maximum when $\frac{7u^6(9-u^2)}{2} = 0$.
Since a, b, c are positive, u cannot be 0, and the only possible value for u is 3.
Since $u \le 3$, the above inequality is true.

Problem 20 *'Murray Klamkin, IMO 1983'* (Hassan Al-Sibyani): Let a, b, c be the lengths of the sides of a triangle. Prove that:

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \ge 0$$

First Solution (Popa Alexandru): Use Ravi substitution $a = x + y$, $b = y + z$, $c = z + x$ then the inequality becomes :

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \ge x + y + z \ ,$$

true by Cauchy-Schwartz.

Second Solution (geniusbliss): we know from triangle inequality that $b \ge (a-c)$ and $c \ge (b-a)$ and $a \ge (c-b)$
therefore,
$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \ge a^2(a-c)(a-b) + b^2(b-a)(b-c) + c^2(c-b)(c-a) \ge 0$
and the last one is schur's inequality for $r = 2$ so proved with the equality holding when $a = b = c$ or for and equilateral triangle

Third Solution (Popa Alexandru): The inequality is equivalent with :

$$\frac{1}{2}((a-b)^2(a+b-c)(b+c-a) + (b-c)^2(b+c-a)(a+c-b) + (c-a)^2(a+c-b)(a+b-c)) \ge 0$$

Problem 21 *'Popa Alexandru'* (Popa Alexandru): Let $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Show that :

$$1 \geq \sqrt[3]{xyz} + \frac{2}{3(x+y+z)}$$

First Solution (Endrit Fejzullahu):
By Am-Gm $x+y+z \geq 3\sqrt[3]{xyz}$,then
$\frac{2}{3(x+y+z)} + \sqrt[3]{xyz} \leq \frac{2}{9}\sqrt[3]{xyz} + \sqrt[3]{xyz}$
Since $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$,then $\frac{1}{3} \leq \sqrt[3]{xyz} \leq \frac{2}{3}$
Let $a = \sqrt[3]{xyz}$,then $a + \frac{2}{9a} \leq 1 \iff 9a^2 - 9a + 2 \leq 0 \iff 9\left(a - \frac{1}{3}\right)\left(a - \frac{2}{3}\right) \leq 0$,we're done since $\frac{1}{3} \leq a \leq \frac{2}{3}$

Second Solution (Popa Alexandru): Let $s = a + b + c$. By AM-GM it is enough to prove that :

$$1 - \frac{s}{3} - \frac{2}{3s} \geq 0 \Leftrightarrow (s-1)(2-s) \geq 0$$

The last one is true since $x, y, z \in \left[\frac{1}{3}, \frac{2}{3}\right]$

Problem 22 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c be side lengths of a triangle,and β is the angle between a and c.Prove that

$$\frac{b^2+c^2}{a^2} > \frac{2\sqrt{3}c\sin\beta - a}{b+c}$$

Solution (Endrit Fejzullahu): According to the Weitzenbock's inequality we have
$a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ and $S = \frac{ac\sin\beta}{2}$
Then
$b^2 + c^2 \geq 2\sqrt{3}ac\sin\beta - a^2$,dividing by a^2 , we have
$\frac{b^2+c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a}$
Since $a < b + c \implies \frac{b^2+c^2}{a^2} \geq \frac{2\sqrt{3}c\sin\beta - a}{a} > \frac{2\sqrt{3}c\sin\beta - a}{b+c}$

Problem 23 *'Dinu Serbanescu, Junior TST 2002, Romania'* (Hassan Al-Sibyani): If $a, b, c \in (0, 1)$ Prove that:

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$$

First Solution (FantasyLover):
Since $a, b, c \in (0, 1)$, let us have $a = \sin^2 A, b = \sin^2 B, c = \sin^2 C$ where $A, B, C \in \left(0, \frac{\pi}{2}\right)$.
Then, we are to prove that $\sin A \sin B \sin C + \cos A \cos B \cos C < 1$.
Now noting that $\sin C, \cos C < 1$, we have $\sin A \sin B \sin C + \cos A \cos B \cos C < \sin A \sin B + \cos A \cos B = \cos(A - B) \leq 1$. $\blacksquare$

Second Solution (Popa Alexandru): Cauchy-Schwartz and AM-GM works fine :

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} = \sqrt{a}\sqrt{bc} + \sqrt{1-a}\sqrt{(1-b)(1-c)} \leq$$

$$\sqrt{a + (1-a)}\sqrt{bc + (1-b)(1-c)} = \sqrt{bc + (1-b)(1-c)} < 1$$

Third Solution (Redwane Khyaoui):

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \leq \frac{a + bc}{2} + \frac{1 - a + (1-b)(1-c)}{2}$$

So we only need to prove that $\dfrac{a + bc}{2} + \dfrac{1 - a + (1-b)(1-c)}{2} \leq 1$
$\Leftrightarrow \dfrac{1}{b} + \dfrac{1}{c} \geq 2$ which is true since $a, b \in [0, 1]$

Problem 24 *'Hojoo Lee'* (FantasyLover): For all positive real numbers a, b, c, prove the following:

$$\frac{1}{\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}} - \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \frac{1}{3}$$

First Solution (Popa Alexandru): Using p, q, r substitution ($p = a + b + c$, $q = ab + bc + ca$, $r = abc$) the inequality becomes :

$$\frac{3(p + q + r + 1)}{2p + q + r} \geq \frac{9 + 3r}{q} \Leftrightarrow pq + 2q^2 \geq 6pr + 9r$$

which is true because is well-known that $pq \geq 9r$ and $q^2 \geq 3pr$

Second Solution (Endrit Fejzullahu): After expanding the inequality is equivalent with :

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right)$$

This is true by Chebyshev's inequality , so we're done .

Problem 25 *'Mihai Opincariu'* (Popa Alexandru): Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{ab}{a^2 + b^2 + \sqrt{c}} + \frac{bc}{b^2 + c^2 + \sqrt{a}} + \frac{ca}{c^2 + a^2 + \sqrt{b}} \leq 1$$

First Solution (FantasyLover):
We have $a^2+b^2+\sqrt{c} \geq 2ab+\sqrt{c} = \frac{2}{c}+\sqrt{c}$.
Hence, it suffices to prove that $\sum_{\text{cyc}} \frac{\frac{1}{a}}{\frac{2}{a}+\sqrt{a}} = \sum_{\text{cyc}} \frac{1}{2+a\sqrt{a}} \leq 1$.
Reducing to a common denominator, we prove that

$$\sum_{\text{cyc}} \frac{1}{2+a\sqrt{a}} = \frac{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}+12}{4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+2(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca})+9} \leq 1$$

.
Rearranging, it remains to prove that $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca} \geq 3$.
Applying AM-GM, we have $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca} \geq 3\sqrt[3]{a^2b^2c^2\sqrt{a^2b^2c^2}} = 3$, and we are done. ■

Second Solution (Popa Alexandru):

$$LHS \leq \sum_{cyc} \frac{ab}{2ab+\sqrt{c}} = \sum_{cyc} \frac{1}{2+c\sqrt{c}} = \sum_{cyc} \frac{1}{2+\frac{x}{y}} \leq RHS$$

Problem 26 *'Korea 2006 First Examination'* (FantasyLover): x, y, z are real numbers satisfying the condition $3x+2y+z=1$. Find the maximum value of

$$\frac{1}{1+|x|}+\frac{1}{1+|y|}+\frac{1}{1+|z|}$$

.

Solution (dgreenb801):
We can assume x,y, and z are all positive, because if one was negative we could just make it positive, which would allow us to lessen the other two variables, making the whole sum larger.
Let $3x=a$, $2y=b$, $z=c$, then $a+b+c=1$ and we have to maximize
$\frac{3}{a+3}+\frac{2}{b+2}+\frac{1}{c+1}$
Note that
$\left(\frac{3}{a+c+3}+1\right)-\left(\frac{3}{a+3}+\frac{1}{c+1}\right) = \frac{a^2c+ac^2+6ac+6c}{(a+3)(c+1)(a+c+3)} \geq 0$
So for fixed $a+c$, the sum is maximized when $c=0$.
We can apply the same reasoning to show the sum is maximized when $b=0$.
So the maximum occurs when $a=1$, $b=0$, $c=0$, and the sum is $\frac{11}{4}$.

Problem 27 *'Balkan Mathematical Olympiad 2006'* (dgreenb801):

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

for all positive reals.

First Solution (Popa Alexandru):
AM-GM works :

$$(1+abc)\,LHS+3=\sum_{cyc}\frac{1+abc+a+ab}{a+ab}=\sum_{cyc}\frac{1+a}{ab+a}+\sum_{cyc}\frac{b(c+1)}{b+1}\geq\frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc}\geq 6$$

Second Solution (Popa Alexandru): Expanding it is equivalent with :

$$ab(b+1)(ca-1)^2+bc(c+1)(ab-1)^2+ca(a+1)(bc-1)^2\geq 0$$

Problem 28 *'Junior TST 2007, Romania'* (Popa Alexandru): Let $a,b,c>0$ such that $ab+bc+ca=3$. Show that :

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\leq\frac{1}{abc}$$

First Solution (Endrit Fejzullahu): Since $ab+bc+ca=3\implies abc\leq 1$
Then $1+3a-abc\geq 3a$
Then
$\sum\frac{1}{1+3a-abc}\leq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{ab+bc+ca}{3abc}=\frac{1}{abc}$
Second Solution (Popa Alexandru): Using $abc\leq 1$ and the condition we have :

$$\sum_{cyc}\frac{1}{1+a^2(b+c)}\leq\frac{1}{abc}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{1}{abc}+a\left(\frac{1}{b}+\frac{1}{c}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{1}{1+a\left(\frac{1}{b}+\frac{1}{c}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{1}{1+a\left(\frac{3}{abc}-\frac{1}{a}\right)}\leq 1\Leftrightarrow\sum_{cyc}\frac{ab}{3}\leq 1$$

Problem 29 *'Lithuania 1987'* (Endrit Fejzullahu): Let a,b,c be positive real numbers .Prove that

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}$$

First Solution (dgreenb801): By Cauchy,
$\sum\frac{a^3}{a^2+ab+b^2}=\sum\frac{a^4}{a^3+a^2b+ab^2}\geq\frac{(a^2+b^2+c^2)^2}{a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2}$
This is $\geq\frac{a+b+c}{3}$ if
$(a^4+b^4+c^4)+2(a^2b^2+b^2c^2+c^2a^2)\geq(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)+(a^2bc+ab^2c+abc^2)$
This is equivalent to
$(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca)\geq 0$
Which is true as $a^2+b^2+c^2-ab-bc-ca\geq 0\iff(a-b)^2+(b-c)^2+(c-a)^2\geq 0$

Second Solution (Popa Alexandru):
Since :

$$\frac{a^3-b^3}{a^2+ab+b^2}=a-b$$

and similars we get :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}.$$

Now it remains to prove :

$$\frac{1}{2} \cdot \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b}{6}$$

which is trivial .

Third Solution (Popa Alexandru): We have :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}$$

Then we need to prove :

$$\frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3}$$

We have :

$$\frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2} - \frac{a+b+c}{3} = \frac{1}{2} \sum_{cyc} \left(\frac{a^3+b^3}{a^2+ab+b^2} - \frac{a+b}{3} \right) =$$

$$\frac{1}{2} \sum_{cyc} \frac{3a^3+3b^3-a^3-a^2b-ab^2-b^3-a^2b-ab^2}{3(a^2+ab+b^2)} = \frac{1}{2} \sum_{cyc} \frac{2(a-b)^2(a+b)}{3(a^2+ab+b^2)} \geq 0$$

Fourth Solution (Mohamed El-Alami): We have :

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2} \sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2}$$

Since $2(a^2+ab+b^2) \leq 3(a^2+b^2)$ we have :

$$LHS = \sum_{cyc} \frac{a^3+b^3}{2(a^2+ab+b^2)} \geq \sum_{cyc} \frac{a^3+b^3}{3(a^2+b^2)} \geq \frac{a+b+c}{3}$$

Problem 30 '—' (dgreenb801): Given $ab+bc+ca=1$ Show that:

$$\frac{\sqrt{3a^2+b^2}}{ab} + \frac{\sqrt{3b^2+c^2}}{bc} + \frac{\sqrt{3c^2+a^2}}{ca} \geq 6\sqrt{3}$$

Solution (Hoang Quoc Viet):
Let's make use of Cauchy-Schwarz as demonstrated as follows

$$\frac{\sqrt{(3a^2+b^2)(3+1)}}{2ab} \geq \frac{3a+b}{2ab}$$

Thus, we have the following estimations

$$\sum_{cyc} \frac{\sqrt{3a^2+b^2}}{ab} \geq 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Finally, we got to prove that

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt{3}$$

However, from the given condition, we derive

$$abc \leq \frac{1}{3\sqrt{3}}$$

and

$$\sum_{cyc} \frac{1}{a} \geq 3\sqrt[3]{\frac{1}{abc}} \geq 3\sqrt{3}$$

Problem 31 *'Komal Magazine'* (Hoang Quoc Viet): Let a, b, c be real numbers. Prove that the following inequality holds

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2$$

Solution (Popa Alexandru): Cauchy-Schwartz gives :

$$(a^2+2)(b^2+2) = (a^2+1)(1+b^2)+a^2+b^2+3 \geq (a+b)^2+\frac{1}{2}(a+b)^2+3 = \frac{3}{2}((a+b)^2+2)$$

And Cauchy-Schwartz again

$$(a^2+2)(b^2+2)(c^2+2) \geq \frac{3}{2}((a+b)^2+2)(2+c^2) \geq \frac{3}{2}(\sqrt{2}(a+b)+\sqrt{2}c)^2 = RHS$$

Problem 32 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c \geq 0$ and $a+b+c=1$. Prove that :

$$\frac{a}{\sqrt{b^2+3c}}+\frac{b}{\sqrt{c^2+3a}}+\frac{c}{\sqrt{a^2+3b}} \geq \frac{1}{\sqrt{1+3abc}}$$

Solution (Endrit Fejzullahu):
Using Holder's inequality
$\left(\sum_{cyc} \frac{a}{\sqrt{b^2+3c}}\right)^2 \cdot \sum_{cyc} a(b^2+3c) \geq (a+b+c)^3 = 1$
It is enough to prove that
$1+3abc \geq \sum_{cyc} a(b^2+3c)$
Homogenise $(a+b+c)^3 = 1$,

Also after Homogenising $\sum_{cyc} a(b^2+3c) = a^2b + b^2c + c^2a + 9abc + 3\sum_{sym} a^2b$
$(a+b+c)^3 = a^3+b^3+c^3+6abc+3\sum_{sym} a^2b$
It is enough to prove that
$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$
By AM-GM
$a^3+a^3+b^3 \geq 3a^2b$
$b^3+b^3+c^3 \geq 3b^2c$
$c^3+c^3+a^3 \geq 3c^2a$
Then $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$,done

Problem 33 *'Apartim De'* (Apartim De): If a, b, c, d be positive reals then prove that:

$$\frac{a^2+b^2}{ab+b^2} + \frac{b^2+c^2}{bc+c^2} + \frac{c^2+d^2}{cd+d^2} \geq \sqrt[3]{\frac{54a}{(a+d)}}$$

Solution (Aravind Srinivas):
Write the LHS as $\dfrac{(\frac{a}{b})^2+1}{(\frac{a}{b})+1}$+ two other similar termsfeel lazy to write them down. This is a beautiful application of Jensen's as the function for positve real t such that $f(t) = \dfrac{t^2+1}{t+1}$ is convex since $\left(\dfrac{t-1}{t+1}\right)^2 \geq 0$.
Thus, we get that $LHS \geq \dfrac{\left(\frac{(\sum \frac{a}{b} - \frac{d}{a}}{3}\right)^2+1}{\sum \frac{a}{b} - \frac{d}{a} + 3}$
I would like to write $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} = K$ for my convenience with latexing.
so we have $\dfrac{\left(\frac{K}{3})^2+1\right)}{K+3} = \dfrac{K+\frac{9}{K}}{1+\frac{3}{K}} \geq \dfrac{6}{1+\frac{3}{K}}$
This is from $K + \dfrac{9}{K} \geq 6$ by AM GM.
Now $K = \sum \dfrac{a}{b} - \dfrac{d}{a} \geq 3\left(\dfrac{d}{a}\right)^{\frac{1}{3}}$ by AM GM.
Thus, we have $\dfrac{6}{1+\frac{3}{K}} \geq \dfrac{6a^{\frac{1}{3}}}{a^{\frac{1}{3}}+d^{\frac{1}{3}}} \geq \dfrac{3a^{\frac{1}{3}}}{\frac{a+d}{2}^{\frac{1}{3}}} = \left(\dfrac{54a}{a+d}\right)^{\frac{1}{3}}$

Problem 34 *'—'* (Raghav Grover): If a and b are non negative real numbers such that $a \geq b$. Prove that

$$a + \frac{1}{b(a-b)} \geq 3$$

First Solution (Dimitris Charisis):
If $a \geq 3$ the problem is obviously true.
Now for $a < 3$ we have :

$a^2b - ab^2 - 3ab + 3b^2 + 1 \geq 0 \Longleftrightarrow ba^2 - (b^2 + 3b)a + 3b^2 + 1 \geq 0$
It suffices to prove that $D \leq 0 \Longleftrightarrow b^4 + 6b^3 + 9b^2 - 12b^3 - 4b \Longleftrightarrow b(b-1)^2(b-4) \leq 0$ which is true.

Second Solution (dgreenb801):
$a + \dfrac{1}{b(a-b)} = b + (a-b) + \dfrac{1}{b(a-b)} \geq 3$ by AM-GM

Third Solution (geniusbliss):
since $a \geq b$ we have $\frac{a}{2} \geq \sqrt{b(a-b)}$ square this and substitute in this denominator we get,
$\dfrac{a}{2} + \dfrac{a}{2} + \dfrac{4}{a^2} \geq 3$ by AM-GM so done.

Problem 35 *'Vasile Cirtoaje'* (Dimitris Charisis):

$$(a^2 - bc)\sqrt{b+c} + (b^2 - ca)\sqrt{c+a} + (c^2 - ab)\sqrt{a+b} \geq 0$$

First Solution (Popa Alexandru): Denote $\frac{a+b}{2} = x^2, \ldots$, then the inequality becomes :

$$\sum_{cyc} xy(x^3 + y^3) \geq \sum_{cyc} x^2y^2(x+y)$$

which is equivalent with :

$$\sum_{cyc} xy(x+y)(x-y)^2 \geq 0$$

Second Solution (Popa Alexandru): Set $A = \sqrt{b+c}$ and similars . Therefore the inequality rewrites to :

$$A(a^2 - bc) + B(b^2 - ca) + C(c^2 - ab) \geq 0$$

We have that :

$$2\sum_{cyc} A(a^2 - bc) = \sum_{cyc} A[(a-b)(a+c) + (a-c)(a+b)] =$$

$$= \sum_{cyc} A(a-b)(a+c) + \sum_{cyc} B(b-a)(b+c) = \sum_{cyc}(a-b)[A(a+c) - B(b+c)] =$$

$$= \sum_{cyc}(a-b)\frac{A^2(a+c)^2 - B^2(b+c)^2}{A(a+c) + B(b+c)} = \sum_{cyc} \frac{(a-b)^2(a+c)(b+c)}{A(a+c) + B(b+c)} \geq 0$$

Problem 36 *'Cezar Lupu'* (Popa Alexandru): Let $a, b, c > 0$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \sqrt{abc}$. Prove that:

$$abc \geq \sqrt{3(a+b+c)}$$

Solution (Endrit Fejzullahu):
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\sqrt{abc} \iff ab+bc+ca=abc\sqrt{abc}$
By Am-Gm
$(ab+bc+ca)^2 \geq 3abc(a+b+c)$
Since ab + bc + ca = abc $\sqrt{abc}, we have$
(abc) $^3 \geq 3abc(a+b+c) \implies abc \geq \sqrt{3(a+b+c)}$

Problem 37 *'Pham Kim Hung'* (Endrit Fejzullahu): Let a,b,c,d be positive real numbers satisfying $a+b+c+d=4$.Prove that

$$\frac{1}{11+a^2}+\frac{1}{11+b^2}+\frac{1}{11+c^2}+\frac{1}{11+d^2}\leq\frac{1}{3}$$

First Solution (Apartim De):
$f(x)=\dfrac{1}{11+x^2} \Leftrightarrow f''(x)=\dfrac{6(x^2-\frac{11}{3})}{(11+x^2)^3}$
$(x^2-\dfrac{11}{3})=\left(x-\sqrt{\dfrac{11}{3}}\right)\left(x+\sqrt{\dfrac{11}{3}}\right)$
If x $\in\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$, $f''(x)<0$

Thus within the interval $\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$, the quadratic polynomial is negative

thereby making $f''(x)<0$, and thus $f(x)$ is concave within $\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$.

Let $a\leq b\leq c\leq d$. If all of $a,b,c,d\in\left(0,\sqrt{\dfrac{11}{3}}\right)$,

Then by Jensen, $f(a)+f(b)+f(c)+f(d)\leq 4f\left(\dfrac{a+b+c+d}{4}\right)=4f(1)=\dfrac{4}{12}=\dfrac{1}{3}$

$f'(x)=\dfrac{-2x}{(11+x^2)^3}<0$ (for all positive x)

At most 2 of a,b,c,d(namely c& d) can be greater than $\sqrt{\dfrac{11}{3}}$

In that case,

$f(a)+f(b)+f(c)+f(d)<f(a-1)+f(b-1)+f(c-3)+f(d-3)<4f\left(\dfrac{a+b+c+d-8}{4}\right)=4f(-1)=\dfrac{4}{12}=\dfrac{1}{3}$

QED

Second Solution (Popa Alexandru): Rewrite the inequality in the following form

$$\sum_{cyc}\left(\frac{1}{11+a^2}-\frac{1}{12}\right)\geq 0$$

or equivalently

$$\sum_{cyc}(1-a)\cdot\frac{a+1}{a^2+11}\geq 0$$

Notice that if (a, b, c, d) is arranged in an increasing order then

$$\frac{a+1}{a^2+11} \geq \frac{b+1}{b^2+11} \geq \frac{c+1}{c^2+11} \geq \frac{d+1}{d^2+11}$$

The desired results follows immediately from the Chebyshev inequality.

Problem 38 *'Crux Mathematicorum'* (Apartim De): Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2$$

.
When does equality occur?

Solution (Virgil Nicula):

Proof. I"ll use the remarkable linear-angled identities $\left\| \begin{array}{ll} a^2+b^2+c^2 = 2\cdot\left(p^2-r^2-4Rr\right) & (1) \\ 4S = \left(b^2+c^2-a^2\right)\cdot\tan A & (2) \\ \sin 2A + \sin 2B + \sin 2C = \frac{2S}{R^2} & (3) \\ \cos A + \cos B + \cos C = 1 + \frac{r}{R} & (4) \end{array} \right\|$. Therefore,

$$\sum a^2 = \sum\left(b^2+c^2-a^2\right) \stackrel{(2)}{=} 4S\cdot\sum\frac{\cos A}{\sin A} = 8S\cdot\sum\frac{\cos^2 A}{\sin 2A} \stackrel{C.B.S.}{\geq} 8S\cdot\frac{\left(\sum\cos A\right)^2}{\sum\sin 2A} \stackrel{(3)\wedge(4)}{=} 8S\cdot\frac{\left(1+\frac{r}{R}\right)^2}{\frac{2S}{R^2}} =$$
$$= 4(R+r)^2 \Longrightarrow \boxed{a^2+b^2+c^2 \geq 4(R+r)^2} \stackrel{(1)}{\Longrightarrow} 2\cdot\left(p^2-r^2-4Rr\right) \geq 4(R+r)^2 \Longrightarrow \boxed{p^2 \geq 2R^2+8Rr+3r^2}.$$

Problem 39 *'Russia 1978'* (Endrit Fejzullahu): Let $0 < a < b$ and $x_i \in [a, b]$.Prove that

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

Solution (Popa Alexandru):
We will prove that if $a_1, a_2, \ldots, a_n \in [a, b] (0 < a < b)$ then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \frac{(a+b)^2}{4ab}n^2$$

$$P = (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = \left(\frac{a_1}{c} + \frac{a_2}{c} + \cdots + \frac{a_n}{c}\right)\left(\frac{c}{a_1} + \frac{c}{a_2} + \cdots + \frac{c}{a_n}\right) \leq$$
$$\leq \frac{1}{4}\left(\frac{a_1}{c} + \frac{c}{a_1} + \frac{a_2}{c} + \frac{c}{a_2} + \cdots + \frac{a_n}{c} + \frac{c}{a_n}\right)^2$$

Function $f(t) = \frac{c}{t} + \frac{t}{c}$ have its maximum on $[a, b]$ in a or b. We will choose c such that $f(a) = f(b), c = \sqrt{ab}$. Then $f(t) \leq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$. Then

$$P \leq n^2 \left(\sqrt{\frac{a}{b}} + \sqrt{b\frac{b}{a}} \right)^2 \cdot \frac{1}{4} = n^2 \frac{(a+b)^2}{4ab}$$

Problem 40 *'Vasile Cartoaje'* (keyree10): a, b, c are non-negative reals. Prove that

$$81abc \cdot (a^2 + b^2 + c^2) \leq (a + b + c)^5$$

Solution (Popa Alexandru):

$$81abc(a^2 + b^2 + c^2) \leq 27 \frac{(ab + bc + ca)^2}{a + b + c}(a^2 + b^2 + c^2) \leq (a + b + c)^5$$

By p, q, r the last one is equivalent with :

$$p^6 - 27q^2p^2 + 54q^3 \geq 0 \Leftrightarrow (p^2 - 3q)^2 \geq 0$$

Problem 41 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c > 0$ such that $a^3 + b^3 + 3c = 5$. Prove that :

$$\sqrt{\frac{a+b}{2c}} + \sqrt{\frac{b+c}{2a}} + \sqrt{\frac{c+a}{2b}} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Solution (Sayan Mukherjee):
$a^3 + 1 + 1 + b^3 + 1 + 1 + 3c = 9 \geq 3a + 3b + 3c \implies a + b + c \leq 3$(AM-GM)
Also $abc \leq \left(\frac{a+b+c}{3}\right) = 1$(AM-GM)
Applying CS on the LHS;
$\left[\sum \sqrt{\frac{a+b}{2c}}\right]^2 \leq (a + b + c) \left(\sum \frac{1}{a}\right)$
It is left to prove that $\sum \frac{1}{a} \geq \sum a$
But this $\implies \sum \frac{1}{a} \geq 3$
Which is perfextly true, as from AM-GM on the LHS;
$\sum \frac{1}{a} \geq 3\sqrt[3]{\frac{1}{abc}} \geq 3\, [\because abc \leq 1]$

Problem 42 *'Sayan Mukherjee'* (Sayan Mukherjee): Let $a, b, c > 0$ PT: If a, b, c satisfy $\sum \frac{1}{a^2+1} = \frac{1}{2}$ Then we always have:

$$\sum \frac{1}{a^3+2} \leq \frac{1}{3}$$

Solution (Endrit Fejzullahu):

$$a^3 + a^3 + 1 \geq 3a^2 \implies a^3 + a^3 + 1 + 3 \geq 3a^2 + 3 \implies \frac{1}{3(a^2+1)} \geq \frac{1}{2(a^3+2)} \implies \frac{1}{3} \geq \sum \frac{1}{a^3+2}$$

Problem 43 *'Russia 2002'* (Endrit Fejzullahu): Let a, b, c be positive real numbers with sum 3.Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

Solution (Sayan Mukherjee):
$2(ab + bc + ca) = 9 - (a^2 + b^2 + c^2)$
as $a + b + c = 9$. Hence the inequality
$\implies 2 \sum \sqrt{a} \geq 9 - (a^2 + b^2 + c^2)$
$\implies \sum (a^2 + \sqrt{a} + \sqrt{a}) \geq 9$
Perfectly true from AGM as :
$a^2 + \sqrt{a} + \sqrt{a} \geq 3a$

Problem 44 *'India 2002'* (Raghav Grover): For any natural number n prove that

$$\frac{1}{2} \leq \frac{1}{n^2+1} + \frac{1}{n^2+2} + ... + \frac{1}{n^2+n} \leq \frac{1}{2} + \frac{1}{2n}$$

Solution (Sayan Mukherjee):
$\sum_{k=1}^{n} \frac{k}{n^2+k} = \sum_{k=1}^{n} \frac{k^2}{n^2k+k^2} \geq \frac{\sum_{k=1}^{n} k}{n^2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2} = \frac{3(n^2+n)}{2(3n^2+2n+1)} > \frac{1}{2}$
As it is equivalent to :$3n^2 + 3n > 3n^2 + 2n + 1 \implies n > 1$
For the 2nd part;
$\sum \frac{k}{n^2+k} = \sum 1 - \frac{n^2}{n^2+k} = n - n^2 \sum \frac{1}{n^2+k}$ and then use AM-HM for $\sum \frac{1}{n^2+k}$ So we get the desired result.

Problem 45 *'—'* (Sayan Mukherjee): For $a, b, c > 0; a^2 + b^2 + c^2 = 1$ find P_{min} if:

$$P = \sum_{cyc} \frac{a^2b^2}{c^2}$$

Solution (Endrit Fejzullahu):
Let $x = \dfrac{ab}{c}, y = \dfrac{bc}{a} and z = \dfrac{ca}{b}$
Then Obviously By Am-Gm
$x^2 + y^2 + z^2 \geq xy + yz + zx = a^2 + b^2 + c^2 = 1$,then $P_{min} = 1$

Problem 46 *'Pham Kim Hung'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $a+b+c = 3$ .Prove that

$$\frac{a^2}{a+2b^2} + \frac{b^2}{b+2c^2} + \frac{c^2}{c+2a^2} \geq 1$$

Solution (Popa Alexandru): We start with a nice use of AM-GM :

$$\frac{a^2}{a+2b^2} = \frac{a^2 + 2ab^2 - 2ab^2}{a+2b^2} = \frac{a(a+2b^2)}{a+2b^2} - \frac{2ab^2}{a+2b^2} \geq a - \frac{2}{3}\sqrt[3]{a^2b^2}$$

Suming the similars we need to prove :

$$\sqrt[3]{a^2b^2} + \sqrt[3]{b^2c^2} + \sqrt[3]{c^2a^2} \leq 3$$

By AM-GM :

$$\sum_{cyc} \sqrt[3]{a^2b^2} \leq \sum_{cyc} \frac{2ab+1}{3} = \frac{1}{3}\left(2\sum_{cyc} ab + 3\right) \leq 3 \Leftrightarrow ab + bc + ca \leq 3 \Leftrightarrow 3(ab+bc+ca) \leq (a+b+c)^2$$

Problem 47 *'mateforum.ro'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c \leq \frac{3}{2}$. Prove that :

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2}$$

Solution:
WLOG $a \geq b \geq c$
Then By Chebyshev's inequality we have
$RHS \geq \dfrac{1}{3} \cdot LHS \cdot \left(\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a}\right)$
It is enough to show that
$\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a} \geq 3$
By Cauchy Schwartz

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{9}{2(a+b+c)} \geq 3 \iff a+b+c \leq \frac{3}{2}, \text{ done}$$

Problem 48 *'USAMO 2003'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers. Prove that:

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+a+c)^2}{2b^2+(a+c)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$$

Solution (Popa Alexandru):
Suppose $a+b+c=3$
The inequality is equivalent with :

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2} \leq 8$$

For this we prove :

$$\frac{(a+3)^2}{2a^2+(3-a)^2} \leq \frac{4}{3}a+\frac{4}{3} \Leftrightarrow 3(4a+3)(a-1)^2 \geq 0$$

done.

Problem 49 *'Marius Mainea'* (Popa Alexandru): Let $x, y, z > 0$ such that $x+y+z=xyz$. Prove that :

$$\frac{x+y}{1+z^2}+\frac{y+z}{1+x^2}+\frac{z+x}{1+y^2} \geq \frac{27}{2xyz}$$

Solution (socrates): Using Cauchy-Schwarz and AM-GM inequality we have:

$$\frac{x+y}{1+z^2}+\frac{y+z}{1+x^2}+\frac{z+x}{1+y^2}=\frac{(x+y)^2}{x+y+(x+y)z^2}+\frac{(y+z)^2}{y+z+(y+z)x^2}+\frac{(z+x)^2}{z+x+(z+x)y^2} \geq \frac{((x+y)+(y+z)+(z+x))^2}{2(x+y+z)+\sum x^2(y+z)}=$$

$$\frac{4(x+y+z)^2}{2xyz+\sum x^2(y+z)}=\frac{4(x+y+z)^2}{\prod(x+y)} \geq 4(x+y+z)^2 \cdot \frac{27}{8} \frac{1}{(x+y+z)^3}=\frac{27}{2xyz}$$

Problem 50 '—' (socrates): Let $a, b, c > 0$ such that $ab+bc+ca=1$. Prove that :

$$abc(a+\sqrt{a^2+1})(b+\sqrt{b^2+1})(c+\sqrt{c^2+1}) \leq 1$$

Solution (dgreenb801): Note that $\sqrt{a^2+1}=\sqrt{a^2+ab+bc+ca}=\sqrt{(a+b)(a+c)}$
Also, by Cauchy, $(a+\sqrt{(a+b)(a+c)})^2 \leq (a+(a+b))(a+(a+c))=(2a+b)(2a+c)$
So after squaring both sides of the inequality, we have to show

$$1=(ab+bc+ca)^6 \geq a^2b^2c^2(2a+b)(2a+c)(2b+a)(2b+c)(2c+a)(2c+b)=$$

$$(2ac + bc)(2ab + bc)(2bc + ac)(2ab + ac)(2cb + ab)(2ca + ab)$$

which is true by AM-GM.

Problem 51 *'Asian Pacific Mathematics Olympiad 2005'* (dgreenb801): Let $a, b, c > 0$ such that $abc = 8$. Prove that:

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Solution (Popa Alexandru): By AM-GM :

$$\sqrt{a^3+1} \leq \frac{a^2+2}{2}.$$

Then we have :

$$\sum_{cyc} \frac{a^2}{\sqrt{(a^3+1)(b^3+1)}} \geq 4\sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)}$$

So we need to prove

$$\sum_{cyc} \frac{a^2}{(a^2+2)(b^2+2)} \geq \frac{1}{3}.$$

which is equivalent with :

$$a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2+b^2+c^2) \geq 72.$$

, true by AM-GM

Problem 52 *'Lucian Petrescu'* (Popa Alexandru): Prove that in any acute-angled triangle ABC we have :

$$\frac{a+b}{\cos C} + \frac{b+c}{\cos A} + \frac{c+a}{\cos B} \geq 4(a+b+c)$$

First Solution (socrates): The inequality can be rewritten as

$$\frac{ab(a+b)}{a^2+b^2-c^2} + \frac{bc(b+c)}{b^2+c^2-a^2} + \frac{ca(c+a)}{c^2+a^2-b^2} \geq 2(a+b+c)$$

or

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2} + \frac{c^2}{c^2+a^2-b^2}\right) \geq 2(a+b+c)$$

Applying Cauchy Schwarz inequality we get

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2} + \frac{c^2}{c^2+a^2-b^2}\right) \geq \sum a\frac{(b+c)^2}{(a^2+b^2-c^2)+(c^2+a^2-b^2)}$$

or

$$\sum_{cyclic} a\left(\frac{b^2}{a^2+b^2-c^2} + \frac{c^2}{c^2+a^2-b^2}\right) \geq \sum \frac{(b+c)^2}{2a}$$

So, it is enough to prove that
$$\sum \frac{(b+c)^2}{2a} \geq 2(a+b+c)$$
which is just CS as above.

Second Solution (Mateescu Constantin):
From the law of cosinus we have $a = b\cos C + c\cos B$and $c = b\cos A + a\cos B$.
$\Longrightarrow \dfrac{a+c}{\cos B} = a + c + \dfrac{b(\cos A + \cos C)}{\cos B} = a + c + 2R\tan B(\cos A + \cos C)$. Then:
$\sum \dfrac{a+c}{\cos B} = 2(a+b+c) + 2R\sum \tan B \cdot (\cos A + \cos C)$ and the inequality becomes:
$2R\sum \tan B(\cos A + \cos C) \geq 2\sum a = 4R\sum \sin A$
$\Longleftrightarrow \sum \tan B\cos A + \sum \tan B\cos C \geq 2\sum \sin A = 2\sum \tan A\cos A$
Wlog assume that $A \leq B \leq C$. Then $\cos A \geq \cos B \geq \cos C$ and $\tan A \leq \tan B \leq \tan C$, so
$\sum \tan A\cos A \leq \sum \tan B\cos A$ and $\sum \tan A\cos A \leq \sum \tan B\cos C$, according to rearrangement inequality
.
Adding up these 2 inequalities yields the conclusion.

Third Solution (Popa Alexandru):
$$LHS \geq \frac{(2(a+b+c))^2}{(a+b)\cos C + (b+c)\cos A + (c+a)\cos B} = 4(a+b+c)$$

Problem 53 '—' (socrates): Given $x_1, x_2, .., x_n > 0$ such that $\sum_{i=1}^{n} x_i = 1$, prove that
$$\sum_{i=1}^{n} \frac{x_i + n}{1 + x_i^2} \leq n^2$$

Solution (Hoang Quoc Viet):
Without loss of generality, we may assume that
$$x_1 \geq x_2 \geq \cdots \geq x_n$$
Therefore, we have
$$nx_1 - 1 \geq nx_2 - 1 \geq \cdots \geq nx_n - 1$$
and
$$\frac{x_1}{x_1^2+1} \geq \frac{x_2}{x_2^2+1} \geq \cdots \geq \frac{x_n}{x_n^2+1}$$
Hence, by Chebyshev inequality, we get
$$\sum_{i=1}^{n} \frac{(nx_i - 1)x_i}{x_i^2+1} \geq \frac{1}{n}\left[n\left(\sum_{i=1}^{n} x_i\right) - n\right]\left(\sum_{i=1}^{n} \frac{x_i}{x_i^2+1}\right) = 0$$
Thus, we get the desired result, which is
$$\sum_{i=1}^{n} \frac{x_i + n}{1 + x_i^2} \leq n^2$$

Problem 54 *'Hoang Quoc Viet'* (Hoang Quoc Viet): Let a, b, c be positive reals satisfying $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^3}{2b^2 + c^2} + \frac{b^3}{2c^2 + a^2} + \frac{c^3}{2a^2 + b^2} \geq 1$$

Solution (Hoang Quoc Viet):
Using Cauchy Schwartz, we get

$$\sum_{cyc} \frac{a^3}{2b^2 + c^2} \geq \frac{(a^2 + b^2 + c^2)^2}{2\sum_{cyc} ab^2 + \sum_{cyc} ac^2}$$

Hence, it suffices to prove that

$$ab^2 + bc^2 + ca^2 \leq 3$$

and

$$a^2b + b^2c + c^2a \leq 3$$

However, applying Cauchy Schwartz again, we obtain

$$a(ab) + b(bc) + c(ca) \leq \sqrt{(a^2 + b^2 + c^2)\left((ab)^2 + (bc)^2 + (ca)^2\right)}$$

In addition to that, we have

$$(ab)^2 + (bc)^2 + (ca)^2 \leq \frac{(a^2 + b^2 + c^2)^2}{3}$$

Hence, we complete our proof.

Problem 55 *'Iran 1998'* (saif): Let $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ prove that:

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{x - 1} + \sqrt{z - 1}$$

First Solution (beautifulliar):
Note that you can substitute $\sqrt{x-1} = a, \sqrt{y-1} = b, \sqrt{z-1} = c$ then you need to prove that $\sqrt{a^2 + b^2 + c^2 + 3} \geq a + b + c$ while you have $\dfrac{1}{a^2+1} + \dfrac{1}{b^2+1} + \dfrac{1}{c^2+1} = 2$ or equivalently $a^2b^2 + b^2c^2 + c^2a^2 + 2a^2b^2c^2 = 1$. next, substitute $ab = \cos x, bc = \cos y, ca = \cos z$ where x, y, z are angles of triangle. since you need to prove that $\sqrt{a^2 + b^2 + c^2 + 3} \geq a + b + c$ then you only need to prove that $3 \geq 2(ab + bc + ca)$ or $\cos x + \cos B + \cos C \leq \dfrac{3}{2}$ which is trivial.

Second Solution (Sayan Mukherjee):
$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 2 \implies \sum \dfrac{x-1}{x} = 1$
Hence $(x + y + z) \sum \dfrac{x-1}{x} \geq \left(\sum \sqrt{x-1}\right)^2$ (From CS)
$\implies \sqrt{x + y + z} \geq \sum \sqrt{x - 1}$

Problem 56 *'IMO 1998'* (saif): Let $a_1, a_2, ..., a_n > 0$ such that $a_1 + a_2 + ... + a_n < 1$. prove that

$$\frac{a_1.a_2...a_n(1 - a_1 - a_2 - ... - a_n)}{(a_1 + a_2 + ... + a_n)(1 - a_1)(1 - a_2)...(1 - a_n)} \le \frac{1}{n^{n-1}}$$

Solution (beautifulliar):
Let $a_{n+1} = 1 - a_1 - a_2 - \cdots - a_n$ the we arrive at $\frac{a_1 a_2 \dots a_n a_{n+1}}{(1 - a_1)(1 - a_2) \dots (1 - a_n)} \le \frac{1}{n^{n+1}}$(it should be n^{n+1} right?) which follows from am-gm $a_1 + a_2 + \cdots + a_n \ge n \sqrt[n]{a_1 a_2 \dots a_n}, a_2 + a_3 + \cdots + a_{n+1} \ge n \sqrt[n]{a_2 a_3 \dots a_{n+1}}, a_1 + a_3 + \cdots + a_{n+1} \ge n \sqrt[n]{a_1 a_3 \dots a_{n+1}}$, and so on... you will find it easy.

Problem 57 *'—'* (beautifulliar): Let n be a positive integer. If $x_1, x_2, \dots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = 0$ and also $y = \max\{x_1, x_2, \dots, x_n\}$ and also $z = \min\{x_1, x_2, \dots, x_n\}$, prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 + nyz \le 0$$

Solution (socrates):
$(x_i - y)(x_i - z) \le 0 \ \forall i = 1, 2, ..., n$ so $\sum_{i=1}^{n}(x_i^2 + yz) \le \sum_{i=1}^{n}(y + z)x_i = 0$ and the conclusion follows.

Problem 58 *'Pham Kim Hung'* (Endrit Fejzullahu): Suppose that x, y, z are positive real numbers and $x^5 + y^5 + z^5 = 3$.Prove that

$$\frac{x^4}{y^3} + \frac{y^4}{z^3} + \frac{z^4}{x^3} \ge 3$$

Solution (Popa Alexandru):
By a nice use of AM-GM we have :

$$10\left(\frac{x^4}{y^3} + \frac{y^4}{z^3} + \frac{z^4}{x^3} + 3(x^5 + y^5 + z^5)^2 \ge 19(\sqrt[19]{x^{100}} + \sqrt[19]{y^{100}} + \sqrt[19]{z^{100}})$$

So it remains to prove :

$$3 + 19(\sqrt[19]{x^{100}} + \sqrt[19]{y^{100}} + \sqrt[19]{z^{100}}) \ge 20(x^5 + y^5 + z^5)$$

which is true by AM-GM .

Problem 59 *'China 2003'* (bokagadha): x, y, and z are positive real numbers such that $x + y + z = xyz$. Find the minimum value of:

$$x^7(yz - 1) + y^7(xz - 1) + z^7(xy - 1)$$

Solution (Popa Alexandru):
By AM-GM and the condition you get

$$xyz \geq 3\sqrt{3}$$

Also observe that the condition is equivalent with

$$yz - 1 = \frac{y+z}{x}$$

So the

$$LHS = x^6(y+z) + y^6(z+x) + z^6(x+y) \geq 6\sqrt[6]{x^{14}y^{14}z^{14}} \geq 216\sqrt{3}$$

Problem 60 *'Austria 1990'* (Rofler):

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{...\sqrt{N}}}}} < 3 - - - \forall N \in \mathbb{N}^{\geq 2}$$

Solution (Brut3Forc3): We prove the generalization $\sqrt{m\sqrt{(m+1)\sqrt{\dots\sqrt{N}}}} < m+1$, for $m+2$. For $m = N$, this is equivalent to $\sqrt{N} < N+1$, which is clearly true. We now induct from $m = N$ down. Assume that $\sqrt{(k+1)\sqrt{(k+2)\dots\sqrt{N}}} < k+2$. Multiplying by k and taking the square root gives $\sqrt{k\sqrt{(k+1)\sqrt{\dots\sqrt{N}}}} < \sqrt{k(k+2)} < k+1$, completing the induction.

Problem 61 *'Tran Quoc Anh'* (Sayan Mukherjee): Given $a, b, c \geq 0$ Prove that:

$$\sum_{cyc} \sqrt{\frac{a+b}{b^2+4bc+c^2}} \geq \frac{3}{\sqrt{a+b+c}}$$

Solution (Hoang Quoc Viet): Using Cauchy inequality, we get

$$\sum_{cyc} \sqrt{\frac{a+b}{b^2+4bc+c^2}} \geq \sum_{cyc} \sqrt{\frac{2(a+b)}{3(b+c)^2}} \geq \sum_{cyc} 3\sqrt[6]{\frac{8}{27(a+b)(b+c)(c+a)}}$$

However, we have

$$(a+b)(b+c)(c+a) \leq \frac{8(a+b+c)^3}{27}$$

Hence, we complete our proof here.

Problem 62 *'Popa Alexandru'* (Popa Alexandru): Let $a, b, c > 0$ such that $a + b + c = 1$. Show that :

$$\frac{a^2 + ab}{1 - a^2} + \frac{b^2 + bc}{1 - b^2} + \frac{c^2 + ca}{1 - c^2} \geq \frac{3}{4}$$

First Solution (Endrit Fejzullahu): Let $a + b = x$, $b + c = y$, $c + a = z$,given inequality becomes

$$\sum \frac{x}{y} + \sum \frac{x}{x + y} \geq \frac{9}{2}$$

Then By Cauchy Schwartz

$$LHS = \sum_{cyc} \frac{x}{y} + \sum_{cyc} \frac{x}{x + y} \geq \frac{(\sum x)^4}{(\sum xy)(\sum xy + \sum x^2)} \geq \frac{8(\sum x)^4}{(\sum xy + (\sum x)^2)^2} \geq \frac{9}{2}$$

Second Solution (Hoang Quoc Viet): Without too many technical terms, we have

$$\left(\sum_{cyc} \frac{x + y}{4y} + \sum_{cyc} \frac{x}{x + y}\right) \geq \sum_{cyc} \sqrt{\frac{x}{y}} \geq 3$$

Therefore, it is sufficient to check that

$$\sum_{cyc} \frac{x}{y} \geq 3$$

which is Cauchy inequality for 3 positive reals.

Problem 63 *'India 2007'* (Sayan Mukherjee): For positive reals a, b, c. Prove that:

$$(a + b + c)^2(ab + bc + ca)^2 \leq 3(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ac + a^2)$$

First Solution (Endrit Fejzullahu): I've got an SOS representation of :

$$RHS - LHS = \frac{1}{2}((x+y+z)^2(x^2y^2+y^2z^2+z^2x^2-xyz(x+y+z))+(xy+yz+zx)(x^2+y^2+z^2-xy-yz-zx))$$

So I may assume that the inequality is true for reals

Second Solution (Popa Alexandru): Since

$$a^2 + ab + b^2 \geq \frac{3}{4}(a - b)^2 \Leftrightarrow (a - b)^2 \geq 0$$

It'll be enough to prove that :

$$\frac{81}{64}(a + b)^2(b + c)^2(c + a)^2 \geq (a + b + c)^2(ab + bc + ca)^2$$

The last one is famous and very easy by AM-GM.

Problem 64 *'Popa Alexandru'* (Endrit Fejzullahu): Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$. Show that :

$$\frac{3}{16abc} \geq a+b+c \geq \frac{3}{2} \geq 12abc$$

First Solution (Apartim De): By AM-GM, $\dfrac{(a+b)(b+c)(c+a)}{8} \geq abc \Leftrightarrow \dfrac{1}{8} \geq abc \Leftrightarrow \dfrac{2}{3} \geq \dfrac{16}{3}abc$ By AM-GM,
$(a+b)+(b+c)+(c+a) \geq 3$
$\Leftrightarrow (a+b+c) \geq \dfrac{3}{2} > \dfrac{2}{3}$
Lemma:We have for any positive reals x, y, z and vectors $\overrightarrow{MA}, \overrightarrow{MB}, \overrightarrow{MC}$
$\left(x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}\right)^2 \geq 0$
$\Leftrightarrow (x+y+z)(xMA^2 + yMB^2 + zMC^2) \geq \left(xyAB^2 + yzBC^2 + zxCA^2\right)$
Now taking $x = y = z = 1$ and M to be the circumcenter of the triangle with sides p, q, r such that $pqr = 1$, and the area of the triangle=Δ, we have by the above lemma,
$9R^2 \geq p^2 + q^2 + r^2 \geq 3\,(pqr)^{\frac{2}{3}} = 3 \Leftrightarrow 3R^2 \geq 1 \Leftrightarrow 16\boldsymbol{\Delta^2} \leq 3$
$\Leftrightarrow (p+q+r)(p+q-r)(q+r-p)(r+p-q) \leq 3$
Now plugging in the famous Ravi substitution i.e,
$p = (a+b); q = (b+c); r = (c+a)$
$\Leftrightarrow (a+b+c) \leq \dfrac{3}{16abc}$

Second Solution (Popa Alexandru): Remember that $8(a+b+c)(ab+bc+ca) \leq 9(a+b)(b+c)(c+a)$ and $a+b+c \geq \sqrt{3(ab+bc+ca)}$ So

$$\frac{9}{8} \geq (a+b+c)(ab+bc+ca) \geq (ab+bc+ca)\sqrt{3(ab+bc+ca)}$$

$$\Leftrightarrow \frac{81}{64} \geq 3(ab+bc+ca)^3 \Leftrightarrow ab+bc+ca \leq \frac{3}{4}$$

Now we use :

$$3abc(a+b+c) \leq (ab+bc+ca)^2 \leq \frac{9}{16} \Rightarrow \frac{3}{16abc} \geq a+b+c$$

For the second recall for :

$$(a+b+c)^3 \geq a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

So

$$(a+b+c)^3 \geq a^3+b^3+c^3+3$$

Now Muirhead shows that :

$$8(a^3+b^3+c^3) \geq 3(a+b)(b+c)(c+a) \Rightarrow a^3+b^3+c^3 \geq \frac{3}{8}$$

There we have :

$$(a+b+c)^3 \geq \frac{3}{8} + 3 \Leftrightarrow a+b+c \geq \frac{3}{2}$$

For the last one by AM-GM we have :

$$(a+b)(b+c)(c+a) \geq 8abc \Rightarrow 8abc \leq 1$$

and the conclusion follows , done.

Problem 65 *'IMO 1988 shortlist'* (Apartim De): In the plane of the acute angled triangle ΔABC, L is a line such that u, v, w are the lengths of the perpendiculars from A, B, C respectively to L. Prove that

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\Delta$$

where Δ is the area of the triangle.

Solution (Hassan Al-Sibyani): Consider a Cartesian system with the x-axis on the line BC and origin at the foot of the perpendicular from A to BC, so that A lies on the y-axis. Let A be $(0, \alpha), B(-\beta, 0), C(\gamma, 0)$, where $\alpha, \beta, \gamma > 0$ (because ABC is acute-angled). Then
$\tan B = \dfrac{\alpha}{\beta}$
$\tan C = \dfrac{\alpha}{\gamma}$
$\tan A = -\tan(B + C) = \dfrac{\alpha(\beta + \gamma)}{\alpha^2 - \beta\gamma}$
here $\tan A > 0$,so $\alpha^2 > \beta\gamma$. Let L have equation $x \cos\theta + y \sin\theta + p = 0$
Then
$u^2 \tan A + v^2 \tan B + w^2 \tan C$
$= \dfrac{\alpha(\beta + \gamma)}{\alpha^2 - \beta\gamma}(\alpha \sin\theta + p)^2 + \dfrac{\alpha}{\beta}(-\beta \cos\theta + p)^2 + \dfrac{\alpha}{\gamma}(\gamma \cos\theta + p)^2$
$= \alpha^2 \sin^2\theta + 2\alpha p \sin\theta + p^2)\dfrac{\alpha(\beta + \gamma)}{\alpha^2 - \beta\gamma} + \alpha(\beta + \gamma)\cos^2\theta + \dfrac{\alpha(\beta + \gamma)}{\beta\gamma}p^2$
$= \dfrac{\alpha(\beta + \gamma)}{\beta\gamma(\alpha^2 - \beta\gamma)}(\alpha^2 p^2 + 2\alpha p\beta\gamma \sin\theta + \alpha^2\beta\gamma \sin^2\theta + \beta\gamma(\alpha^2 - \beta\gamma)\cos^2\theta)$
$= \dfrac{\alpha(\beta + \gamma)}{\beta\gamma(\alpha^2 - \beta\gamma)}[(\alpha p + \beta\gamma \sin\theta)^2 + \beta\gamma(\alpha^2 - \beta\gamma)] \geq \alpha(\beta + \gamma) = 2\Delta$
with equality when $\alpha p + \beta\gamma \sin\theta = 0$, i.e., if and only if L passes through $(0, \beta\gamma/\alpha)$, which is the orthocenter of the triangle.

Problem 66 *'—'* (Hassan Al-Sibyani): For positive real number a, b, c such that $abc \leq 1$, Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

Solution (Endrit Fejzullahu): It is easy to prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a + b + c}{\sqrt[3]{abc}}$$

and since $abc \leq 1$ then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$,as desired .

Problem 67 *'Endrit Fejzullahu'* (Endrit Fejzullahu): Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$.Find the minimal value of :

$$\sum_{cyc} \frac{a^4}{(b + 1)(c + 1)(d + 1)}$$

Solution (Sayan Mukherjee): From AM-GM,

$$P=\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}+\frac{b+1}{16}+\frac{c+1}{16}+\frac{d+1}{16}\right]\geq\sum_{cyc}\left[\frac{4a}{8}\right]$$

$$\Longrightarrow\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right]\geq\frac{a+b+c+d}{2}-\frac{3}{16}(a+b+c+d)-4\cdot\frac{3}{16}$$

$$\therefore\sum_{cyc}\left[\frac{a^4}{(b+1)(c+1)(d+1)}\right]\geq\frac{4}{2}-\frac{6}{4}=2-\frac{3}{2}=\frac{1}{2}$$

So $P_{min}=\dfrac{1}{2}$

Problem 68 *'Sayan Mukherjee'* (Sayan Mukherjee): Prove that

$$\sum_{cyc}\frac{x}{7+z^3+y^3}\leq\frac{1}{3}\forall 1\geq x,y,z\geq 0$$

Solution (Toang Huc Khein):
$\sum\dfrac{x}{7+y^3+z^3}\leq\sum\dfrac{x}{6+x^3+y^3+z^3}\leq\sum\dfrac{x}{3(x+y+z)}=\dfrac{1}{3}$ because $x^3+y^3+z^3+6\geq 3(x+y+z)\Leftrightarrow$ $\sum(x-1)^2(x+2)\geq 0$

Problem 69 *'Marius Mainea'* (Toang Huc Khein): Let $x,y,z>0$ with $x+y+z=1$. Then :

$$\frac{x^2-yz}{x^2+x}+\frac{y^2-zx}{y^2+y}+\frac{z^2-xy}{z^2+z}\leq 0$$

First Solution (Endrit Fejzullahu):

$$\frac{x^2-yz}{x^2+x}=1-\frac{x+yz}{x^2+x}$$

Inequality is equivalent with :

$$\sum_{cyc}\frac{x+yz}{x^2+x}\geq 3$$

$$\sum_{cyc}\frac{x+yz}{x^2+x}=\sum_{cyc}\frac{1}{x+1}+\sum_{cyc}\frac{yz}{x^2+x}$$

By Cauchy-Schwarz inequality

$$\sum_{cyc}\frac{1}{x+1}\geq\frac{9}{4}$$

And

$$\sum_{cyc} \frac{yz}{x^2+x} \geq \frac{(xy+yz+zx)^2}{x^2yz+y^2xz+z^2xy+3xyz} = \frac{(xy+yz+zx)^2}{4xyz} \geq \frac{3}{4} \iff (xy+yz+zx)^2 \geq 3xyz$$

This is true since $x+y+z=1$ and $(xy+yz+zx)^2 \geq 3xyz(x+y+z)$

Second Solution (Endrit Fejzullahu): Let us observe that by Cauchy-Schwartz we have :

$$\sum_{cyc} \frac{x+yz}{x^2+x} = \sum_{cyc} \frac{x(x+y+z)+yz}{x(x+1)} = \sum_{cyc} \frac{(x+y)(x+z)}{x(x+y+x+z)} = \sum_{cyc} \frac{1}{\frac{x}{x+y}+\frac{x}{x+z}} \geq \frac{(1+1+1)^2}{3} = \frac{9}{3} = 3$$

Then we can conclude that :

$$LHS = \sum_{cyc} \frac{x^2+x}{x^2+x} - \sum_{cyc} \frac{x+yz}{x^2+x} \leq 3-3=0$$

Problem 70 *'Claudiu Mandrila'* (Endrit Fejzullahu): Let $a,b,c>0$ such that $abc=1$. Prove that :

$$\frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b} \geq \frac{a^7}{b^7+c^7} + \frac{b^7}{c^7+a^7} + \frac{c^7}{a^7+b^7}$$

Solution (Sayan Mukherjee): Since $abc=1$ so, we have:

$$\sum \frac{a^{10}}{b+c} = \sum \frac{a^7}{b^3c^3(b+c)} \geq \sum \frac{a^7}{\frac{1}{64}(b+c)^6 \cdot (b+c)}$$

So we are only required to prove that: $(b+c)^7 \leq 2^6b^7 + 2^6c^7$
But, from Holder;
$(b^7+c^7)^{\frac{1}{7}}(1+1)^{\frac{6}{7}} \geq b+c$ Hence we are done

Problem 71 *'Hojoo Lee, Crux Mathematicorum'* (Sayan Mukherjee): Let $a,b,c \in \mathbb{R}^+$;Prove that:

$$\frac{2}{abc}(a^3+b^3+c^3) + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \geq 33$$

First Solution (Endrit Fejzullahu): Inequality is equivalent with :

$$\Big((a+b+c)(a^2+b^2+c^2) - 9abc\Big)\Big((a-b)^2+(b-c)^2+(c-a)^2\Big) \geq 0$$

Second Solution (Popa Alexandru):

$$LHS - RHS = \left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} - 3\right)\left(2 - \frac{18abc}{(a+b+c)(a^2+b^2+c^2)}\right) \geq 0$$

Problem 72 *'Cezar Lupu'* (Endrit Fejzullahu): Let a, b, c be positive real numbers such that $a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. Show that :

$$\frac{a^3c}{b(c+a)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

First Solution (Sayan Mukherjee): Observe that $a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$ (1) from AM-GM. Also from rearrangement inequality:

$$\begin{bmatrix} a; b; c \\ \frac{1}{b}; \frac{1}{c}; \frac{1}{a} \end{bmatrix} \geq \begin{bmatrix} a; b; c \\ \frac{1}{c}; \frac{1}{a}; \frac{1}{b} \end{bmatrix} \implies \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

... (2)
Hence we rewrite LHS of the inequality in the form:

$$\sum_{cyc} \frac{a^2}{b(a+c)\cdot\frac{1}{ac}} = \sum_{cyc} \frac{a^2}{\frac{b}{a}+\frac{b}{c}} \geq \frac{(a+b+c)^2}{\sum_{cyc}\frac{b}{c}+\sum_{cyc}\frac{c}{b}} \geq \frac{(a+b+c)^2}{2\sum_{cyc}\frac{a}{b}} \geq \frac{(a+b+c)}{2} \geq \frac{3}{2}$$

(From (1) (2))
QED.

Second Solution (Popa Alexandru): Holder gives :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} 2(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)^3.$$

Therefore we have :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} \geq \frac{(a+b+c)^3}{2(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} \geq \frac{3}{2}$$

Third Solution (Popa Alexandru): We have that :

$$\sum_{cyc} \frac{a^3c}{b(c+a)} = \sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)}$$

By Cauchy-Schwartz we get :

$$\left(\sum_{cyc} \frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)\right)\left(\sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)}\right) \geq \left(\sum_{cyc}\frac{a}{b}\right)^2$$

So:

$$\sum_{cyc} \frac{\left(\frac{a}{b}\right)^2}{\frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right)} \geq \frac{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{a}{b}\right)}{2\sum_{cyc}\frac{1}{ab}}$$

It'll be enough to prove that :

$$\sum_{cyc}\frac{a}{b} \geq \sum_{cyc}\frac{1}{ab}$$

By contradiction method we assume that $\sum_{cyc} \frac{a}{b} < \sum_{cyc} \frac{1}{ab}$ which is equivalent with $\sum_{cyc} ab^2 < \sum_{cyc} a$. Suming this with the condition $2\sum_{cyc} a \geq 2\sum_{cyc} \frac{a}{b}$, we will have that

$$3\sum_{cyc} a > a\left(b^2+\frac{2}{b}\right)+b\left(c^2+\frac{2}{c}\right)+c\left(a^2+\frac{2}{a}\right)$$

But using AM-GM as $x^2+\frac{2}{x} = x^2+\frac{1}{x}+\frac{1}{x} \geq 3\sqrt[3]{x^2\cdot\frac{1}{x}\cdot\frac{1}{x}} = 3$, we get $3(a+b+c) > 3(a+b+c)$, contradiction .

Problem 73 *'Popa Alexandru'* (Sayan Mukherjee): For $x, y, z > 0$ and $x+y+z = 1$ Find P_{min} if :

$$P = \frac{x^3}{(1-x)^2}+\frac{y^3}{(1-y)^2}+\frac{z^3}{(1-z)^2}$$

Solution (Zhero):
$(3x-1)^2 \geq 0 \iff \frac{x^3}{(1-x)^2} \geq x-\frac{1}{4}$. Adding these up yields that the RHS is $\geq 1-\frac{3}{4} = \boxed{\frac{1}{4}}$

Problem 74 *'Hlawka'* (Zhero): Let $x, y,$ and z be vectors in $\mathbb{R}^n$. Show that

$$|x+y+z|+|x|+|y|+|z| \geq |x+y|+|y+z|+|z+x|$$

First Solution (Dimitris Charisis): I will solve this inequality for $x, y, z \in \mathbb{C}$.
First observe that $|x+y+z|^2+|x|^2+|y|^2+|z|^2 = |x+y|^2+|y+z|^2+|z+x|^2 \boxed{1}$
To prove this just use the well-known identity $|z|^2 = z\cdot\bar{z}$.
Then our inequality is:
$|x+y+z|^2+|x|^2+|y|^2+|z|^2+2(|x+y+z|)(|x|+|y|+|z|)+|xy|+|yz|+|xz| \geq \sum|x+y|^2+2\sum|x+y \parallel y+z|$
From $\boxed{1}$ we only need to prove that:
$(|x+y+z|)(|x|+|y|+|z|)+|xy|+|yz|+|xz| \geq \sum|x+y||y+z|$
But $|(x+y)(y+z)| = |y(x+y+z)+xz| \leq |y(x+y+z)|+|yz|$.
Cyclic we have the result

Second Solution (Popa Alexandru): Use Hlawka's identity :

$$(|a|+|b|+|c|-|b+c|-|c+a|-|a+b|+|a+b+c|)\cdot(|a|+|b|+|c|+|a+b+c|) =$$

$$= (|b|+|c|-|b+c|)\cdot(|a|-|b+c|+|a+b+c|)+(|c|+|a|-|c+a|)\cdot(|b|-|c+a|+|a+b+c|)+(|a|+|b|-|a+b|)\cdot(|c|-|a+b|+|a+b+c|) \geq 0$$

and the conclusion follows .

Problem 75 *'Greek Mathematical Olympiad'* (Dimitris Charisis): If $x, y, z \in \mathbb{R}$ so that $x^2 + 2y^2 + z^2 = \frac{2}{5}a^2, a > 0$,prove that

$$|x - y + z| \leq a$$

Solution (Popa Alexandru): The inequality is equivalent with :

$$a^2 \geq x^2 + y^2 + z^2 - 2yz - 2xy + 2zx$$

Since the condition this rewrites to :

$$\frac{5}{2}x^2 + 5y^2 + \frac{5}{2}z^2 \geq x^2 + y^2 + z^2 - 2yz - 2xy + 2zx$$

which is equivalent with :

$$(x - y)^2 + \left(\frac{x}{\sqrt{2}} + y\sqrt{2}\right)^2 + \left(\frac{z}{\sqrt{2}} + y\sqrt{2}\right)^2 \geq 0$$

Problem 76 *'Vasile Cirtoaje and Mircea Lascu, Junior TST 2003, Romania'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers so that $abc = 1$. Prove that:

$$1 + \frac{3}{a + b + c} \geq \frac{6}{ab + ac + bc}$$

First Solution (Endrit Fejzullahu): Given inequality is equivalent with :

$$ab + bc + ca \geq \frac{6(a + b + c)}{a + b + c + 3} \implies (ab + bc + ca)^2 \geq \frac{36(a + b + c)^2}{(a + b + c + 3)^2}$$

By the AM-GM inequality we know that :

$$(ab + bc + ca)^2 \geq 3abc(a + b + c) = 3(a + b + c)$$

It is enough to prove that

$$3(a + b + c) \geq \frac{36(a + b + c)^2}{(a + b + c + 3)^2} \iff (a + b + c - 3)^2 \geq 0$$

Second Solution (Dimitris Charisis): Setting $a = \dfrac{1}{x}, b = \dfrac{1}{y}, c = \dfrac{1}{z}$ we have to prove that:
$1 + \dfrac{3}{xy + yz + zx} \geq \dfrac{6}{x + y + z}$
But $\dfrac{1}{xy + yz + zx} \geq \dfrac{3}{(x + y + z)^2}$
So we have to prove that:
$1 + \dfrac{9}{(x + y + z)^2} \geq \dfrac{6}{x + y + z}$.
Now setting $x + y + z = k$, the inequality becomes:

$(k-3)^2 \geq 0$ which is true.

Third Solution (Popa Alexandru): We have

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$$

Then

$$1 + \frac{3}{a+b+c} \geq 1 + \frac{9}{(ab+ac+bc)^2} \geq \frac{6}{ab+ac+bc}$$

Problem 77 *'Vasile Pop, Romania 2007 Shortlist'* (Endrit Fejzullahu): Let $a, b, c \in [0,1]$. Show that :

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + abc \leq \frac{5}{2}$$

Solution (Dimitris Charisis):
$bc \leq abc$
So $\dfrac{a}{1+bc} \leq \dfrac{a}{1+abc}$
cyclic we have that $\sum \dfrac{a}{1+bc} \leq \dfrac{a+b+c}{1+abc}$
So our problem reduces to:
$\dfrac{a+b+c}{1+abc} + abc \leq \dfrac{5}{2}.\boxed{1}$. But $a+b+c \leq abc+2$
Because:
$(1-ab)(1-c) \geq 0 \Longleftrightarrow 1-c-ab+abc \geq 0 \Longleftrightarrow 2+abc \geq ab+c+1$.
So we have to prove that $ab+c+1 \geq a+b+c$
But $(a-1)(b-1) \geq 0 \Longleftrightarrow ab-a-b+1 \geq 0 \Longleftrightarrow ab+1 \geq a+b \Longleftrightarrow ab+c+1 \geq a+b+c$
So $\dfrac{a+b+c}{1+abc} \leq \dfrac{abc+2}{abc+1} = 1 + \dfrac{1}{abc+1}$
Finally our inequality reduces to:
$\dfrac{1}{abc+1} + abc \leq \dfrac{3}{2}$.
Setting $abc = x \leq 1$ we have:
$\dfrac{1}{x+1} + x \leq \dfrac{3}{2} \Longleftrightarrow 2x^2 - x - 1 \leq 0 \Longleftrightarrow (x-1)(2x+1) \leq 0$ which is true.

Problem 78 *'Greek 2004'* (Dimitris Charisis): Find the best constant M so that the inequality:

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq M(xy+yz+zx)^2$$

for all $x, y, z \in \mathbb{R}$.

Solution (Endrit Fejzullahu):
Setting $x = y = z = 1$,we see that $M \leq \dfrac{2}{3}$ .We prove that the inequality is true for $M = \dfrac{2}{3}$

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq \frac{2}{3}(xy+yz+zx)^2$$

After squaring $(xy+yz+zx)^2$ on the RHS ,Inequality is equivalent with :

$$x^4+y^4+z^4+2\left(x^4+y^4+z^4-(xy)^2-(yz)^2-(zx)^2\right)\geq xyz(x+y+z)$$

It is obvious that $x^4+y^4+z^4\geq(xy)^2+(yz)^2+(zx)^2$,so it is enough to prove that :

$$x^4+y^4+z^4\geq xyz(x+y+z)$$

$$x^4+y^4+z^4\geq(xy)^2+(yz)^2+(zx)^2\geq x^2yz+xy^2z+xyz^2=xyz(x+y+z)$$

The last one is true ,setting $xy=a, yz=b, zx=c$,it becomes $a^2+b^2+c^2\geq ab+bc+ca$,which is trivial

Problem 79 *'Dragoi Marius and Bogdan Posa'* (Endrit Fejzullahu): Let $a,b,c>0$ auch that $abc=1$, Prove that :

$$a^2(b^5+c^5)+b^2(a^5+c^5)+c^2(b^5+a^5)\geq 2(a+b+c)$$

Solution (Sayan Mukherjee):
Lemma:
Note that
$a^5+a^5+a^5+b^5+b^5\geq 3a^3b^2$
$a^5+a^5+b^5+b^5+b^5\geq 3a^2b^3$
Summing up;
$a^5+b^5\geq a^2b^2(a+b)$
So $LHS=\sum a^2(b^5+c^5)\geq a^2b^2c^2(c+b)=2(a+b+c)$ from $abc=1$ and our lemma above.

Problem 80 *'Russia 1995'* (Sayan Mukherjee): For positive x,y prove that:

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{x^2+y^4}$$

Solution (FantasyLover):
By AM-GM, we have $x^4+y^2\geq 2x^2y$ and $x^2+y^4\geq 2xy^2$.
Hence, $\dfrac{x}{x^4+y^2}+\dfrac{y}{x^2+y^4}\leq\dfrac{x}{2x^2y}+\dfrac{y}{2xy^2}=\dfrac{1}{xy}$, and we are done. ■

Problem 81 *'Adapted after an IMO problem'* (FantasyLover): Let a,b,c be the side lengths of a triangle with semiperimeter of 1. Prove that

$$1<ab+bc+ca-abc\leq\frac{28}{27}$$

Solution (Sayan Mukherjee): Since $a,b,c\rightarrow$ sides of a triangle; write:
$a=x+y; b=y+z; c=z+x$ so that $x+y+z=1$
The inequality is equivalent to :

$1 < \sum(1-x)(1-y) - \prod(x+y) \le \frac{28}{27} \implies 1 < 3 - 2\sum x + xyz = 1 + xyz \le \frac{28}{27}$ Since $xyz > 0$ So the left side is proved. And, since $x+y+z=1$ and $xyz \le \left(\frac{x+y+z}{3}\right)^3 = \frac{1}{27}$ hence the right side is proved

Problem 82 *'Latvia 2002'* (Sayan Mukherjee): Let $a,b,c,d > 0$; $\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+a^4} = 1$ Prove that:

$$abcd \ge 3$$

First Solution (Zhero):
Let $w = \frac{1}{1+a^4}, x = \frac{1}{1+b^4}, y = \frac{1}{1+c^4}$, and $z = \frac{1}{1+d^4}$.
Then $a = \sqrt[4]{\frac{1}{w} - 1}, b = \sqrt[4]{\frac{1}{x} - 1}, c = \sqrt[4]{\frac{1}{y} - 1}$, and $d = \sqrt[4]{\frac{1}{z} - 1}$.
Let $f(n) = -\ln(\sqrt[4]{\frac{1}{n} - 1})$. It is easy to verify that f has exactly one inflection point. Since we see seek to minimize $f(w) + f(x) + f(y) + f(z)$, by the inflection point theorem, it suffices to minimize it in the case in which $w = x = y$. In other words, in our original inequality, it suffices to minimize this in the case that $a = b = c$.
Let $p = a^4, q = b^4, r = c^4$, and $s = d^4$. Then we want to show that $pqrs \ge 81$ when $\sum_{cyc} \frac{1}{1+p} = 1$. But we only need to check this in the case in which $p = q = r$, that is, when $s = \frac{3}{p-2}$. In other words, we want to show that $\frac{3p^3}{p-2} \ge 81 \iff 3p^3 \ge 81p - 162 \iff 3(x-3)^2(x+6) \ge 0$, as desired.
Second Solution (Sayan Mukherjee):
$\frac{d^4}{1+d^4} = 1 - \frac{1}{1+d^4} = \sum_{a,b,c} \frac{1}{1+a^4} \ge \frac{3}{\sqrt[3]{\prod_{a,b,c}(1+a^4)}}$(AM-GM)
Similarly we get three ther relations and multiplying we get:

$$\frac{a^4b^4c^4d^4}{\prod_{a,b,c,d}(1+a^4)} \ge \frac{3^4}{\prod_{a,b,c,d}(1+a^4)} \implies abcd \ge 3$$

Problem 83 *'Vasile Cartoaje and Mircea Lascu'* (Zhero): Let a,b,c,x,y, and z be positive real numbers such that $a+x \ge b+y \ge c+zanda+b+c = x+y+z$. Show that

$$ay + bx \ge ac + xz.$$

Solution (mathinequs):
$a(y-c) + x(b-z) = a(a-x) + (a+x)(b-z)$
Now i used :
$a(a-x) = \frac{(a-x)^2 + a^2 - x^2}{2}$

So i got that :
$a(a-x)+(a+x)(b-z)=\dfrac{(a-x)^2+(a+x)(a-x+2(b-z))}{2}$
So i have to prove that : $(a+x)(a-x+2(b-z))\geq 0$
but $a+x$ is greater or equal than 0 and $a-x+2(b-z)=b+y-c-z$ which is greater or equal than 0.

Problem 84 *'Nicolae Paun'* (mathinequs): Let a,b,c,x,y,z reals with $a+b+c=x^2+y^2+z^2=1$. To prove:

$$a(x+b)+b(y+c)+c(z+a)<1$$

Solution (Mharchi Abdelmalek): Using AM-GM inequality we have: $a^2+x^2+b^2+y^2+c^2+z^2\geq 2ax+2by+2cz$
Hence we have :
$1=\dfrac{(a+b+c)^2+x^2+y^2+z^2}{2}=\dfrac{1}{2}\left(\sum_{cyc}(a^2+x^2)\right)+ab+bc+ca\geq ax+by+cz+ab+bc+ca=$
$=a(x+b)+b(y+c)+c(z+a)$

Problem 85 *'Asian Pacific Mathematics Olympiad'* (Mharchi Abdelmalek): Let x,y,z, be a positive real numbers . Prove that:

$$\left(\frac{x}{y}+1\right)\left(\frac{y}{z}+1\right)\left(\frac{z}{x}+1\right)\geq 2+\frac{2(x+y+z)}{\sqrt[3]{xyz}}$$

Solution (mathinequs):
Bashing the left hand side we need to prove :
$\sum\dfrac{x}{y}+\sum\dfrac{y}{x}\geq\dfrac{2(x+y+z)}{\sqrt[3]{xyz}}$
But this is AM-GM as :
$2\dfrac{x}{y}+\dfrac{y}{z}\geq\dfrac{3x}{\sqrt[3]{xyz}}$
Done.

Problem 86 *'mateforum.ro'* (mathinequs): Let $x,y,z>0$ such that $xy+yz+zx=1$. Show:

$$\frac{1}{x+x^3}+\frac{1}{y+y^3}+\frac{1}{z+z^3}\geq\frac{9\sqrt{3}}{4}$$

Solution (Apartim De):
The condition $xy+yz+zx=1$ accomodates the substitution
$x=\cot A, y=\cot B,\ z=\cot C,$ where A,B,C are the angles of ΔABC.
We obtain the equivalent inequality
$\otimes\sum_{cyc}\dfrac{1}{\cot A(1+\cot^2 A)}\geqslant\dfrac{9\sqrt{3}}{4}$

$\Leftrightarrow \sum_{cyc} \tan A \sin^2 A \geqslant \frac{9\sqrt{3}}{4}$
Let $f(x) = \tan x \sin^2 x$
• $f'(x) = 2\sin^2 x + \tan^2 x > 0, \forall x \Leftrightarrow f(x)$ is increasing
$f''(x) = 2\sin x \left[2\cos x + \frac{1}{\cos^3 x}\right] \begin{cases} > 0, \forall x \in (0, \pi/2) \\ < 0, \forall x \in (\pi/2, \pi) \\ > 0, \forall x \in (\pi, 3\pi/2) \end{cases}$
If the ΔABC be acute , by Jensen,
$\sum_{cyc} f(A) \geqslant 3f\left(\frac{\pi}{3}\right) = \frac{9\sqrt{3}}{4}$
If the ΔABC be obtuse-angled(at A) ,
$f(A) + f(B) + f(C) = f\left(A + \frac{\pi}{2}\right) + f(B - k) + f\left(C - \left(\frac{\pi}{2} - k\right)\right)$
$\geqslant 3f\left(\frac{\left(A + \frac{\pi}{2}\right) + (B - k) + \left(C - \left(\frac{\pi}{2} - k\right)\right)}{3}\right) = 3f\left(\frac{\pi}{3}\right)$
for some suitable$k \leqslant \min\{B, C\}$

Problem 87 '—' (Apartim De): a, b, c are sides of a triangle.Prove that

$$\frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \frac{b}{\sqrt{2c^2 + 2a^2 - b^2}} + \frac{c}{\sqrt{2a^2 + 2b^2 - c^2}} \geqslant \sqrt{3}$$

First Solution (Endrit Fejzullahu): According to the Holder's inequality we have :

$$\left(\sum_{cyc} \frac{a}{\sqrt{2b^2 + 2c^2 - a^2}}\right)^2 \left(\sum_{cyc} a(2b^2 + 2c^2 - a^2)\right) \geq (a + b + c)^3$$

We only need to prove that

$$(a + b + c)^3 \geq 3\sum_{cyc} a(2b^2 + 2c^2 - a^2)$$

It can be rewritten :

$$3\left(abc - (a + c - b)(b + c - a)(a + b - c)\right) + 2\left(a^3 + b^3 + c^3 - 3abc\right) \geq 0$$

By Schur $abc \geq (a + c - b)(b + c - a)(a + b - c)$ and by AM-GM $a^3 + b^3 + c^3 \geq 3abc$,so we're done !

Second Solution (geniusbliss):

$$\sum_{cyclic} \frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \sum_{cyclic} \frac{a}{\sqrt{2b^2 + 2c^2 - a^2}} + \sum_{cyclic} \frac{3\sqrt{3}a(2b^2 + 2c^2 - a^2)}{(a + b + c)^3} \geq 3\sum_{cyclic} \frac{\sqrt{3}a}{a + b + c} = 3\sqrt{3}$$

and finish off the problem

Problem 88 *'Pavel Novotn, Slovakia'* (Endrit Fejzullahu): Let a,b,c,d be positive real numbers such that $abcd = 1$ and $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$. Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$$

First Solution (mathinequs): It's AM-GM ! $\frac{a}{d}+2\frac{a}{b}+\frac{b}{c} \geq 4a$ Suming the other and with condition : $3(a+b+c+d)+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} > 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} \geq 4(a+b+c+d)$

Second Solution (socrates): The second condition implies that

$$a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq \frac{a^2}{ab}+\frac{b^2}{bc}+\frac{c^2}{cd}+\frac{d^2}{da} \geq \frac{(a+b+c+d)^2}{ab+bc+cd+da}$$

so $ab+bc+cd+da > a+b+c+d$. Now, it is

$$\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d} = \frac{(bc)^2}{abc^2}+\frac{(cd)^2}{bcd^2}+\frac{(da)^2}{cda^2}+\frac{(ab)^2}{dab^2} = \frac{(bc)^2}{\frac{c}{d}}+\frac{(cd)^2}{\frac{d}{a}}+\frac{(da)^2}{\frac{a}{b}}+\frac{(ab)^2}{\frac{b}{c}} >$$

$$\frac{(bc+cd+da+ab)^2}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}} > \frac{(a+b+c+d)^2}{a+b+c+d} = a+b+c+d$$

completing the proof.

Problem 89 '—' (geniusbliss): Let $a_1, a_2, a_3 ..., a_n$ and $b_1, b_2, b_3, ..., b_n$ be real numbers such that - $a_1 \geq \frac{a_1+a_2}{2} \geq \frac{a_1+a_2+a_3}{3} \geq .... \geq \frac{a_1+a_2+a_3+...+a_n}{n}$, $b_1 \geq \frac{b_1+b_2}{2} \geq \frac{b_1+b_2+b_3}{3} \geq .... \geq \frac{b_1+b_2+b_3+...+b_n}{n}$, then prove that -

$$a_1b_1+a_2b_2+a_3b_3+...+a_nb_n \geq \frac{1}{n}\cdot(a_1+a_2+a_3+...+a_n)\cdot(b_1+b_2+b_3+...+b_n)$$

Solution (geniusbliss): For each $k1, 2, ..., n$,to we denote $S_k = a_1+a_2+a_3+...+a_k$ and $b_{n+1} = 0$
Then by Abel's Formulae, we have
$\sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n}(b_i - b_{i+1})S_i = \sum_{i=1}^{n} i(b_i - b_{i+1})\frac{S_i}{i}$
According to Abel's Formulae,
$\sum_{i=1}^{n} a_ib_i = (S_1-\frac{S_2}{2})(b_1-b_2)+(\frac{S_2}{2}-\frac{S_3}{3})(b_1+b_2-2b_3)+........+(\frac{S_{n-1}}{n-1}-\frac{S_n}{n})(\sum_{i=1}^{n-1}-(n-1)b_n)+\frac{1}{n}(\sum_{i=1}^{n} a_1)(\sum_{i=1}^{n} b_1)$
By Hypothesis, we have $\frac{S_1}{1} \geq \frac{S_2}{2} \geq ..... \geq \frac{S_n}{n}$ so its enough to prove that
$\sum_{i=1}^{k} b_i \geq kb_{k+1}$
This one comes directly from the hypothesis
$\frac{1}{k}\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k+1} b_i$

Problem 90 *'mateforum.ro'* (mathinequs): Let a, b, c positives with $\sum \frac{1}{1+a^2} = 2$. Show :

$$abc(a+b+c-abc) \leq \frac{5}{8}$$

Solution (socrates): Multiplying out we get $2a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 = 1$.
Moreover,

$$abc(a+b+c-abc) = \frac{1}{2}\left(2abc(a+b+c) - 2a^2b^2c^2\right) = \frac{1}{2}\left(2abc(a+b+c) + a^2b^2 + b^2c^2 + c^2a^2 - 1\right) = \frac{1}{2}\left((ab+bc+ca)^2 - 1\right)$$

so it is enough to prove that $ab + bc + ca \leq \frac{3}{2}$
Now put $a^2 = \frac{x}{y+z}, b^2 = \frac{y}{x+z}, c^2 = \frac{z}{x+y}$, $x, y, z > 0$. Then
$ab = \sqrt{\frac{xy}{(y+z)(x+z)}} \leq \frac{1}{2}\left(\frac{x}{x+z} + \frac{y}{y+z}\right)$ and similarly
$bc = \sqrt{\frac{yz}{(x+z)(x+y)}} \leq \frac{1}{2}\left(\frac{z}{x+z} + \frac{y}{x+y}\right)$
$ac = \sqrt{\frac{xz}{(y+z)(x+y)}} \leq \frac{1}{2}\left(\frac{x}{x+y} + \frac{z}{y+z}\right)$
Adding these, we are done.

Problem 91 *'Komal'* (Hassan Al-Sibyani): For arbitrary real numbers a, b, c. Prove that:

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

First Solution (Dimitris Charisis): By Minkowski's inequality we have:
$LHS \geq \sqrt{(a+b+c)^2 + (a+b+c-3)^2}$.
Setting $a + b + c = x$ our inequality becomes:

$$\sqrt{x^2 + (x-3)^2} \geq \frac{3\sqrt{2}}{2} \Longleftrightarrow 2x^2 - 6x + \frac{9}{2} \geq 0 \Longleftrightarrow x^2 - 3x + \frac{9}{4} \geq 0 \Longleftrightarrow \left(x - \frac{3}{2}\right)^2 \geq 0$$

,which is true....

Second Solution (Mharchi Abdelmalek): we have the flowing well-know inequality for all real numbers a, b :
$\sqrt{a^2 + (1-b)^2} \geq \frac{|a+1-b|}{\sqrt{2}}$
Simarly for the other numbers.Hence we have

$$\text{LHS} \geq \frac{|a-b+1| + |b-c+1| + |c-a+1|}{\sqrt{2}} \geq \frac{|a-b+b-c+c-a+3|}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Problem 92 *'Asian Pacific Mathematics Olympiad 1996'* (Dimitris Charisis): For a, b, c sides of a triangle prove that:

$$\sqrt{a+b-c}+\sqrt{a-b+c}+\sqrt{-a+b+c} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

First Solution (geniusbliss): substitute $a = x + y, b = y + z, c = z + x$ we have to prove then,
$\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} \geq \sqrt{2x}+\sqrt{2y}+\sqrt{2z}$ which is true because squaring both sides,
we have to prove $\sum_{cyc} \sqrt{(x+y)(y+z)} \geq 2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}$ which is true by Cauchy - $(x+y)(y+z) \geq (\sqrt{xy}+\sqrt{yz})^2$ and
similarly for others ,equality holds for $x = y = z$

Second Solution (Redwane Khyaoui): First of all , we set $a = \dfrac{x+y}{2}$ and $b = \frac{y+z}{2}$ and $c = \frac{z+x}{2}$
the inequality become into : $\sqrt{2}(\sqrt{x}+\sqrt{y}+\sqrt{z}) \leq \sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}$
then we use the well-known inequality : $\dfrac{1}{2}(\sqrt{x}+\sqrt{y})^2 \leq x+y$ which means $\sqrt{2(x+y)} \geq \sqrt{x}+\sqrt{y}$
and then summing the three inequalities , gives inequality desired .

Problem 93 *'geniusbliss'* (geniusbliss): Prove that for x, y, z positive reals such as $xz \geq y^2, xy \geq z^2$ the follwing inequality holds-

$$(x^2-yz)^2 \geq \frac{27}{8}(xz-y^2)(xy-z^2)$$

Solution (geniusbliss): this obviously implies the LHS is also positive.
make this substitution - $a = \dfrac{x}{y}, b = \dfrac{y}{z}, c = \dfrac{z}{x}$,
the inequality gets transformed into
$8(a-c)^2 \geq 27(a-b)(b-c)$ or
$2((a-c)^2)\frac{1}{3} \geq 3((a-b)(b-c))^{\frac{1}{3}}$ we have $a \geq b \geq c$,
multiply both sides by $(a-c)^{\frac{1}{3}}$, and we get,
$2(a-c) \geq 3((a-b)(b-c)(a-c))^{\frac{1}{3}}$ but this is just AM-GM if we write the LHS as $2(a-c) = a-b+b-c+a-c$
so we are done and the inequality is true with equality holding when $a = b = c$

Problem 94 *'Kvant 1988'* (Mharchi Abdelmalek): Let $a, b, c \geq 0$ such that $a^4+b^4+c^4 \leq 2(a^2b^2+b^2c^2+c^2a^2)$
Prove that :

$$a^2+b^2+c^2 \leq 2(ab+bc+ca)$$

Solution (Hassan Al-Sibyani): The condition

$$\sum a^4 \leq 2\sum a^2b^2$$

is equivalent to

$$(a+b+c)(a+b-c)(b+c-a) \geq 0$$

In any of the cases $a = b+c, b = c+a, c = a+b$, the inequality

$$\sum a^2 \leq 2\sum ab$$

is clear. So, suppose $a \neq b+c, b \neq c+a, c \neq a+b$. Because at most one of the numbers $b+c-a, c+a-b, a+b-c$ is negative and their products is non-negative, all of them are positive. Thus, we may assume that:

$$a^2 < ab+ac, b^2 < bc+ba, c^2 < ca+cb$$

and the conclusion follows.

Problem 95 *'Junior Balkan Mathematical Olympiad 2002 Shortlist'* (Hassan Al-Sibyani): Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

First Solution (Redwane Khyaoui): If $a \geq b \geq c$ then we put $a = tb$ and $b = t'c$ so $a = tt'c$ with $t, t' \geq 1$
th inequality become into $t^5t'^3 + t'^5.t^2 + 1 \geq t^4t'^3 + t'^4.t^2 + tt'$
and we consider the fonction $f(t) = t^5t'^3 + t'^5.t^2 + 1 - t^4t'^3 + t'^4.t^2 + tt'$
so $f'(t) = 5t'^3t^4 - 4t'^3t^3 + 2t(t'^5 - t'^4) - tt' + 1$
then , since we know that $t, t' \geq 1$ so $f'(t) \geq 0$ which means f is increasing function ,
$t \geq 1 \to f(t) \geq f(1) = t'^5 - t'^4 - t' + 1$
setting again $g(t') = t'^5 - t'^4 - t' + 1$ so $g'(t) = 5t'^4 - 4t'^3 - 1 \geq 0$ because $t' \geq 1$
whic means that g is increasing fonction , so $t' \geq 1 \to g(t) \geq g(1) = 0$
and finally $t \geq 1 \to f(t) \geq f(1) = t'^5 - t'^4 - t' + 1 \geq 0$

Second Solution (Mharchi Abdelmalek):

$$\text{LHS}-\text{RHS} \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{a^3}{b^2}-\frac{2a^2}{b}+a\right)+\sum_{cyc}\left(\frac{a^2}{b}-2a+b\right) \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{\sqrt{a^3}}{b}-\sqrt{a}\right)^2+\sum_{cyc}\left(\frac{(a-b)^2}{b}\right) \geq 0$$

Which is Obviously true!

Problem 96 *'Hojoo Lee'* (Redwane Khyaoui): If $a+b+c = 1$ then prouve that

$$\frac{a}{a+bc}+\frac{b}{b+ac}+\frac{\sqrt{abc}}{c+ab} \leq 1+\frac{3\sqrt{3}}{4}$$

Solution (Endrit Fejzullahu): Let $x = \sqrt{\frac{ab}{c}}, y = \sqrt{\frac{bc}{a}}, z = \sqrt{\frac{ca}{b}}$.We see that $xy+yz+zx = 1$,and $a = zx, b = xy$ and $c = yz$
Then we can write : (A,B,C are angles of an acute angled triangle)

$$x = \tan\frac{A}{2}\tan\frac{B}{2}, y = \tan\frac{B}{2}\tan\frac{C}{2}, z = \tan\frac{C}{2}\tan\frac{A}{2}$$

Then $a = \tan^2 \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$, and so on ...
After making the substitutions ,and using trigonometric transformations formulas ,the given inequality becomes :

$$\frac{1}{1+\tan^2 \frac{C}{2}} + \frac{1}{1+\tan^2 \frac{B}{2}} + \frac{\tan \frac{A}{2}}{1+\tan^2 \frac{A}{2}} \leq + \frac{3\sqrt{3}}{4} \iff \cos C + \cos B + \sin A \leq \frac{3\sqrt{3}}{2}$$

$$\cos C + \cos B + \sin C \leq \sin \frac{C}{2} + \sin C < \frac{3\sqrt{3}}{2} \iff 12x^2 < 27 + 16x^4$$

where $x = \sin \dfrac{C}{2}$

Problem 97 *'—'* (Endrit Fejzullahu): If $a_1, a_2, ..., a_n$ are positive real numbers such that $a_1 + a_2 + ... + a_n = 1$.Prove that

$$\sum_{i=1}^{n} \frac{a_i}{2-a_i} \geq \frac{n}{2n-1}$$

Solution (Mharchi Abdelmalek): Using The Cauchy Shwarz inequality
we have

$$\sum_{i=1}^{n} \frac{a_i}{2-a_i} = \sum_{i=1}^{n} \frac{a_i^2}{2a_i - a_i^2} \geq \frac{(a_1+a_2...+a_n)^2}{2(a_1+a_2...+a_n) - \sum a_i^2} = \frac{1}{2-\sum a_i^2} \geq \frac{1}{2-\frac{1}{n}} = \frac{n}{2n-1}$$

Problem 98 *'MOSP 2001'* (Mharchi Abdelmalek): Prove that if $a, b, c > 0$ have product 1 then :

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1)$$

Solution (Endrit Fejzullahu): Since $abc = 1$
Then

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - 1 \geq 4(a+b+c-1) \iff ab+bc+ca + \frac{3}{a+b+c} \geq 4$$

By the Am-Gm inequality

$$3\left(\frac{ab+bc+ca}{3}\right) + \frac{3}{a+b+c} \geq 4\sqrt[4]{\frac{3(ab+bc+ca)^3}{27(a+b+c)}}$$

It is enough to prove that
$3(ab+bc+ca)^3 \geq 27(a+b+c) or (ab+bc+ca)^3 \geq 9(a+b+c)$
By AM-GM ,we know that $(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$, so it suffices to prove $ab+bc+ca \geq 3$,which is obvious .

Problem 99 *'—'* (Endrit Fejzullahu): Let m, n be natural numbers .Prove that

$$\sin^{2m} x \cdot \cos^{2n} y \le \frac{m^n n^n}{(m+n)^{m+n}}$$

Solution (socrates): Use AM-GM (or weighted AM-GM) to get

$$1 = \sin^2 x + \cos^2 x = m \cdot \frac{\sin^2 x}{m} + n \cdot \frac{\cos^2 x}{n} =$$
$$\underbrace{\frac{\sin^2 x}{m} + \frac{\sin^2 x}{m} + ... + \frac{\sin^2 x}{m}}_{m\ times} + \underbrace{\frac{\cos^2 x}{n} + \frac{\cos^2 x}{n} + ... \frac{\cos^2 x}{n}}_{n\ times} \ge$$
$$\ge (m+n) \sqrt[m+n]{(\tfrac{\sin^2 x}{m})^m (\tfrac{\cos^2 x}{n})^n}$$

etc...

Problem 100 *'Mircea Lascu'* (Mircea Lascu): Let $a, b, c > 0$. Prove that

$$\frac{c}{a} + \frac{a}{b+c} + \frac{b}{c} \ge 2$$

Solution (Vo Quoc Ba Can): Write the inequality as

$$\frac{c}{a} + \frac{a}{b+c} + \frac{b+c}{c} \ge 3$$

In this form, we can see immediately that it is a direct consequence of the AM-GM Inequality.

Inequalities From Around the World 1995-2005

Solutions to 'Inequalities through problems' by Hojoo Lee

AUTORS: Mathlink Members

EDITOR: Ercole Suppa

Teramo, 28 March 2011 - Version 1

Introduction

The aim of this work is to provide solutions to problems on inequalities proposed in various countries of the world in the years 1990-2005.

In the summer of 2006, after reading Hoojoo Lee's nice book, *Topics in Inequalities - Theorem and Techniques*, I developed the idea of demonstrating all the inequalities proposed in chapter 5, subsequently reprinted in the article *Inequalities Through Problems* by the same author. After a hard and tiresome work lasting over two months, thanks also to the help I mustered from specialised literature and from the http://www.mathlinks.ro website, I finally managed to bring this ambitious project to an end.

To many inequalities I have offered more than one solution and I have always provided the source and the name of the author. In the contents I have also marked with an asterisk all the solutions which have been devised by myself. Furthermore I corrected the text of the problems 5, 11, 32, 79, 125, 140, 159 which seems to contain some typos (I think !).

I would greatly appreciate hearing comments and corrections from my readers. You may email me at

Ercole Suppa
ercsuppa@gmail.com

To Readers

This book is addressed to challenging high schools students who take part in mathematical competitions and to all those who are interested in inequalities and would like improve their skills in nonroutine problems. I heartily encourage readers to send me their own alternative solutions of the proposed inequalities: these will be published in the definitive version of this book. **Enjoy!**

Acknowledgement

I'm indebted to ***Hojoo Lee***, ***Vasile Cîrtoaje***, ***Massimo Gobbino***, ***Darij Grinberg*** and many other contributors of Mathlinks Forum for their nice solutions. Without their valuable help this work would not have been possible.

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1 Years 2001 $\sim$ 2005

1. (**BMO 2005, Proposed by Dušan Djukić, Serbia and Montenegro**)
$(a, b, c > 0)$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}$$

First Solution. (*Andrei, Chang Woo-JIn - ML Forum*)
Rewrite the initial inequality to:

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a} \geq \frac{4(a-b)^2}{a+b+c}$$

This is equivalent to

$$(a+b+c)\left(\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a}\right) \geq 4(a-b)^2$$

Using Cauchy one can prove

$$(a+b+c)\left[\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a}\right] \geq 4\left[\max(a,b,c) - \min(a,b,c)\right]^2$$

In fact

$$(a+b+c)\left(\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a}\right) \geq (|a-b| + |b-c| + |c-a|)^2$$

WLOG[1] assume $|c-a| = \max(|a-b|, |b-c|, |c-a|)$. Then, we get

$$|a-b| + |b-c| \geq |c-a|$$

[1] Without loss of generality.

So

$$|a-b|+|b-c|+|c-a| \geq 2|c-a|$$

Therefore,

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right) \geq 4\max\left(|a-b|,|b-c|,|c-a|\right)^2$$

Equality hold if and only if one of two cases occur : $a=b=c$ or $c=\omega b$, $a=\omega c$, where $\omega=\frac{\sqrt{5}-1}{2}$. $\square$

Second Solution. (*Ciprian - ML Forum*) With Lagrange theorem (for 3 numbers) we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{1}{a+b+c}\cdot\left[\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab}+\frac{\left(ab-c^2\right)^2}{ac}\right]$$

So we have to prove that

$$\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab}+\frac{\left(ab-c^2\right)^2}{ac} \geq 4\left(a-b\right)^2$$

But $\frac{\left(ab-c^2\right)^2}{ac} \geq 0$ and

$$\frac{\left(ac-b^2\right)^2}{bc}+\frac{\left(bc-a^2\right)^2}{ab} \geq \frac{\left(ac-b^2-bc+a^2\right)^2}{b\left(a+c\right)}=\frac{\left(a-b\right)^2\left(a+b+c\right)^2}{b\left(a+c\right)}$$

By AM-GM we have

$$b\left(a+c\right) \leq \frac{\left(a+b+c\right)^2}{4} \quad \Longrightarrow \quad \frac{\left(a-b\right)^2\left(a+b+c\right)^2}{b\left(a+c\right)} \geq 4\left(a-b\right)^2$$

Then we get

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{4(a-b)^2}{a+b+c}$$

$\square$

Remark. The Binet-Cauchy identity

$$\left(\sum_{i=1}^{n} a_i c_i\right)\left(\sum_{i=1}^{n} b_i d_i\right)-\left(\sum_{i=1}^{n} a_i d_i\right)\left(\sum_{i=1}^{n} b_i c_i\right)=\sum_{1\leq i<j\leq n}\left(a_i b_j-a_j b_i\right)\left(c_i d_j-c_j d_i\right)$$

by letting $c_i = a_i$ and $d_i = b_i$ gives the Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2$$

It implies the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Equality holds if and only if $a_k b_j = a_j b_k$ for all $1 \le k, j \le n$.

2. (**Romania 2005, Cezar Lupu**) $(a, b, c > 0)$

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Solution. (*Ercole Suppa*) By using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{b+c}}{a}\frac{1}{\sqrt{b+c}}\right)^2 \le \\ &\le \left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right) \le \quad \text{(Cauchy-Schwarz)} \\ &\le \left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\end{aligned}$$

Therefore

$$\left(\sum_{\text{cyc}} \frac{b+c}{a^2}\right) \ge \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

□

3. (**Romania 2005, Traian Tamaian**) $(a, b, c > 0)$

$$\frac{a}{b+2c+d} + \frac{b}{c+2d+a} + \frac{c}{d+2a+b} + \frac{d}{a+2b+c} \ge 1$$

First Solution. (*Ercole Suppa*) From the Cauchy-Schwartz inequality we have

$$(a+b+c+d)^2 \leq \sum_{\text{cyc}} \frac{a}{b+2c+d} \sum_{\text{cyc}} a\,(b+2c+d)$$

Thus in order to prove the requested inequality is enough to show that

$$\frac{(a+b+c+d)^2}{\sum_{\text{cyc}} a\,(b+2c+d)} \geq 1$$

The last inequality is equivalent to

$$\begin{aligned}
&(a+b+c+d)^2 - \sum_{\text{cyc}} a\,(b+2c+d) \geq 0 && \iff \\
&a^2+b^2+c^2+d^2-2ac-2bd \geq 0 && \iff \\
&(a-c)^2+(b-d)^2 \geq 0
\end{aligned}$$

which is true. □

Second Solution. (*Ramanujan - ML Forum*) We set $S = a+b+c+d$. It is

$$\begin{aligned}
&\frac{a}{b+2c+d}+\frac{b}{c+2d+a}+\frac{c}{d+2a+b}+\frac{d}{a+2b+c}= \\
=&\frac{a}{S-(a-c)}+\frac{b}{S-(b-d)}+\frac{c}{S+(a-c)}+\frac{d}{S+(b-d)}
\end{aligned}$$

But

$$\frac{a}{S-(a-c)}+\frac{c}{S+(a-c)}=\frac{(a+c)S+(a-c)^2}{S^2-(a-c)^2} \geq \frac{a+c}{S} \tag{1}$$

and

$$\frac{b}{S-(b-d)}+\frac{d}{S+(b-d)} \geq \frac{b+d}{S} \tag{2}$$

Now from (1) and (2) we get the result. □

4. (**Romania 2005, Cezar Lupu**) $\left(a+b+c \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c},\ a,b,c>0\right)$

$$a+b+c \geq \frac{3}{abc}$$

Solution. (*Ercole Suppa*)
From the well-known inequality $(x+y+x)^2 \geq 3(xy+yz+zx)$ it follows that

$$\begin{aligned}(a+b+c)^2 \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &\geq \\ &\geq 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \\ &= \frac{3(a+b+c)}{abc}\end{aligned}$$

Dividing by $a+b+c$ we have the desidered inequality. □

5. (**Romania 2005, Cezar Lupu**) $(1=(a+b)(b+c)(c+a),\ a,b,c>0)$

$$ab+bc+ca \leq \frac{3}{4}$$

Solution. (*Ercole Suppa*) From the identity

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$$

we have

$$ab+bc+ca = \frac{1+abc}{a+b+c} \tag{1}$$

From AM-GM inequality we have

$$1 = (a+b)(b+c)(c+a) \geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca} = 8abc \quad\Longrightarrow\quad abc \leq \frac{1}{8} \tag{2}$$

and

$$2\frac{a+b+c}{3} = \frac{(a+b)+(b+c)+(c+a)}{3} \geq \sqrt[3]{(a+b)(b+c)(c+a)} = 1 \quad\Longrightarrow$$

$$a+b+c \geq \frac{3}{2} \tag{3}$$

From (1),(2),(3) it follows that

$$ab+bc+ca = \frac{1+abc}{a+b+c} \leq \frac{1+\frac{1}{8}}{\frac{3}{2}} = \frac{3}{4}$$

□

6. (**Romania 2005, Robert Szasz - Romanian JBMO TST**) ($a+b+c = 3,\ a, b, c > 0$)

$$a^2b^2c^2 \geq (3-2a)(3-2b)(3-2c)$$

First Solution. (*Thazn1 - ML Forum*) The inequality is equivalent to

$$(a+b+c)^3(-a+b+c)(a-b+c)(a+b-c) \leq 27a^2b^2c^2$$

Let $x = -a+b+c$, $y = a-b+c$, $z = a+b-c$ and note that at most one of x, y, z can be negative (since the sum of any two is positive). Assume $x, y, z \geq 0$ if not the inequality will be obvious. Denote $x+y+z = a+b+c, x+y = 2c$, etc. so our inequality becomes

$$64xyz(x+y+z)^3 \leq 27(x+y)^2(y+z)^2(z+x)^2$$

Note that

$$9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx)$$

and

$$(xy+yz+zx)^2 \geq 3xyz(x+y+z)$$

Combining these completes our proof! □

Second Solution. (*Fuzzylogic - Mathlink Forum*) As noted in the first solution, we may assume a, b, c are the sides of a triangle. Multiplying LHS by $a+b+c$ and RHS by 3, the inequality becomes

$$16\Delta^2 \leq 3a^2b^2c^2$$

where Δ is the area of the triangle. That is equivalent to $R^2 \geq \frac{1}{3}$ since $\Delta = \frac{abc}{4R}$, where R is the circumradius. But this is true since

$$R = \frac{a+b+c}{2(\sin A + \sin B + \sin C)}$$

and

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

by Jensen. □

Third Solution. (*Harazi - Mathlink Forum*) Obviously, we may assume that a, b, c are sides of a triangle. Write Schur inequality in the form

$$\frac{9abc}{a+b+c} \geq a(b+c-a)+b(c+a-b)+c(a+b-c)$$

and apply AM-GM for the RHS. The conclusion follows. $\square$

7. (**Romania 2005**) ($abc \geq 1,\ a, b, c > 0$)

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq 1$$

First Solution. (*Virgil Nicula - ML Forum*)
The inequality is equivalent with the relation

$$\sum a^2(b+c)+2abc \geq 2(a+b+c)+2 \tag{1}$$

But

$$2abc \geq 2 \tag{2}$$

and

$$\begin{aligned} 3+\sum a^2(b+c) = \sum(a^2b+a^2c+1) &\geq \\ \geq 3\sum \sqrt[3]{a^2b \cdot a^2c \cdot 1} &\geq \\ \geq 3\sum a \cdot \sqrt[3]{abc} &\geq \\ \geq 3\sum a &= \\ = 2\sum a + \sum a &\geq \\ \geq 2\sum a + 3\sqrt[3]{abc} &\geq \\ \geq 2\sum a + 3 & \end{aligned}$$

Thus we have

$$\sum a^2(b+c) \geq 2\sum a \tag{3}$$

From the sum of the relations (2) and (3) we obtain (1). $\square$

Second Solution. (*Gibbenergy - ML Forum*)
Clear the denominator, the inequality is equivalent to:

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \geq 2+2(a+b+c)$$

Since $abc \geq 1$ so $a+b+c \geq 3$ and $2abc \geq 2$. It remains to prove that

$$a^2(b+c)+b^2(a+c)+c^2(a+b) \geq 2(a+b+c)$$

It isn't hard since

$$\begin{aligned}
\sum(a^2b+a^2c+1)-3 &\geq \sum 3\sqrt[3]{a^4bc}-3 \geq \\
&\geq \sum 3\sqrt[3]{a^3}-3 = \\
&= 3\sum a - 3 \geq \\
&\geq 2(a+b+c)+(a+b+c-3) \geq \\
&\geq 2(a+b+c)
\end{aligned}$$

□

Third Solution. (*Sung-yoon Kim - ML Forum*)
Let be $abc = k^3$ with $k \geq 1$. Now put $a = kx^3$, $b = ky^3$, $c = kz^3$, and we get $xyz = 1$. So

$$\begin{aligned}
\sum \frac{1}{1+a+b} &= \sum \frac{1}{1+k(x^3+y^3)} \leq \\
&\leq \sum \frac{1}{1+x^3+y^3} \leq \\
&\leq \sum \frac{1}{xyz+x^2y+xy^2} = \\
&= \sum \frac{1}{xy} \cdot \frac{1}{x+y+z} = \\
&= \left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\frac{1}{x+y+z} = \\
&= \frac{1}{xyz} = 1
\end{aligned}$$

and we are done. □

Remark 1. The problem was proposed in Romania at IMAR Test 2005, Juniors Problem 1. The same inequality with $abc = 1$, was proposed in Tournament of the Town 1997 and can be proved in the following way:

Solution. (*See* [66], *pag.* 161)
By AM-GM inequality

$$a+b+c \geq 3\sqrt[3]{abc} \geq 3 \quad \text{and} \quad ab+bc+ca \geq 3\sqrt[3]{ab\cdot bc\cdot ca} \geq 3$$

Hence

$$(a+b+c)(ab+bc+ca-2) \geq 3$$

which implies

$$2\left(a+b+c\right)\leq ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)$$

Therefore

$$\begin{aligned}&\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}-1=\\=&\frac{2+2a+2b+2c-\left(a+b\right)\left(b+c\right)\left(c+a\right)}{\left(1+a+b\right)\left(1+b+c\right)\left(1+c+a\right)}=\\=&\frac{2a+2b+2c-ab\left(a+b\right)-bc\left(b+c\right)-ca\left(c+a\right)}{\left(1+a+b\right)\left(1+b+c\right)\left(1+c+a\right)}\leq 0\end{aligned}$$

□

Remark 2. A similar problem was proposed in USAMO 1997 (problem 5) *Prove that, for all positive real numbers a, b, c we have*

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{1}{abc}$$

The inequality can be proved with the same technique employed in the third solution (see problem 87).

8. (**Romania 2005, Unused**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{b^2(c+1)}+\frac{b}{c^2(a+1)}+\frac{c}{a^2(b+1)}\geq\frac{3}{2}$$

First Solution. (*Arqady - ML Forum*) Let $a=\frac{x}{z}$, $b=\frac{y}{x}$ and $c=\frac{z}{y}$, where $x>0$, $y>0$ and $z>0$. Hence, using the Cauchy-Schwarz inequality in the Engel form, we have

$$\begin{aligned}\sum_{\text{cyc}}\frac{a}{b^2(c+1)}&=\sum_{\text{cyc}}\frac{x^3}{yz(y+z)}=\\&=\sum_{\text{cyc}}\frac{x^4}{xyz(y+z)}\geq\\&\geq\frac{(x^2+y^2+z^2)^2}{2xyz(x+y+z)}\end{aligned}$$

Id est, remain to prove that

$$\frac{(x^2+y^2+z^2)^2}{2xyz(x+y+z)}\geq\frac{3}{2}$$

which follows easily from Muirhead theorem. In fact

$$\begin{aligned}\frac{\left(x^2+y^2+z^2\right)^2}{2xyz\,(x+y+z)} \geq \frac{3}{2} &\qquad \Longleftrightarrow \\ 2\sum_{\text{cyc}} x^4 + 4\sum_{\text{cyc}} x^2y^2 \geq 6\sum_{\text{cyc}} x^2yz &\qquad \Longleftrightarrow \\ \sum_{\text{sym}} x^4 + 2\sum_{\text{sym}} x^2y^2 \geq 3\sum_{\text{sym}} x^2yz &\end{aligned}$$

and

$$\sum_{\text{sym}} x^4 \geq \sum_{\text{sym}} x^2yz \quad , \quad \sum_{\text{sym}} x^2y^2 \geq \sum_{\text{sym}} x^2yz$$

□

Second Solution. (*Travinhphuctk14 - ML Forum*) Let $a = \frac{x}{z}$, $b = \frac{y}{x}$ and $c = \frac{z}{y}$, where $x > 0$, $y > 0$ and $z > 0$. We need prove

$$\sum_{\text{cyc}} \frac{a}{b^2(c+1)} = \sum_{\text{cyc}} \frac{x^3}{yz(y+z)} \geq \frac{3}{2}$$

We have $x^3 + y^3 \geq xy(x+y)$, …, etc. Thus the desidered inequality follows from Nesbit inequality:

$$\frac{x^3}{yz(y+z)} + \frac{y^3}{xz(x+z)} + \frac{z^3}{xy(x+y)} \geq \frac{x^3}{y^3+z^3} + \frac{y^3}{z^3+x^3} + \frac{z^3}{x^3+y^3} \geq \frac{3}{2}$$

□

9. (**Romania 2005, Unused**) $(a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ a,b,c>0)$

$$\frac{a^3c}{b(c+a)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

Solution. (*Zhaobin - ML Forum*) First use Holder or the generalized Cauchy inequality. We have:

$$\left(\frac{a^3c}{b(a+c)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)}\right)(2a+2b+2c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)^3$$

so:

$$\frac{a^3c}{b(a+c)} + \frac{b^3a}{c(a+b)} + \frac{c^3b}{a(b+c)} \geq \frac{a+b+c}{2}$$

but we also have:

$$a+b+c \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

so the proof is over. $\square$

10. (**Romania 2005, Unused**) $(a+b+c=1,\ a,b,c>0)$

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

First Solution. (*Ercole Suppa*) By Cauchy-Schwarz inequality we have

$$1 = (a+b+c)^2 = \left(\sum_{\text{cyc}} \frac{\sqrt{a}}{\sqrt[4]{b+c}} \sqrt{a}\sqrt[4]{b+c}\right)^2 \leq$$

$$\leq \left(\sum_{\text{cyc}} \frac{a}{\sqrt{b+c}}\right)\left(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}\right)$$

Therefore

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{b+c}}\right) \geq \frac{1}{a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}} \tag{1}$$

Since $a+b+c=1$ we have

$$\begin{aligned} a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b} &= \left(\sum_{\text{cyc}} \sqrt{a}\sqrt{a\,(b+c)}\right) \leq \\ &\leq \sqrt{a+b+c}\sqrt{2ab+2bc+2ca} = \quad \text{(Cauchy-Schwarz)} \\ &= \sqrt{\frac{2}{3}}\sqrt{3ab+3bc+3ac} \leq \\ &\leq \sqrt{\frac{2}{3}}\sqrt{(a+b+c)^2} = \sqrt{\frac{2}{3}} \end{aligned}$$

and from (1) we get the result.

$\square$

Second Solution. (*Ercole Suppa*) Since $a+b+c=1$ we must prove that:

$$\frac{a}{\sqrt{1-a}}+\frac{b}{\sqrt{1-b}}+\frac{c}{\sqrt{1-c}}\geq\sqrt{\frac{3}{2}}$$

The function $f(x)=\frac{x}{\sqrt{1-x}}$ is convex on interval $[0,1]$ because

$$f''(x)=\frac{1}{4}(1-x)^{-\frac{5}{2}}(4-x)\geq 0\ ,\ \forall x\in[0,1]$$

Thus, from Jensen inequality, follows that

$$\sum_{\text{cyc}}\frac{a}{\sqrt{b+c}}=f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{1}{3}\right)=\sqrt{\frac{3}{2}}$$

□

11. (**Romania 2005, Unused**) $(ab+bc+ca+2abc=1,\ a,b,c>0)$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq\frac{3}{2}$$

Solution. (*See* [4], *pag.* 10, *problem* 19)
Set $x=\sqrt{ab}$, $y=\sqrt{bc}$, $z=\sqrt{ca}$, $s=x+y+z$. The given relation become

$$x^2+y^2+z^2+2xyz=1$$

and, by AM-GM inequality, we have

$$\begin{aligned}
s^2-2s+1&=(x+y+z)^2-2(x+y+z)+1=\\
&=1-2xyz+2(xy+xz+yz)-2(x+y+z)+1=\\
&=2(xy+xz+yz-xyz-x-y-z+1)=\\
&=2(1-x)(1-y)(1-z)\leq && \text{(AM-GM)}\\
&\leq 2\left(\frac{1-x+1-y+1-z}{3}\right)^3=2\left(\frac{3-s}{3}\right)^3
\end{aligned}$$

Therefore

$$2s^3+9s^2-27\leq 0\quad\Leftrightarrow\quad(2s-3)(s+3)^2\leq 0\quad\Leftrightarrow\quad s\leq\frac{3}{2}$$

and we are done. □

12. (**Chzech and Slovak 2005**) ($abc = 1,\ a, b, c > 0$)

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}$$

Solution. (*Ercole Suppa*) The given inequality is equivalent to

$$4(ab + bc + ca) + 4(a + b + c) \geq 3(abc + ab + bc + ca + a + b + c + 1)$$

that is, since $abc = 1$, to

$$ab + bc + ca + a + b + c \geq 6$$

The latter inequality is obtained summing the inequalities

$$a + b + c \geq 3\sqrt[3]{abc} = 3$$

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3$$

which are true by AM-GM inequality. □

13. (**Japan 2005**) ($a + b + c = 1,\ a, b, c > 0$)

$$a\,(1 + b - c)^{\frac{1}{3}} + b\,(1 + c - a)^{\frac{1}{3}} + c\,(1 + a - b)^{\frac{1}{3}} \leq 1$$

First Solution. (*Darij Grinberg - ML Forum*) The numbers $1+b-c$, $1+c-a$ and $1 + a - b$ are positive, since $a + b + c = 1$ yields $a < 1$, $b < 1$ and $c < 1$. Now use the weighted Jensen inequality for the function $f(x) = \sqrt[3]{x}$, which is concave on the positive halfaxis, and for the numbers $1 + b - c$, $1 + c - a$ and $1 + a - b$ with the respective weights a, b and c to get

$$\frac{a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b}}{a+b+c} \leq$$
$$\leq \sqrt[3]{\frac{a\,(1+b-c) + b\,(1+c-a) + c\,(1+a-b)}{a+b+c}}$$

Since $a + b + c = 1$, this simplifies to

$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq$$
$$\leq \sqrt[3]{a\,(1+b-c) + b\,(1+c-a) + c\,(1+a-b)}$$

But

$$\sum_{\text{cyc}} a\left(1+b-c\right) = (a+ab-ca)+(b+bc-ab)+(c+ca-bc) = a+b+c = 1$$

and thus

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+c\sqrt[3]{1+a-b} \leq \sqrt[3]{1} = 1$$

and the inequality is proven.

$\square$

Second Solution. (*Kunny - ML Forum*) Using A.M-G.M.

$$\sqrt[3]{1+b-c} = \sqrt[3]{1\cdot 1\cdot (1+b-c)} \leqq \frac{1+1+(1+b-c)}{3} = 1+\frac{b-c}{3}$$

Therefore by $a+b+c=1$ we have

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+c\sqrt[3]{1+a-b} \leq$$

$$\leqq a\left(1+\frac{b-c}{3}\right)+b\left(1+\frac{c-a}{3}\right)+c\left(1+\frac{a-b}{3}\right) = 1$$

$\square$

Third Solution. (*Soarer - ML Forum*) By Holder with $p=\frac{3}{2}$ and $q=3$ we have

$$\begin{aligned}\sum_{\text{cyc}} a\left(1+b-c\right)^{\frac{1}{3}} &= \sum_{\text{cyc}} a^{\frac{2}{3}}\left[a\left(1+b-c\right)\right]^{\frac{1}{3}} \leq \\ &\leq \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}}\left[\sum_{\text{cyc}} a\left(1+b-c\right)\right]^{\frac{1}{3}} = \\ &= \left(\sum_{\text{cyc}} a\right)^{\frac{2}{3}}\left(\sum_{\text{cyc}} a\right)^{\frac{1}{3}} = \sum_{\text{cyc}} a = 1\end{aligned}$$

$\square$

14. (**Germany 2005**) $(a+b+c=1,\ a,b,c>0)$

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}$$

First Solution. (*Arqady - ML Forum*)

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\quad\Leftrightarrow$$

$$\sum\left(\frac{2a}{b}-\frac{2a}{b+c}-1\right)\geq 0\quad\Leftrightarrow$$

$$\sum_{\text{cyc}}(2a^4c^2-2a^2b^2c^2)+\sum_{\text{cyc}}(a^3b^3-a^3b^2c-a^3c^2b+a^3c^3)\geq 0$$

which is obviously true.

Second Solution. (*Darij Grinberg - ML Forum*) The inequality

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq 2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})$$

can be transformed to

$$\frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

or equivalently to

$$\frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)}\geq\frac{3}{2}$$

We will prove the last inequality by rearrangement. Since the number arrays

$$\left(\frac{ab}{c};\ \frac{bc}{a};\ \frac{ca}{b}\right)$$

and

$$\left(\frac{1}{a+b};\ \frac{1}{b+c};\ \frac{1}{c+a}\right)$$

are oppositely sorted (in fact, e. g., if $c\geq a\geq b$, we have $\frac{ab}{c}\leq\frac{bc}{a}\leq\frac{ca}{b}$ and $\frac{1}{a+b}\geq\frac{1}{b+c}\geq\frac{1}{c+a}$), we have

$$\frac{ab}{c}\cdot\frac{1}{b+c}+\frac{bc}{a}\cdot\frac{1}{c+a}+\frac{ca}{b}\cdot\frac{1}{a+b}\geq\frac{ab}{c}\cdot\frac{1}{a+b}+\frac{bc}{a}\cdot\frac{1}{b+c}+\frac{ca}{b}\cdot\frac{1}{c+a}$$

i.e.

$$\frac{ab}{c\,(b+c)}+\frac{bc}{a\,(c+a)}+\frac{ca}{b\,(a+b)}\geq\frac{ab}{c\,(a+b)}+\frac{bc}{a\,(b+c)}+\frac{ca}{b\,(c+a)}$$

Hence, in order to prove the inequality

$$\frac{ab}{c\,(b+c)}+\frac{bc}{a\,(c+a)}+\frac{ca}{b\,(a+b)}\geq\frac{3}{2}$$

it will be enough to show that

$$\frac{ab}{c\,(a+b)}+\frac{bc}{a\,(b+c)}+\frac{ca}{b\,(c+a)}\geq\frac{3}{2}$$

But this inequality can be rewritten as

$$\frac{ab}{ca+bc}+\frac{bc}{ab+ca}+\frac{ca}{bc+ab}\geq\frac{3}{2}$$

which follows from Nesbitt.

□

Third Solution. (*Hardsoul and Darij Grinberg - ML Forum*) The inequality

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq 2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})$$

can be transformed to

$$\frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$$

or equivalently to

$$\frac{ab}{c(b+c)}+\frac{bc}{a(c+a)}+\frac{ca}{b(a+b)}\geq\frac{3}{2}$$

Now by Cauchy to

$$\left(\sqrt{\frac{ab}{c(b+c)}},\sqrt{\frac{bc}{a(c+a)}},\sqrt{\frac{ca}{b(a+b)}}\right)$$

and

$$\left(\sqrt{b+c},\sqrt{a+c},\sqrt{a+b}\right)$$

we have

$$LHS\cdot(2a+2b+2c)\geq\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)^2$$

To establish the inequality $LHS \geq \frac{3}{2}$ it will be enough to show that

$$\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)^2 \geq 3\left(a+b+c\right)$$

Defining $\sqrt{\frac{bc}{a}}=x$, $\sqrt{\frac{ca}{b}}=y$, $\sqrt{\frac{ab}{c}}=z$, we have

$$yz=\sqrt{\frac{ca}{b}}\sqrt{\frac{ab}{c}}=\sqrt{\frac{ca}{b}\cdot\frac{ab}{c}}=\sqrt{a^2}=a$$

and similarly $zx=b$ and $xy=c$, so that the inequality in question,

$$\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)^2 \geq 3\left(a+b+c\right)$$

takes the form

$$\left(x+y+z\right)^2 \geq 3\left(yz+zx+xy\right)$$

what is trivial because

$$\left(x+y+z\right)^2-3\left(yz+zx+xy\right)=\frac{1}{2}\cdot\left(\left(y-z\right)^2+\left(z-x\right)^2+\left(x-y\right)^2\right)$$

$\square$

Fourth Solution. (*Behzad - ML Forum*) With computation we get that the inequality is equivalent to:

$$2(\sum a^3b^3+\sum a^4b^2) \geq 6a^2b^2c^2+\sum a^3b^2c+a^3bc^2$$

which is obvious with Muirhead and AM-GM.

$\square$

15. (**Vietnam 2005**) $(a,b,c>0)$

$$\left(\frac{a}{a+b}\right)^3+\left(\frac{b}{b+c}\right)^3+\left(\frac{c}{c+a}\right)^3 \geq \frac{3}{8}$$

Solution. (*Ercole Suppa*) In order to prove the inequality we begin with the following Lemma

LEMMA. Given three real numbers $x,y,z \geq 0$ such that $xyz=1$ we have

$$\frac{1}{\left(1+x\right)^2}+\frac{1}{\left(1+y\right)^2}+\frac{1}{\left(1+z\right)^2} \geq \frac{3}{4}$$

Proof. WLOG we can assume that $xy \geq 1$, $z \leq 1$. The problems 17 yields

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy} = \frac{z}{z+1}$$

Thus it is easy to show that

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} \geq \frac{z}{z+1} + \frac{1}{(1+z)^2} \geq \frac{3}{4}$$

and the lemma is proved. $\square$

The power mean inequality implies

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} \geq \sqrt{\frac{a^2+b^2+c^2}{3}} \quad \Leftrightarrow$$
$$a^3+b^3+c^3 \geq \frac{1}{\sqrt{3}}\left(a^2+b^2+c^2\right)^{3/2} \tag{1}$$

Thus setting $x = \frac{b}{a}$, $y = \frac{c}{b}$, $z = \frac{a}{c}$, using (1) and the Lemma we have:

$$\begin{aligned}
LHS &= \left(\frac{a}{a+b}\right)^3 + \left(\frac{b}{b+c}\right)^3 + \left(\frac{c}{c+a}\right)^3 \geq \\
&\geq \frac{1}{\sqrt{3}}\left[\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2\right]^{3/2} \geq \\
&\geq \frac{1}{\sqrt{3}}\left[\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2}\right]^{3/2} \geq \\
&\geq \frac{1}{\sqrt{3}}\left(\frac{3}{4}\right)^{3/2} = \frac{3}{8}
\end{aligned}$$

$\square$

Remark. The lemma can be proved also by means of problem 17 with $a = x$, $b = y$, $c = z$, $d = 1$.

16. (**China 2005**) $(a+b+c = 1,\ a, b, c > 0)$

$$10(a^3+b^3+c^3) - 9(a^5+b^5+c^5) \geq 1$$

Solution. (*Ercole Suppa*) We must show that for all $a, b, c > 0$ with $a+b+c = 1$ results:

$$\sum_{\text{cyc}} \left(10a^3 - 9a^5\right) \geq 1$$

The function $f(x) = 10x^3 - 9x^5$ is convex on $[0, 1]$ because

$$f''(x) = 30x\left(2 - 3x^2\right) \geq 0, \quad \forall x \in [0, 1]$$

Therefore, since $f\left(\frac{1}{3}\right) = \frac{1}{3}$, the Jensen inequality implies

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right) = 3 \cdot f\left(\frac{1}{3}\right) = 1$$

□

17. (**China 2005**) $(abcd = 1,\ a, b, c, d > 0)$

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1$$

First Solution. (*Lagrangia - ML Forum*) The source is Old and New Inequalities [4]. The one that made this inequality is Vasile Cartoaje.
I will post a solution from there:

The inequality obviously follows from:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}$$

and

$$\frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{1}{(1+cd}$$

Only the first inequality we are going to prove as the other one is done in the same manner: it's same as $1 + ab(a^2 + b^2) \geq a^2b^2 + 2ab$ which is true as

$$1 + ab(a^2 + b^2) - a^2b^2 - 2ab \geq 1 + 2a^2b^2 - a^2b^2 - 2ab = (ab - 1)^2$$

This is another explanation:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} - \frac{1}{1+ab} = \frac{ab(a-b)^2 + (ab-1)^2}{(1+a)^2(1+b)^2ab)} \geq 0$$

Then, the given expression is greater than

$$1/(1+ab) + 1/(1+cd) = 1$$

with equality if $a = b = c = d = 1$.

□

Second Solution. (*Iandrei - ML Forum*) I've found a solution based on an idea from the hardest inequality I've ever seen (it really is impossible, in my opinon!). First, I'll post the original inequality by Vasc, from which I have taken the idea.
Let $a, b, c, d > 0$ be real numbers for which $a^2 + b^2 + c^2 + d^2 = 1$. Prove that the following inequality holds:

$$(1 - a)(1 - b)(1 - c)(1 - d) \geq abcd$$

I'll leave its proof to the readers. A little historical note on this problem: it was proposed in some *Gazeta Matematica Contest* in the last years and while I was still in high-school and training for mathematical olympiads, I tried to solve it on a very large number of occasions, but failed. So I think I will always remember its difficult and smart solution, which I'll leave to the readers.

Now, let us get back to our original problem:

Let $a, b, c, d > 0$ with $abcd = 1$. Prove that:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1$$

Although this inequality also belongs to Vasc (he published it in the Gazeta Matematica), it surprisingly made the China 2005 TSTs, thus confirming (in my opinion) its beauty and difficulty. Now, on to the solution:
Let

$$x = \frac{1}{1+a},\ y = \frac{1}{1+b},\ z = \frac{1}{1+c},\ t = \frac{1}{1+d}$$

Then

$$abcd = 1 \Rightarrow \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} = 1 \Rightarrow (1-x)(1-y)(1-z)(1-t) = xyzt$$

We have to prove that $x^2+y^2+z^2+t^2 \geq 1$. We will prove this by contradiction. Assume that $x^2 + y^2 + z^2 + t^2 < 1$.
Keeping in mind that $x^2+y^2+z^2+t^2 < 1$, let us assume that $(1-x)(1-y) \leq zt$ and prove that it is not true (I'm talking about the last inequality here, which we assumed to be true). Upon multiplication with 2 and expanding, this gives:

$$1 - 2(x + y) + 1 + 2xy \leq 2zt$$

This implies that

$$2zt > x^2 + y^2 + z^2 + t^2 - 2(x + y) + 1 + 2xy$$

So, $2zt > (x+y)^2 - 2(x+y) + 1 + z^2 + t^2$, which implies $(z-t)^2 + (x+y-1)^2 < 0$, a contradiction. Therefore, our original assumption implies $(1 - x)(1 - y) > zt$. In a similar manner, it is easy to prove that $(1 - z)(1 - t) > xy$. Multiplying

the two, we get that $(1-x)(1-y)(1-z)(1-t) > xyzt$, which is a contradiction with the original condition $abcd = 1$ rewritten in terms of x, y, z, t.
Therefore, our original assumption was false and we indeed have

$$x^2 + y^2 + z^2 + t^2 \geq 1$$

$\square$

18. (**China 2005**) $(ab + bc + ca = \frac{1}{3},\ a, b, c \geq 0)$

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \leq 3$$

First Solution. (*Cuong - ML Forum*) Our inequality is equivalent to:

$$\sum \frac{a}{3a(a+b+c)+2} \geq \frac{\frac{1}{3}}{a+b+c}$$

Since $ab + bc + ca = \frac{1}{3}$, by Cauchy we have:

$$\begin{aligned} LHS \geq & \frac{(a+b+c)^2}{3\left(a+b+c\right)\left(a^2+b^2+c^2\right)+2\left(a+b+c\right)} = \\ = & \frac{a+b+c}{3\left(a^2+b^2+c^2\right)+2\left(a+b+c\right)} = \\ = & \frac{a+b+c}{3\left(a^2+b^2+c^2+2ab+2ac+2bc\right)} = \\ = & \frac{a+b+c}{3\left(a+b+c\right)^2} = \frac{\frac{1}{3}}{a+b+c} \end{aligned}$$

$\square$

Second Solution. (*Billzhao - ML Forum*) Homogenizing, the inequality is equivalent to

$$\sum_{\text{cyc}} \frac{1}{a(a+b+c)+2(ab+bc+ca)} \leq \frac{1}{ab+bc+ca}$$

Multiply both sides by $2(ab + bc + ca)$ we have

$$\sum_{\text{cyc}} \frac{2(ab+bc+ca)}{a(a+b+c)+2(ab+bc+ca)} \leq 2$$

Subtracting from 3, the above inequality is equivalent to

$$\sum_{\text{cyc}} \frac{a(a+b+c)}{a(a+b+c)+2(ab+bc+ca)} \geq 1$$

Now by Cauchy we have:

$$LHS = (a+b+c)\sum_{\text{cyc}} \frac{a^2}{a^2\left(a+b+c\right)+2a\left(ab+bc+ca\right)} \geq$$

$$\geq \frac{\left(a+b+c\right)^3}{\sum_{\text{cyc}}\left[a^2\left(a+b+c\right)+2a\left(ab+bc+ca\right)\right]} = 1$$

□

19. (**Poland 2005**) $(0 \leq a, b, c \leq 1)$

$$\frac{a}{bc+1}+\frac{b}{ca+1}+\frac{c}{ab+1} \leq 2$$

First Solution. (*See* [25] *pag.* 204 *problem* 95) WLOG we can assume that $0 \leq a \leq b \leq c \leq 1$. Since $0 \leq (1-a)(1-b)$ we have

$$a+b \leq 1+ab \leq 1+2ab \quad \Rightarrow$$
$$a+b+c \leq a+b+1 \leq 2\left(1+ab\right)$$

Therefore

$$\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab} \leq \frac{a}{1+ab}+\frac{b}{1+ab}+\frac{c}{1+ab} \leq \frac{a+b+c}{1+ab} \leq 2$$

□

Second Solution. (*Ercole Suppa*) We denote the LHS with $f\left(a,b,c\right)$. The function f is defined and continuous on the cube $C=[0,1]\times[0,1]\times[0,1]$ so, by Wierstrass theorem, f assumes its maximum and minimum on C. Since f is convex with respect to all variables we obtain that f take maximum value in one of vertices of the cube. Since f is symmetric in a, b, c it is enough compute the values $f(0,0,0), f(0,0,1), f(0,1,1), f(1,1,1)$. It's easy verify that f take maximum value in $(0,1,1)$ and $f\left(0,1,1\right)=2$. The convexity of f with respect to variable a follows from the fact that

$$f\left(x,b,c\right)=\frac{x}{bc+1}+\frac{b}{cx+1}+\frac{c}{bx+1}$$

is the sum of three convex functions. Similarly we can prove the convexity with respect to b and c. □

20. (**Poland 2005**) ($ab + bc + ca = 3,\ a, b, c > 0$)

$$a^3 + b^3 + c^3 + 6abc \geq 9$$

Solution. (*Ercole Suppa*) Since $ab + bc + ca = 1$ by Mac Laurin inequality we have:

$$\frac{a+b+c}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} = 1 \tag{1}$$

By Schur inequality we have

$$\sum_{\text{cyc}} a\,(a-b)\,(a-c) \geq 0 \quad \Longrightarrow$$
$$a^3 + b^3 + c^3 + 3abc \geq \sum_{\text{sym}} a^2 b$$

and, from (1), it follows that

$$\begin{aligned} a^3 + b^3 + c^3 + 6abc &\geq a^2b + a^2c + abc + b^2a + b^2c + abc + c^2a + c^2b + abc = \\ &= (a+b+c)\,(ab+bc+ca) = \\ &= 3\,(a+b+c) \geq 9 \end{aligned}$$

□

21. (**Baltic Way 2005**) ($abc = 1,\ a, b, c > 0$)

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \geq 1$$

Solution. (*Sailor - ML Forum*) We have

$$\sum \frac{a}{a^2+2} \leq \sum \frac{a}{2a+1}$$

We shall prove that $\sum \frac{a}{2a+1} \leq 1$ or

$$\sum \frac{2a}{2a+1} \leq 2 \quad \Longleftrightarrow \quad \sum \frac{1}{2a+1} \geq 1$$

Clearing the denominator we have to prove that:

$$2\sum a \geq 6$$

wich is true by AM-GM. □

22. (**Serbia and Montenegro 2005**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}(a+b+c)}$$

Solution. (*Ercole Suppa*) Putting

$$x = \frac{a}{a+b+c}, \qquad y = \frac{b}{a+b+c}, \qquad z = \frac{c}{a+b+c}$$

the proposed inequality is exctly that of problem 10. $\square$

23. (**Serbia and Montenegro 2005**) $(a + b + c = 3,\ a, b, c > 0)$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

Solution. (*Suat Namly - ML Forum*) From AM-GM inequality we have

$$a^2 + \sqrt{a} + \sqrt{a} \geq 3a$$

By the same reasoning we obtain

$$b^2 + \sqrt{b} + \sqrt{b} \geq 3b$$

$$c^2 + \sqrt{c} + \sqrt{c} \geq 3c$$

Adding these three inequalities, we obtain

$$\begin{aligned} a^2 + b^2 + c^2 + 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) &\geq 3(a+b+c) = \\ &= (a+b+c)^2 = \\ &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \end{aligned}$$

from which we get the required result. $\square$

24. (**Bosnia and Hercegovina 2005**) $(a + b + c = 1, a, b, c > 0)$

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}$$

Solution. (*Ercole Suppa*) From the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left(a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\right)^2 &= \left(\sum_{\text{cyc}}\sqrt{a}\sqrt{ab}\right)^2 \leq \\ &\leq \left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}} ab\right) = \\ &= (ab+bc+ca) \leq \\ &\leq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}\end{aligned}$$

Extracting the square root yields the required inequality. □

25. (**Iran 2005**) $(a,b,c>0)$

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Solution. (*Ercole Suppa*) After setting

$$x=\frac{a}{b}, \qquad y=\frac{b}{c}, \qquad z=\frac{c}{a}$$

the inequality become

$$\begin{aligned}&(x+y+z)^2 \geq +x+y+z+xy+xz+yz \qquad \Longleftrightarrow \\ &x^2+y^2+z^2+xy+xz+yz \geq 3+x+y+z\end{aligned}$$

where $xyz=1$. From AM-GM inequality we have

$$xy+xz+yz \geq 3\sqrt[3]{x^2y^2z^2}=3 \tag{1}$$

On the other hand we have

$$\begin{aligned}&\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2 \geq 0 \qquad \Longrightarrow \\ &x^2+y^2+z^2 \geq x+y+z+\frac{3}{4} \geq x+y+z\end{aligned} \tag{2}$$

Adding (1) and (2) yields the required result. □

26. (**Austria 2005**) $(a,b,c,d>0)$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{d^3} \geq \frac{a+b+c+d}{abcd}$$

Solution. (*Ercole Suppa*) WLOG we can assume that $a \geq b \geq c \geq d$ so

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \geq \frac{1}{d}$$

Since the RHS can be written as

$$\frac{a+b+c+d}{abcd} = \frac{1}{bcd} + \frac{1}{acd} + \frac{1}{abd} + \frac{1}{abc}$$

from the rearrangement (applied two times) we obtain

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{1}{bcd} + \frac{1}{acd} + \frac{1}{abd} + \frac{1}{abc} = \frac{a+b+c+d}{abcd}$$

□

27. (**Moldova 2005**) $(a^4 + b^4 + c^4 = 3,\ a, b, c > 0)$

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ca} \leq 1$$

First Solution. (*Anto - ML Forum*) It is easy to prove that :

$$\sum \frac{1}{4-ab} \leq \sum \frac{1}{4-a^2} \leq \sum \frac{a^4+5}{18}$$

The first inequality follows from :

$$\frac{2}{4-ab} \leq \frac{1}{4-a^2} + \frac{1}{4-b^2}$$

The second :

$$\frac{1}{4-a^2} \leq \frac{a^4+5}{18}$$

is equivalent to :

$$0 \leq 4a^4 + 2 - a^6 - 5a^2 \Longleftrightarrow 0 \leq (a^2-1)^2(2-a^2)$$

which is true since $a^4 \leq 3$ and as a result $a^2 \leq 2$.

Thus

$$\sum_{\text{cyc}} \frac{1}{4-ab} \leq \sum_{\text{cyc}} \frac{a^4+5}{18} = \frac{a^4+b^4+c^4+15}{18} = \frac{3+15}{18} = 1$$

□

Second Solution. (*Treegoner - ML Forum*) By applying AM-GM inequality, we obtain

$$LHS \le \sum \frac{1}{4-\sqrt{\frac{a^4+b^4}{2}}} = \sum \frac{1}{4-\sqrt{\frac{3-c^4}{2}}}$$

Denote

$$u = \frac{3-c^4}{2} \quad , \quad v = \frac{3-c^4}{2} \quad , \quad w = \frac{3-c^4}{2}$$

Then $0 < u, v, w < \frac{3}{2}$ and $u+v+w = 3$. Let $f(u) = \frac{1}{4-\sqrt{u}}$. Then

$$f'(u) = \frac{1}{2\sqrt{u}(4-\sqrt{u})^2}$$

$$f''(u) = \frac{-u^{\frac{-3}{4}}(1-\frac{3}{4}u^{\frac{1}{2}})}{(4u^{\frac{1}{4}}-u^{\frac{3}{4}})^3}$$

Hence $f''(u) < 0$ for every $0 < u < \frac{3}{2}$. By apply Karamata 's inequality for the function that is concave down, we obtain the result. $\square$

28. (**APMO 2005**) $(abc = 8,\ a, b, c > 0)$

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \ge \frac{4}{3}$$

First Solution. (*Valiowk, Billzhao - ML Forum*) Note that

$$\frac{a^2+2}{2} = \frac{(a^2-a+1)+(a+1)}{2} \ge \sqrt{(a^2-a+1)(a+1)} = \sqrt{a^3+1}$$

with equality when $a = 2$. Hence it suffices to prove

$$\frac{a^2}{(a^2+2)(b^2+2)} + \frac{b^2}{(b^2+2)(c^2+2)} + \frac{c^2}{(c^2+2)(a^2+2)} \ge \frac{1}{3}$$

and this is easily verified. In fact clearing the denominator, we have

$$3\sum_{\text{cyc}} a^2(c^2+2) \ge (a^2+2)(b^2+2)(c^2+2)$$

Expanding, we have

$$6a^2 + 6b^3 + 6c^3 + 3a^2b^2 + 3b^2c^2 + 3c^2a^2 \ge$$
$$\ge a^2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 + 4a^2 + 4b^2 + 4c^2 + 8$$

Recalling that $abc = 8$, the above is equivalent to

$$2a^2 + 2b^2 + 2c^2 + a^2b^2 + b^2c^2 + c^2a^2 \geq 72$$

But $2a^2 + 2b^2 + 2c^2 \geq 24$ and $a^2b^2 + b^2c^2 + c^2a^2 \geq 48$ through AM-GM. Adding gives the result.

$\square$

Second Solution. (*Official solution*) Observe that

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}$$

In fact, this is equivalent to $(2+x^2)^2 \geq 4(1+x^3)$, or $x^2(x-2)^2 \geq 0$. Notice that equality holds if and only if if $x = 2$. Then

$$\begin{aligned}
&\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \\
&\geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+c^2)} \geq \\
&\geq \frac{2 \cdot S(a,b,c)}{36 + S(a,b,c)} = \\
&= \frac{2}{1 + \frac{36}{S(a,b,c)}}
\end{aligned}$$

where

$$S(a,b,c) = 2\left(a^2+b^2+c^2\right) + (ab)^2 + (bc)^2 + (ca)^2$$

By AM-GM inequality, we have

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{(abc)^2} = 12$$

$$(ab)^2 + (bc)^2 + (ca)^2 \geq 3\sqrt[3]{(abc)^4} = 48$$

The above inequalities yield

$$S(a,b,c) = 2\left(a^2+b^2+c^2\right) + (ab)^2 + (bc)^2 + (ca)^2 \geq 72$$

Therefore

$$\frac{2}{1 + \frac{36}{S(a,b,c)}} \geq \frac{2}{1 + \frac{36}{72}} = \frac{4}{3}$$

which is the required inequality. Note that the equalitiy hold if and only if $a = b = c = 2$. $\square$

29. (**IMO 2005**) $(xyz \geq 1, x, y, z > 0)$

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

First Solution. (*See* [32], *pag.* 26) It's equivalent to the following inequality

$$\left(\frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1\right) + \left(\frac{y^2 - y^5}{y^5 + z^2 + x^2} + 1\right) + \left(\frac{z^2 - z^5}{z^5 + x^2 + y^2} + 1\right) \leq 3$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 3.$$

With the Cauchy-Schwarz inequality and the fact that $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2 \quad \text{or} \quad \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum and $x^2 + y^2 + z^2 \geq xy + yz + zx$ give us

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{xy + yz + zx}{x^2 + y^2 + z^2} \leq 3.$$

$\square$

Second Solution. (*by an IMO 2005 contestant Iurie Boreico from Moldova, see* [32] *pag.* 28). We establish that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)}.$$

It follows immediately from the identity

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} - \frac{x^5 - x^2}{x^3(x^2 + y^2 + z^2)} = \frac{(x^3 - 1)^2 x^2 (y^2 + z^2)}{x^3(x^2 + y^2 + z^2)(x^5 + y^2 + z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyc}} \frac{x^5 - x^2}{x^5 + y^2 + z^2} \geq \frac{1}{x^5 + y^2 + z^2} \sum_{\text{cyc}} \left(x^2 - \frac{1}{x}\right) \geq \frac{1}{x^5 + y^2 + z^2} \sum_{\text{cyc}} (x^2 - yz) \geq 0.$$

$\square$

30. (**Poland 2004**) $(a + b + c = 0, a, b, c \in \mathbb{R})$

$$b^2c^2 + c^2a^2 + a^2b^2 + 3 \geq 6abc$$

First Solution. (*Ercole Suppa*) Since $a+b+c=0$ from the identity

$$(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$$

follows that

$$a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2$$

Then, from the AM-GM inequality we have

$$a^2b^2+b^2c^2+c^2a^2 \geq 3 \cdot \left(\sqrt[3]{a^2b^2c^2}\right)^2$$

By putting $abc = P$ we have

$$\begin{aligned}
&a^2b^2+b^2c^2+c^2a^2+3-6abc \geq \\
&\geq 9\,(abc)^{\frac{4}{3}}+3-6abc = \\
&= 9P^4-6P^3+3 = \\
&= 3\left[\left(P^2-1\right)^2+2P^2\left(P-\frac{1}{2}\right)^2+\frac{3}{2}P^2\right] \geq 0
\end{aligned}$$

□

Second Solution. (*Darij Grinberg - ML Forum*)
For any three real numbers a, b, c, we have

$$\begin{aligned}
&\left(b^2c^2+c^2a^2+a^2b^2+3\right)-6abc = \\
&= (b+1)^2(c+1)^2+(c+1)^2(a+1)^2+(a+1)^2(b+1)^2 \\
&\quad -2\,(a+b+c)\,(a+b+c+bc+ca+ab+2)
\end{aligned}$$

so that, in the particular case when $a+b+c=0$, we have

$$\begin{aligned}
&\left(b^2c^2+c^2a^2+a^2b^2+3\right)-6abc = \\
&= (b+1)^2(c+1)^2+(c+1)^2(a+1)^2+(a+1)^2(b+1)^2
\end{aligned}$$

and thus

$$b^2c^2+c^2a^2+a^2b^2+3 \geq 6abc$$

□

Third Solution. (*Nguyenquockhanh, Ercole Suppa - ML Forum*)
WLOG we can assume that $a > 0$ e $b > 0$. Thus, since $c = -a - b$, we have

$$\begin{aligned}
&a^2b^2 + b^2c^2 + c^2a^2 + 3 - 6abc = \\
&= a^2b^2 + \left(a^2 + b^2\right)(a + b)^2 + 3 + 6ab\,(a + b) \geq \\
&\geq \left(a^2 + b^2\right)\left(a^2 + ab + b^2\right) + ab\left(a^2 + b^2\right) + a^2b^2 + 3 + 6ab\,(a + b) \geq \\
&\geq 3ab\left(a^2 + b^2\right) + 3\left(a^2b^2 + 1\right) + 6ab\,(a + b) = \\
&\geq 3ab\left(a^2 + b^2\right) + 6ab + 6ab\,(a + b) = \\
&= 3ab\left(a^2 + b^2 + 2 + 2a + 2b\right) = \\
&= 3ab\left[(a + 1)^2 + (b + 1)^2\right] \geq 0
\end{aligned}$$

□

31. (**Baltic Way 2004**) ($abc = 1, a, b, c > 0, n \in \mathbb{N}$)

$$\frac{1}{a^n + b^n + 1} + \frac{1}{b^n + c^n + 1} + \frac{1}{c^n + a^n + 1} \leq 1$$

Solution. (*Ercole Suppa*) By setting $a^n = x$, $b^n = y$ e $c^n = z$, we must prove that

$$\frac{1}{1 + x + y} + \frac{1}{1 + y + z} + \frac{1}{1 + z + x} \leq 1$$

where $xyz = 1$. The above inequality is proven in the problem 7. □

32. (**Junior Balkan 2004**) ($(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$)

$$\frac{2\sqrt{2}}{\sqrt{x^2 + y^2}} \geq \frac{x + y}{x^2 - xy + y^2}$$

First Solution. (*Ercole Suppa*) By using the two inequalities

$$x + y \leq \sqrt{2(x^2 + y^2)} \quad , \quad x^2 + y^2 \leq 2(x^2 - xy + y^2)$$

we have:

$$\begin{aligned}
\frac{(x + y)\sqrt{x^2 + y^2}}{x^2 - xy + y^2} &\leq \frac{\sqrt{2(x^2 + y^2)}\sqrt{x^2 + y^2}}{x^2 - xy + y^2} \leq \\
&\leq \frac{\sqrt{2}(x^2 + y^2)}{x^2 - xy + y^2} \leq \\
&\leq \frac{2(x^2 - xy + y^2)\sqrt{2}}{(x^2 - xy + y^2)} = 2\sqrt{2}
\end{aligned}$$

□

Second Solution. (*Darij Grinberg - ML Forum*) You can also prove the inequality by squaring it (in fact, the right hand side of the inequality is obviously ≥ 0; if the left hand side is ≤ 0, then the inequality is trivial, so it is enough to consider the case when it is ≥ 0 as well, and then we can square the inequality); this leads to

$$\frac{(x+y)^2}{(x^2-xy+y^2)^2} \leq \frac{8}{x^2+y^2}$$

This is obviously equivalent to

$$(x+y)^2(x^2+y^2) \leq 8(x^2-xy+y^2)^2$$

But actually, an easy calculation shows that

$$8(x^2-xy+y^2)^2-(x+y)^2(x^2+y^2) = (x-y)^2\left[2(x-y)^2+5x^2+5y^2\right] \geq 0$$

so everything is proven.

□

33. (**IMO Short List 2004**) $(ab+bc+ca=1,\ a,b,c>0)$

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}$$

First Solution. (*Ercole Suppa*) The function $f(x)=\sqrt[3]{x}$ is concave on $(0,+\infty)$. Thus from Jensen inequality we have:

$$\sum_{\text{cyc}} f\left(\frac{1}{a}+6b\right) \leq 3\cdot f\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6a+6b+6c}{3}\right) \tag{1}$$

From the well-know inequality $3(xy+yz+zx) \leq (x+y+z)^2$ we have

$$3abc(a+b+c) = 3(ab\cdot ac+ab\cdot bc+ac\cdot bc) \leq (ab+bc+ca)^2 = 1 \quad \Longrightarrow$$

$$6(a+b+c) \leq \frac{2}{abc} \tag{2}$$

The AM-GM inequality and (2) yields

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+6a+6b+6c\right) \leq \frac{ab+bc+ca}{abc}+\frac{2}{abc}=\frac{3}{abc} \tag{3}$$

Since $f(x)$ is increasing from (1) and (3) we get

$$f\left(\frac{1}{a}+6b\right)+f\left(\frac{1}{b}+6c\right)+f\left(\frac{1}{c}+6a\right) \leq 3\cdot f\left(\frac{1}{abc}\right)=\frac{3}{\sqrt[3]{abc}} \leq \frac{1}{abc}$$

where in the last step we used the AM-GM inequality

$$\frac{3}{\sqrt[3]{abc}} = \frac{3\sqrt[3]{(abc)^2}}{abc} = \frac{3\sqrt[3]{ab \cdot bc \cdot ca}}{abc} \le \frac{3 \cdot \frac{ab+bc+ca}{3}}{abc} = \frac{1}{abc}$$

□

Second Solution. (*Official solution*) By the power mean inequality

$$\frac{1}{3}(u+v+w) \le \sqrt[3]{\frac{1}{3}(u^3+v^3+w^3)}$$

the left-hand side does not exceed

$$\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{a}+6b+\frac{1}{b}+6c+\frac{1}{c}+6a} = \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{ab+bc+ca}{abc}+6(a+b+c)} \qquad (\star)$$

The condition $ab+bc+ca=1$ enables us to write

$$a+b = \frac{1-ab}{c} = \frac{ab-(ab)^2}{abc}, \quad b+c = \frac{bc-(bc)^2}{abc}, \quad c+a = \frac{ca-(ca)^2}{abc}$$

Hence

$$\begin{aligned}\frac{ab+bc+ca}{abc}+6(a+b+c) &= \frac{1}{abc}+3[(a+b)+(b+c)+(c+a)] = \\ &= \frac{4-3\big[(ab)^2+(bc)^2+(ca)^2\big]}{abc}\end{aligned}$$

Now, we have

$$3\big((ab)^2+(bc)^2+(ca)^2\big) \ge (ab+bc+ca)^2 = 1$$

by the well-known inequality $3(u^2+v^2+w^2) \ge (u+v+w)^2$. Hence an upper bound for the right-hand side of $(\star)$ is $3/\sqrt[3]{abc}$. So it suffices to check $3/\sqrt[3]{abc} \le 1/(abc)$, which is equivalent to $(abc)^2 \le 1/27$. This follows from the AM-GM inequality, in view of $ab+bc+ca=1$ again:

$$(abc)^2 = (ab)(bc)(ca) \le \left(\frac{ab+bc+ca}{3}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

Clearly, equality occurs if and only if $a=b=c=1/\sqrt{3}$. □

Third Solution. (*Official solution*)
Given the conditions $a,b,c>0$ and $ab+bc+ca=1$, the following more general result holds true for all $t_1,t_2,t_3>0$:

$$3abc(t_1+t_2+t_3) \le \frac{2}{3}+at_1^3+bt_2^3+ct_3^3. \qquad (1)$$

The original inequality follows from (2) by setting

$$t_1 = \frac{1}{3}\sqrt[3]{\frac{1}{a}+6b}, \quad t_2 = \frac{1}{3}\sqrt[3]{\frac{1}{b}+6c}, \quad t_3 = \frac{1}{3}\sqrt[3]{\frac{1}{c}+6a}.$$

In turn, (1) is obtained by adding up the three inequalities

$$3abct_1 \le \frac{1}{9}+\frac{1}{3}bc+at_1^3, \quad 3abct_2 \le \frac{1}{9}+\frac{1}{3}ca+bt_2^3, \quad 3abct_3 \le \frac{1}{9}+\frac{1}{3}ab+ct_3^3.$$

By symmetry, it suffices to prove the first one of them. Since $1-bc = a(b+c)$, the AM-GM inequality gives

$$(1-bc)+\frac{at_1^3}{bc} = a\left(b+c+\frac{t_1^3}{bc}\right) \ge 3a\sqrt[3]{bc\cdot\frac{t_1^3}{bc}} = 3at_1.$$

Hence $3abct_1 \le bc(1-bc)+at_1^3$, and one more application of the AM-GM inequality completes the proof:

$$\begin{aligned} 3abct_1 &\le bc(1-bc)+at_1^3 = bc\left(\frac{2}{3}-bc\right)+\frac{1}{3}bc+at_1^3 \\ &\le \left(\frac{bc+(\frac{2}{3}-bc)}{2}\right)^2+\frac{1}{3}bc+at_1^3 = \frac{1}{9}+\frac{1}{3}bc+at_1^3. \end{aligned}$$

□

34. (**APMO 2004**) $(a,b,c>0)$

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

First Solution. (*See* [32], *pag.* 14) Choose $A,B,C \in \left(0,\frac{\pi}{2}\right)$ with $a=\sqrt{2}\tan A$, $b=\sqrt{2}\tan B$, and $c=\sqrt{2}\tan C$. Using the well-known trigonometric identity $1+\tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \ge \cos A\cos B\cos C\left(\cos A\sin B\sin C+\sin A\cos B\sin C+\sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\begin{gathered} \cos(A+B+C) = \\ = \cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C. \end{gathered}$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \ge \cos A\cos B\cos C\left(\cos A\cos B\cos C-\cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying the AM-GM inequality and Jensen's inequality, we have

$$\cos A \cos B \cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3 \leq \cos^3 \theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3 \theta (\cos^3 \theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta \quad \text{or} \quad \cos 3\theta - \cos 3\theta = 3\cos \theta - 3\cos^3 \theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4 \theta \left(1 - \cos^2 \theta\right),$$

which follows from the AM-GM inequality

$$\left(\frac{\cos^2 \theta}{2} \cdot \frac{\cos^2 \theta}{2} \cdot \left(1 - \cos^2 \theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2 \theta}{2} + \frac{\cos^2 \theta}{2} + \left(1 - \cos^2 \theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. $\square$

Second Solution. (*See* [32], *pag.* 34) After expanding, it becomes

$$8 + (abc)^2 + 2\sum_{\text{cyc}} a^2b^2 + 4\sum_{\text{cyc}} a^2 \geq 9\sum_{\text{cyc}} ab.$$

From the inequality $(ab-1)^2 + (bc-1)^2 + (ca-1)^2 \geq 0$, we obtain

$$6 + 2\sum_{\text{cyc}} a^2b^2 \geq 4\sum_{\text{cyc}} ab.$$

Hence, it will be enough to show that

$$2 + (abc)^2 + 4\sum_{\text{cyc}} a^2 \geq 5\sum_{\text{cyc}} ab.$$

Since $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$, it will be enough to show that

$$2 + (abc)^2 + \sum_{\text{cyc}} a^2 \geq 2\sum_{\text{cyc}} ab,$$

which is proved in [32], pag.33.

$\square$

Third Solution. (*Darij Grinberg - ML Forum*)
First we prove the auxiliary inequality

$$1 + 2abc + a^2 + b^2 + c^2 \geq 2bc + 2ca + 2ab$$

According to the pigeonhole principle, among the three numbers $a-1$, $b-1$, $c-1$ at least two have the same sign; WLOG, say that the numbers $b-1$ and $c-1$ have the same sign so that $(b-1)(c-1) \geq 0$. Then according to the inequality $x^2 + y^2 \geq 2xy$ for any two reals x and y, we have

$$(b-1)^2 + (c-1)^2 \geq 2(b-1)(c-1) \geq -2(a-1)(b-1)(c-1)$$

Thus

$$\begin{aligned}
&(1 + 2abc + a^2 + b^2 + c^2) - (2bc + 2ca + 2ab) = \\
&= (a-1)^2 + (b-1)^2 + (c-1)^2 + 2(a-1)(b-1)(c-1) \geq \\
&\geq (a-1)^2 \geq 0
\end{aligned}$$

and the lemma is proved. Now, the given inequality can be proved in the following way:

$$\begin{aligned}
&(a^2+2)(b^2+2)(c^2+2) - 9(ab+bc+ca) = \\
&= \frac{3}{2}\left((b-c)^2 + (c-a)^2 + (a-b)^2\right) + 2\left((bc-1)^2 + (ca-1)^2 + (ab-1)^2\right) + \\
&+ (abc-1)^2 + \left((1 + 2abc + a^2 + b^2 + c^2) - (2bc + 2ca + 2ab)\right) \geq 0
\end{aligned}$$

□

Fourth Solution. (*Official solution.*)
Let $x = a+b+c$, $y = ab+bc+ca$, $z = abc$. Then

$$\begin{aligned}
a^2 + b^2 + c^2 &= x^2 - 2y \\
a^2b^2 + b^2c^2 + c^2a^2 &= y^2 - 2xz \\
a^2b^2c^2 &= z^2
\end{aligned}$$

so the inequality to be proved becomes

$$z^2 + 2\left(y^2 - 2xz\right) + 4\left(x^2 - 2y\right) + 8 \geq 9y$$

or

$$z^2 + 2y^2 - 4xz + 4x^2 - 17y + 8 \geq 0$$

Now from $a^2 + b^2 + c^2 \geq ab + bc + ca = y$, we get

$$x^2 = a^2 + b^2 + c^2 + 2y \geq 3y$$

Also

$$\begin{aligned}a^2b^2+b^2c^2+a^2c^2 &= (ab)^2+(bc)^2+(ca)^2 \geq \\ &\geq ab\cdot ac+bc\cdot ab+ac\cdot bc = \\ &= (a+b+c)abc = xz\end{aligned}$$

and thus

$$y^2a^2b^2+b^2c^2+a^2c^2+2xz \geq 3xz$$

Hence

$$\begin{aligned}&z^2+2y^2-4xz+4x^2-17y+8 = \\ &= \left(z-\frac{x}{3}\right)^2+\frac{8}{9}(y-3)^2+\frac{10}{9}\left(y^2-3xz\right)+\frac{35}{9}\left(x^2-3y\right) \geq 0\end{aligned}$$

as required.

□

35. (**USA 2004**) $(a, b, c > 0)$

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$$

Solution. (*See* [11] *pag.* 19) For any positive number x, the quantities x^2-1 and x^3-1 have the same sign. Thus, we have

$$0 \leq (x^3-1)(x^2-1) = x^5-x^3-x^2+1 \quad\Longrightarrow\quad x^5-x^2+3 \geq x^3+2$$

It follows that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a^3+2)(b^3+2)(c^3+2)$$

It suffices to show that

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3 \tag{$\star$}$$

Expanding both sides of inequality $(\star)$ and cancelling like terms gives

$$\begin{aligned}&a^3b^3c^3+3(a^3+b^3+c^3)+2(a^3b^3+b^3c^3+c^3a^3)+8 \geq \\ &\geq 3(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c)+6abc\end{aligned}$$

By AM-GM inequality, we have $a^3+a^3b^3+1 \geq 3a^2b$. Combining similar results, the desidered inequality reduces to

$$a^3b^3c^3+a^3+b^3+c^3+1+1 \geq 6abc$$

which is evident by AM-GM inequality. □

36. (**Junior BMO 2003**) $(x, y, z > -1)$

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$$

Solution. (*Arne - ML Forum*) As $x \leq \frac{1+x^2}{2}$ we have

$$\sum \frac{1+x^2}{1+y+z^2} \geq \sum \frac{2(1+x^2)}{(1+y^2)+2(1+z^2)}.$$

Denoting $1+x^2 = a$ and so on we have to prove that

$$\sum \frac{a}{b+2c} \geq 1$$

but Cauchy tells us

$$\sum \frac{a}{b+2c} \sum a(2b+c) \geq \left(\sum a\right)^2$$

and as

$$\left(\sum a\right)^2 \geq 3(ab+bc+ca) = \sum a(2b+c)$$

we have the result. □

37. (**USA 2003**) $(a, b, c > 0)$

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

First Solution. (*See* [10] *pag.* 21) By multipliyng a, b and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desidered inequality read

$$\frac{(a+3)^2}{2a^2+(3-a)^2}+\frac{(b+3)^2}{2b^2+(3-b)^2}+\frac{(c+3)^2}{2c^2+(3-c)^2}\leq 8$$

Set

$$f(x) = \frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c) \leq 8$. Note that

$$\begin{aligned} f(x) &= \frac{x^2+6x+9}{3\left(x^2-2x+3\right)} = \\ &= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right) = \\ &= \frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right) \leq \\ &\leq \frac{1}{3}\left(1+\frac{8x+6}{2}\right) = \frac{1}{3}(4x+4) \end{aligned}$$

Hence

$$f(a)+f(b)+f(c) \leq \frac{1}{3}(4a+4+4b+4+4c+4) = 8$$

as desidered, with equality if and only if $a=b=c$. □

Second Solution. (*See* [40]) We can assume, WLOG, $a+b+c=1$. Then the first term of LHS is equal to

$$f(a) = \frac{(a+1)^2}{2a^2+(1-a)^2} = \frac{a^2+2a+1}{3a^2-2a+1}$$

(When $a=b=c=\frac{1}{3}$, there is equality. A simple sketch of $f(x)$ on $[0,1]$ shows the curve is below the tangent line at $x=\frac{1}{3}$, which has the equation $y=\frac{12x+4}{3}$). So we claim that

$$\frac{a^2+2a+1}{3a^2-2a+1} \leq \frac{12a+4}{3}$$

for $a<0<1$. This inequality is equivalent to

$$36a^3-15a^2-2a+1=(3a-1)^2(4a+1) \geq 0 \quad , \quad 0<a<1$$

hence is true. Adding the similar inequalities for b and c we get the desidered inequality. □

38. (**Russia 2002**) $(x+y+z=3,\ x,y,z>0)$

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx$$

Solution. (*Ercole Suppa*) See Problem 23. □

39. (**Latvia 2002**) $\left(\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1,\ a,b,c,d>0\right)$

$$abcd \geq 3$$

First Solution. (*Ercole Suppa*) We first prove a lemma:

LEMMA. For any real positive numbers x, y with $xy \geq 1$ we have

$$\frac{1}{x^2+1}+\frac{1}{y^2+1} \geq \frac{2}{xy+1}$$

PROOF. The required inequality follows from the identity

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}-\frac{2}{xy+1}=\frac{(x-y)^2(xy-1)}{(x^2+1)(y^2+1)(xy+1)}$$

the proof of which is immediate. $\square$

In order to prove the required inequality we observe at first that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4} \leq 1 \quad \Longrightarrow \quad a^4b^4 \geq 1 \quad \Longrightarrow \quad a^2b^2 \geq 1$$

Thus by previous lemma we have

$$\frac{1}{1+a^4}+\frac{1}{1+b^4} \geq \frac{2}{a^2b^2+1} \tag{1}$$

and similarly

$$\frac{1}{1+c^4}+\frac{1}{1+d^4} \geq \frac{2}{c^2d^2+1} \tag{2}$$

Since $ab \geq 1$ e $cd \geq 1$ we can add (1) and (2) and we can apply again the lemma:

$$\begin{aligned} 1 &= \frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4} \geq \\ &\geq 2\left(\frac{1}{a^2b^2+1}+\frac{1}{c^2d^2+1}\right) \geq \\ &\geq \frac{4}{abdc+1} \end{aligned}$$

Thus $abcd+1 \geq 4$ so $abcd \geq 3$. $\square$

Second Solution. (*See* [32], *pag.* 14) We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying the AM-GM inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3\left(\cos B \cos C \cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3\left(\cos C \cos D \cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3\left(\cos D \cos A \cos B\right)^{\frac{2}{3}}$$

and

$$\sin^2 D \geq 3\left(\cos A \cos B \cos C\right)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! □

40. (**Albania 2002**) $(a, b, c > 0)$

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq a+b+c+\sqrt{a^2+b^2+c^2}$$

Solution. (*Ercole Suppa*) From AM-GM inequality we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} \geq \frac{3\sqrt[3]{a^2b^2c^2}}{abc} = \frac{3}{\sqrt[3]{abc}} \tag{1}$$

From AM-QM inequality we have

$$a+b+c \leq 3\sqrt{\frac{a^2+b^2+c^2}{3}} \tag{2}$$

From (1) and (2) we get

$$\begin{aligned}
\frac{a+b+c+\sqrt{a^2+b^2+c^2}}{(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} &\leq \frac{\frac{3}{\sqrt{3}}\sqrt{a^2+b^2+c^2}+\sqrt{a^2+b^2+c^2}}{(a^2+b^2+c^2)\frac{3}{\sqrt[3]{abc}}} \leq \\
&\leq \frac{3+\sqrt{3}}{\sqrt{3}}\frac{\sqrt{a^2+b^2+c^2}\sqrt[3]{abc}}{3\left(a^2+b^2+c^2\right)} \leq \\
&\leq \frac{3+\sqrt{3}}{3\sqrt{3}}\frac{\sqrt{a^2+b^2+c^2}\sqrt{\frac{a^2+b^2+c^2}{3}}}{a^2+b^2+c^2} = \\
&= \frac{\sqrt{3}+1}{3\sqrt{3}}
\end{aligned}$$

Therefore

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq a+b+c+\sqrt{a^2+b^2+c^2}$$

□

41. (**Belarus 2002**) $(a,b,c,d>0)$

$$\sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}} \geq \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \sqrt{(a+c)^2+(b+d)^2}$$

Solution. (*Sung-Yoon Kim, BoesFX*) Let $A(0,0)$, $B(a,b)$, $C(-c,-d)$ and let D be the foot of perpendicular from A to BC. Since

$$[ABC] = \frac{1}{2}\left|\det\begin{pmatrix} 0 & 0 & 1 \\ a & b & 1 \\ -c & -d & 1 \end{pmatrix}\right| = \frac{1}{2}|ad-bc|$$

we have that

$$AH = \frac{2\,[ABC]}{BC} = \frac{|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$$

So the inequality becomes:

$$BC + 2\cdot AH \geq AB + AC \geq BC$$

$\angle A$ is obtuse, since $A(0,0)$, B is in quadran I, and C is in the third quadrant. Since $\angle A$ is obtuse, $BD+DC$ must be BC. By triangle inequality,

$$AB+AC \geq BC, \qquad BD+AD \geq AB, \qquad DC+AD \geq AC$$

So, $AB+AC \leq BD+DC+2AD = BC+2AD$ and the inequality is proven.

□

42. (**Canada 2002**) $(a,b,c>0)$

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c$$

First Solution. (*Massimo Gobbino - Winter Campus 2006*) We can assume WLOG that $a \geq b \geq c$. Then from the rearrangement inequality we have

$$a^3 \geq b^3 \geq c^3 \ , \ \frac{1}{bc} \geq \frac{1}{ac} \geq \frac{1}{ab} \ \Rightarrow \ \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}$$

and

$$a^2 \geq b^2 \geq c^2 \ , \ \frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a} \ \Rightarrow \ a + b + c \leq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Therefore

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}$$

□

Second Solution. (*Shobber - ML Forum*) By AM-GM, we have

$$\frac{a^3}{bc} + b + c \geq 3a$$

Sum up and done. □

Third Solution. (*Pvthuan - ML Forum*) The inequality is simple applications of $x^2 + y^2 + z^2 \geq xy + yz + zx$ for a^2, b^2, c^2 and $ab, bc, ca,$. We have

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

□

Fourth Solution. (*Davron - ML Forum*) The inequality

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

can be proved by Muirheads Theorem. □

43. (**Vietnam 2002, Dung Tran Nam**) $(a^2 + b^2 + c^2 = 9, \ a, b, c \in \mathbb{R})$

$$2(a + b + c) - abc \leq 10$$

First Solution. (*Nttu - ML Forum*) We can suppose, WLOG, that

$$|a| \leq |b| \leq |c| \quad \Rightarrow \quad c^2 \geq 3 \quad \Rightarrow \quad 2ab \leq a^2 + b^2 \leq 6$$

We have

$$\begin{aligned}
[2(a+b+c) - abc]^2 &= [2(a+b) + c(2-ab)]^2 \leq \qquad \text{(Cauchy-Schwarz)} \\
&\leq \left[(a+b)^2 + c^2\right]\left[2^2 + (2-ab)^2\right] = \\
&= 100 + (ab+2)^2(2ab-7) \leq 100
\end{aligned}$$

Thus

$$2(a+b+c) - abc \leq 10$$

□

Second Solution. (*See* [4], *pag.* 88, *problem* 93) □

44. (**Bosnia and Hercegovina 2002**) $(a^2 + b^2 + c^2 = 1, a, b, c \in \mathbb{R})$

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \leq \frac{3}{5}$$

Solution. (*Arne - ML Forum*) From Cauchy-Schwartz inequality we have

$$1 = \left(a^2 + b^2 + c^2\right)^2 \leq \left(\sum_{\text{cyc}} \frac{a^2}{1+2bc}\right)\left(\sum_{\text{cyc}} a^2(1+2bc)\right) \tag{1}$$

From GM-AM-QM inequality we have:

$$\begin{aligned}
\left(\sum_{\text{cyc}} a^2(1+2bc)\right) &= a^2 + b^2 + c^2 + 2abc(a+b+c) \leq \\
&\leq 1 + 2\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)^3} \cdot 3\sqrt{\frac{a^2+b^2+c^2}{3}} = \qquad (2) \\
&= 1 + \frac{2}{3}\left(a^2+b^2+c^2\right)^2 = \frac{5}{3}
\end{aligned}$$

The required inequality follows from (1) and (2). □

45. (**Junior BMO 2002**) $(a, b, c > 0)$

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

Solution. (*Silouan, Michael Lipnowski - ML Forum*) From AM-GM inequality we have

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\geq\frac{3}{XY}$$

where $X=\sqrt[3]{abc}$ and $Y=\sqrt[3]{(a+b)(b+c)(c+a)}$. By AM-GM again we have that

$$X\leq\frac{a+b+c}{3}$$

and

$$Y\leq\frac{2a+2b+2c}{3}$$

So

$$\frac{3}{XY}\geq\frac{27}{2(a+b+c)^2}$$

and the result follows. □

46. (**Greece 2002**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\geq\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) From Cauchy-Schwarz inequality, $a^2+b^2+c^2=1$ and the well-knon inequality

$$a^2b^2+b^2c^2+c^2a^2\leq\frac{1}{3}\left(a^2+b^2+c^2\right)^2$$

we have

$$\begin{aligned}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 &\leq\left(\sum_{\text{cyc}}\frac{\sqrt{a}}{\sqrt{b^2+1}}\right)^2\leq\\ &\leq\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left(\sum_{\text{cyc}}a^2b^2+a^2\right)=\\ &=\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left(1+a^2b^2+b^2c^2+c^2a^2\right)\leq\\ &\leq\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\left[1+\frac{1}{3}\left(a^2+b^2+c^2\right)\right]=\\ &=\frac{4}{3}\left(\sum_{\text{cyc}}\frac{a}{b^2+1}\right)\end{aligned}$$

Hence

$$\left(\sum_{\text{cyc}} \frac{a}{b^2+1}\right) \geq \frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

$\square$

47. (**Greece 2002**) ($bc \neq 0,\ \frac{1-c^2}{bc} \geq 0,\ a,b,c \in \mathbb{R}$)

$$10(a^2+b^2+c^2-bc^3) \geq 2ab+5ac$$

Solution. (*Ercole Suppa*) At first we observe that $\frac{1-c^2}{bc} \geq 0$ if and only if $bc\left(1-c^2\right) \geq 0$. Thus:

$$\begin{aligned}&10\left(a^2+b^2+c^2-bc^3\right)-2ab-5ac= \\ &=5(b-c)^2+\frac{5}{2}(a-c)^2+(a-b)^2+10bc\left(1-c^2\right)+4b^2+\frac{5}{2}c^2+\frac{13}{2}a^2 \geq 0\end{aligned}$$

$\square$

48. (**Taiwan 2002**) $\left(a,b,c,d \in \left(0,\frac{1}{2}\right]\right)$

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)} \leq \frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

Solution. (*Liu Janxin - ML Forum*) We first prove two auxiliary inequalities:

Lemma 1. If $a,b \in \left[0,\frac{1}{2}\right[$ we have

$$\frac{a^2+b^2}{ab} \geq \frac{(1-a)^2+(1-b)^2}{(1-a)(1-b)}$$

Proof. Since $1-a-b \geq 0$ (bacause $0 \leq a,b \leq \frac{1}{2}$) we get

$$\frac{a^2+b^2}{ab}-\frac{(1-a)^2+(1-b)^2}{(1-a)(1-b)}=\frac{(1-a-b)(a-b)^2}{ab(1-a)(1-b)} \geq 0$$

Lemma 2. If $a,b,c,d \in \left[0,\frac{1}{2}\right[$ we have

$$\frac{\left(a^2-b^2\right)^2}{abcd} \geq \frac{\left((1-a)^2-(1-b)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

Proof. Since $0 \leq c, d \leq \frac{1}{2}$ we get

$$\frac{(1-c)(1-d)}{cd} \geq 1 \tag{1}$$

Since $0 \leq a, b \leq \frac{1}{2}$ we get

$$\frac{\left(a^2-b^2\right)^2}{ab} - \frac{\left((1-a)^2-(1-b)^2\right)^2}{(1-a)(1-b)} = \frac{(a-b)^4(1-a-b)}{ab(1-a)(1-b)} \geq 0$$

Therefore

$$\frac{\left(a^2-b^2\right)^2}{ab} \geq \frac{\left((1-a)^2-(1-b)^2\right)^2}{(1-a)(1-b)} \tag{2}$$

Multiplying (1) and (2) we have

$$\frac{\left(a^2-b^2\right)^2}{abcd} \geq \frac{\left((1-a)^2-(1-b)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

and the Lemma 2 is proven. $\square$

Now we can prove the required inequality. By Lemma 2, we have

$$\begin{aligned}
&\frac{a^4+b^4+c^4+d^4}{abcd} - \frac{\left(a^2+b^2\right)\left(b^2+c^2\right)}{abcd} = \\
&= \frac{\left(a^2-c^2\right)^2+\left(a^2-d^2\right)^2+\left(b^2-c^2\right)^2+\left(b^2-d^2\right)^2}{2abcd} \geq \\
&\geq \frac{\left((1-a)^2-(1-c)^2\right)^2+\left((1-a)^2-(1-d)^2\right)^2+\left((1-b)^2-(1-c)^2\right)^2+\left((1-b)^2-(1-d)^2\right)^2}{2(1-a)(1-b)(1-c)(1-d)} = \\
&= \frac{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}{(1-a)(1-b)(1-c)(1-d)} - \frac{\left((1-a)^2+(1-b)^2\right)\left((1-c)^2+(1-d)^2\right)}{(1-a)(1-b)(1-c)(1-d)}
\end{aligned}$$

By Lemma 1, we have

$$\frac{\left(a^2+b^2\right)\left(b^2+c^2\right)}{abcd} \geq \frac{\left((1-a)^2+(1-b)^2\right)\left((1-c)^2+(1-d)^2\right)}{(1-a)(1-b)(1-c)(1-d)}$$

Thus, addingthe last two inequalities, we get

$$\frac{a^4+b^4+c^4+d^4}{abcd} \geq \frac{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}{(1-a)(1-b)(1-c)(1-d)}$$

and the desidered inequality follows:

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)} \leq \frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

$\square$

49. (**APMO 2002**) ($\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,\ x,y,z>0$)

$$\sqrt{x+yz}+\sqrt{y+zx}+\sqrt{z+xy}\ge\sqrt{xyz}+\sqrt{x}+\sqrt{y}+\sqrt{z}$$

Solution. (*Suat Namly*) Multiplying by $\sqrt{xyz}$, we have

$$\sqrt{xyz}=\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}$$

So it is enough to prove that

$$\sqrt{z+xy}\ge\sqrt{z}+\sqrt{\frac{xy}{z}}$$

By squaring, this is equivalent to

$$\begin{aligned}
&z+xy\ge z+\frac{xy}{z}+2\sqrt{xy} && \iff\\
&z+xy\ge z+xy\left(1-\frac{1}{x}-\frac{1}{y}\right)+2\sqrt{xy} && \iff\\
&x+y\ge 2\sqrt{xy} && \iff\\
&\left(\sqrt{x}-\sqrt{y}\right)^2\ge 0
\end{aligned}$$

□

50. (**Ireland 2001**) ($x+y=2,\ x,y\ge 0$)

$$x^2y^2(x^2+y^2)\le 2.$$

First Solution. (*Soarer - ML Forum*)

$$x^2y^2(x^2+y^2)=x^2y^2(4-2xy)=2x^2y^2(2-xy)\le 2(1)\left(\frac{xy+2-xy}{2}\right)^2=2$$

□

Second Solution. (*Pierre Bornzstein - ML Forum*) WLOG, we may assume that $x\le y$ so that $x\in[0,1]$. Now

$$x^2y^2(x^2+y^2)=x^2(2-x)^2(x^2+(2-x)^2)=f(x)$$

Straighforward computations leads to

$$f'(x)=4x(1-x)(2-x)(2x^2-6x+4)\ge 0$$

Thus f is increasing on $[0;1]$. Since $f(1)=2$, the result follows. Note that equality occurs if and only if $x=y=1$.

□

Third Solution. (*Kunny - ML Forum*) We can set

$$x = 2\cos^2\theta, y = 2\sin^2\theta$$

so we have

$$x^2y^2(x^2 + y^2) = 2 - 2\cos^4 2\theta \leqq 2$$

$\square$

51. (**BMO 2001**) $(a + b + c \geq abc,\ a, b, c \geq 0)$

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc$$

First Solution. (*Fuzzylogic - ML Forum*) From the well-know inequality

$$(x + y + z)^2 \geq 3(xy + yz + zx)$$

by putting $x = bc, y = ca, z = ab$ we get

$$ab + bc + ca \geq \sqrt{3abc(a + b + c)}$$

Then

$$a^2 + b^2 + c^2 \geq ab + bc + ca \geq \sqrt{3abc(a + b + c)} \geq abc\sqrt{3}$$

$\square$

Second Solution. (*Cezar Lupu - ML Forum*) Let's assume by contradiction that

$$a^2 + b^2 + c^2 < abc\sqrt{3}$$

By applying Cauchy-Schwarz inequality, $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$ and the hipothesys $a + b + c \geq abc$ we get

$$abc < 3\sqrt{3}$$

On the other hand , by AM-GM we have

$$abc\sqrt{3} > a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2}$$

We get from here $abc > 3\sqrt{3}$, a contradiction.

$\square$

Third Solution. (*Cezar Lupu - ML Forum*) We have

$$a + b + c \geq abc \Leftrightarrow \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq 1$$

We shall prove a stronger inequality

$$ab + bc + ca \geq abc\sqrt{3} \Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{3}$$

Now, let us denote $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ and the problems becomes:
If x, y, z are three nonnegative real numbers such that $xy + yz + zx \geq 1$, then the following holds:

$$x + y + z \geq \sqrt{3}$$

But, this last problem follows immediately from this inequality

$$(x + y + z)^2 \geq 3(xy + yz + zx)$$

□

52. (**USA 2001**) $(a^2 + b^2 + c^2 + abc = 4,\ a, b, c \geq 0)$

$$0 \leq ab + bc + ca - abc \leq 2$$

First Solution. (*Richard Stong, see* [9] *pag.* 22) From the given condition, at least one of a, b, c does not exceed 1, say $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0$$

It is easy to prove that the equality holds if and only if (a, b, c) is one of the triples $(2, 0, 0)$, $(0, 2, 0)$ or $(0, 0, 2)$.
To prove the upper bound we first note that some two of three numbers a, b, c are both greater than or equal to 1 or less than or equal to 1. WLOG assume that the numbers with this property are b and c. Then we have

$$(1 - b)(1 - c) \geq 0 \tag{1}$$

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \geq 2bc$ imply

$$a^2 + 2bc + abc \leq 4 \quad \Longleftrightarrow \quad bc(2 + a) \leq 4 - a^2$$

Dividing both sides of the last inequality by $2 + a$ yelds

$$bc \leq 2 - a \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} ab+bc+ac-abc \le ab+2-a+ac(1-b) &= \\ = 2-a(1+bc-b-c) &= \\ = 2-a(1-b)(1-c) &\le 2 \end{aligned}$$

as desidered. The last equality holds if and only if $b=c$ and $a(1-b)(1-c)=0$. Hence, equality for upper bound holds if and only if (a,b,c) is one of the triples $(1,1,1)$, $(0,\sqrt{2},\sqrt{2})$, $(\sqrt{2},0,\sqrt{2})$ and $(\sqrt{2},\sqrt{2},0)$. $\square$

Second Solution. (*See* [62]) Assume WLOG $a \ge b \ge c$. If $c > 1$, then $a^2+b^2+c^2+abc > 1+1+1+1 = 4$, contradiction. So $c \le 1$. Hence $ab+bc+ca \ge ab \ge abc$.
Put $a = u+v$, $b = u-v$, so that $u, v = 0$. Then the equation given becomes

$$(2+c)u^2 + (2-c)v^2 + c^2 = 4$$

So if we keep c fixed and reduce v to nil, then we must increase u. But $ab+bc+ca-abc = (u^2-v^2)(1-c)+2cu$, so decreasing v and increasing u has the effect of increasing $ab+bc+ca-abc$. Hence $ab+bc+ca-abc$ takes its maximum value when $a=b$. But if $a=b$, then the equation gives $a=b=\sqrt{2-c}$. So to establish that $ab+bc+ca-abc \le 2$ it is sufficient to show that $2-c+2c\sqrt{2-c} = 2+c(2-c)$. Evidently we have equality if $c=0$. If c is non-zero, then the relation is equivalent to $2\sqrt{2-c} \le 3-c$ or $(c-1)^2 \ge 0$. Hence the relation is true and we have equality only for $c=0$ or $c=1$. $\square$

53. (**Columbia 2001**) $(x, y \in \mathbb{R})$

$$3(x+y+1)^2 + 1 \ge 3xy$$

Solution. (*Ercole Suppa*) After setting $x=y$ we have

$$3(2x+1)^2 + 1 - 3x^2 \ge 0 \quad \Longleftrightarrow \quad (3x+2)^2 \ge 0 \tag{1}$$

where the equality holds if $x = -\frac{2}{3}$. This suggest the following change of variable

$$3x+2 = a \qquad , \qquad 3y+2 = b$$

Now for all $x, y \in \mathbb{R}$ we have:

$$\begin{aligned}
3\left(x+y+1\right)^2+1-3xy &= 3\left(\frac{a+b-4}{3}+1\right)^2+1-3\frac{\left(a-2\right)\left(b-2\right)}{9} = \\
&= \frac{\left(a+b-1\right)^2}{3}+1-\frac{ab-2a-2b+4}{3} = \\
&= \frac{a^2+b^2+ab}{3} = \\
&= \frac{a^2+b^2+\left(a+b\right)^2}{6} = \\
&= \frac{\left(3x+2\right)^2+\left(3y+2\right)^2+\left[3\left(x+y\right)+4\right]^2}{6} \geq 0
\end{aligned}$$

$\square$

54. (**KMO Winter Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right)} \geq abc+\sqrt[3]{\left(a^3+abc\right)\left(b^3+abc\right)\left(c^3+abc\right)}$$

First Solution. (*See* [32], *pag.* 38) Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)} \geq abc+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$, we obtain the constraint $xyz=1$. It takes the form

$$\sqrt{\left(x+y+z\right)\left(xy+yz+zx\right)} \geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

From the constraint $xyz=1$, we find two identities

$$\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)=\left(\frac{x+z}{z}\right)\left(\frac{y+x}{x}\right)\left(\frac{z+y}{y}\right)=(z+x)(x+y)(y+z),$$

$$\left(x+y+z\right)\left(xy+yz+zx\right)=(x+y)(y+z)(z+x)+xyz=(x+y)(y+z)(z+x)+1.$$

Letting $p=\sqrt[3]{(x+y)(y+z)(z+x)}$, the inequality now becomes $\sqrt{p^3+1} \geq 1+p$. Applying the AM-GM inequality, we have $p \geq \sqrt[3]{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}}=2$. It follows that $(p^3+1)-(1+p)^2=p(p+1)(p-2) \geq 0$. $\square$

Second Solution. (*Based on work by an winter program participant, see* [32] *pag.* 43).

□

55. (**IMO 2001**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) Let T is the left hand side of the inequality. We have

$$\begin{aligned}
(a+b+c)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{a}}{\sqrt[4]{a^2+8bc}} \sqrt{a}\sqrt[4]{a^2+8bc}\right)^2 \leq && \text{(Cauchy-Schwarz)} \\
&\leq T \cdot \left(\sum_{\text{cyc}} a\sqrt{a^2+8bc}\right) = \\
&\leq T \cdot \left(\sum_{\text{cyc}} \sqrt{a}\sqrt{a}\sqrt{a^2+8bc}\right) \leq && \text{(Cauchy-Schwarz)} \\
&\leq T \cdot (a+b+c)^{\frac{1}{2}} \left(\sum_{\text{cyc}} a^3 + 8abc\right)^{\frac{1}{2}} = \\
&= T \cdot (a+b+c)^{\frac{1}{2}} \left(a^3+b^3+c^3+24abc\right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$T \geq \frac{(a+b+c)^{\frac{3}{2}}}{(a^3+b^3+c^3+24abc)^{\frac{1}{2}}} \geq 1$$

where in the last step we used the inequality

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc$$

which is true by BUNCHING, since

$$
\begin{aligned}
&(a+b+c)^3 \geq a^3+b^3+c^3+24abc && \iff \\
&3\left(\sum_{\text{sym}} a^2b\right)+6abc \geq 24abc && \iff \\
&\sum_{\text{sym}} a^2b \geq 6abc && \iff \\
&\sum_{\text{sym}} a^2b \geq \sum_{\text{sym}} abc
\end{aligned}
$$

$\square$

2 Years 1996 ~ 2000

56. (**IMO 2000, Titu Andreescu**) ($abc = 1,\ a, b, c > 0$)

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

Solution. (*See* [32], *pag.* 3) Since $abc = 1$, we make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for x, y, $z > 0$. We rewrite the given inequality in the terms of x, y, z :

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1 \quad \Leftrightarrow$$
$$xyz \geq (y + z - x)(z + x - y)(x + y - z)$$

This is true by Schur inequality. □

Remark. Alternative solutions are in [32], pag. 18, 19.

57. (**Czech and Slovakia 2000**) ($a, b > 0$)

$$\sqrt[3]{2(a+b)\left(\frac{1}{a} + \frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$$

First Solution. (*Massimo Gobbino - Winter Campus 2006*) After setting $a = x^3$ a $b = y^3$ the required inequality become

$$\frac{x}{y} + \frac{y}{x} \leq \sqrt[3]{2\left(x^3 + y^3\right)\left(\frac{1}{x^3} + \frac{1}{y^3}\right)}$$
$$\frac{x^2 + y^2}{xy} \leq \frac{1}{xy}\sqrt[3]{2\left(x^3 + y^3\right)^2}$$
$$\left(x^2 + y^2\right)^3 \leq 2\left(x^3 + y^3\right)^2$$
$$\left(x^2 + y^2\right)^{\frac{1}{2}} \leq 2^{\frac{1}{6}}\left(x^3 + y^3\right)^{\frac{1}{3}}$$
$$\sqrt{\frac{x^2 + y^2}{2}} \leq \sqrt[3]{\frac{x^3 + y^3}{2}}$$

which is true by Power Mean inequality. The equality holds if $x = y$, i.e. if $a = b$. □

Second Solution. (*Official solution.*) Elevating to the third power both members of the given inequality we get the equivalent inequality

$$\frac{a}{b} + 3\sqrt[3]{\frac{a}{b}} + 3\sqrt[3]{\frac{b}{a}} + \frac{b}{a} \geq 4 + 2\frac{a}{b} + 2\frac{b}{a}$$

that is

$$\frac{a}{b} + \frac{b}{a} + 4 \geq 3\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}\right)$$

The AM-GM inequality applied to the numbers $\frac{a}{b}, 1, 1$ implies

$$\frac{a}{b} + 1 + 1 \geq 3\sqrt[3]{\frac{a}{b}}$$

Similarly we have

$$\frac{b}{a} + 1 + 1 \geq 3\sqrt[3]{\frac{b}{a}}$$

Adding the two last inequalities we get the required result. □

58. (**Hong Kong 2000**) ($abc = 1,\ a, b, c > 0$)

$$\frac{1+ab^2}{c^3} + \frac{1+bc^2}{a^3} + \frac{1+ca^2}{b^3} \geq \frac{18}{a^3+b^3+c^3}$$

First Solution. (*Official solution*) Apply Cauchy-Scwarz Inequality, we have

$$\left(\frac{1+ab^2}{c^3} + \frac{1+bc^2}{a^3} + \frac{1+ca^2}{b^3}\right)\left(c^3+a^3+b^3\right) \geq \left(\sum_{\text{cyc}} \sqrt{1+ab^2}\right)^2$$

It remain to prove

$$\sum_{\text{cyc}} \sqrt{1+ab^2} \geq \sqrt{18}$$

The proof goes as follows

$$\begin{aligned}
&\sqrt{1+ab^2} + \sqrt{1+bc^2} + \sqrt{1+ca^2} \geq \\
&\geq \sqrt{(1+1+1)^2 + \left(\sqrt{ab^2} + \sqrt{bc^2} + \sqrt{ca^2}\right)^2} \geq \qquad \text{(Minkowski Ineq)} \\
&\geq \sqrt{9 + \left(3\sqrt{abc}\right)^2} = \qquad \text{(AM-GM Ineq)} \\
&= \sqrt{18}
\end{aligned}$$

□

Second Solution. (*Ercole Suppa*) From AM-HM inequality we have

$$\frac{1}{c^3}+\frac{1}{a^3}+\frac{1}{b^3}\geq\frac{9}{a^3+b^3+c^3} \tag{1}$$

and

$$\frac{ab^2}{c^3}+\frac{bc^2}{a^3}+\frac{ca^2}{b^3}\geq 3\sqrt[3]{\frac{a^3b^3c^3}{a^3b^3c^3}}=\frac{9}{3\sqrt[3]{a^3b^3c^3}}\geq\frac{9}{a^3+b^3+c^3} \tag{2}$$

Adding (1) and (2) we get the required inequality. □

59. (**Czech Republic 2000**) $(m,n\in N,\ x\in[0,1])$

$$(1-x^n)^m+(1-(1-x)^m)^n\geq 1$$

Solution. (*See* [61] *pag.* 83) The given inequality follow from the following most general result:

Let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ be nonnegative real numbers such that $x_i+y_i=1$ for each $i=1,2,\ldots,n$. Prove that

$$(1-x_1x_2\cdots x_n)^m+(1-y_1^m)(1-y_2^m)\cdots(1-y_n^m)\geq 1$$

We use the following probabilistic model suggested by the circumstance that $x_i+y_i=1$. Let n unfair coins. Let x_i be the probability that a toss of the $i-$th coin is a head $(i=1,2,\ldots,n)$. Then the probability that a toss of this coin is a tail equals $1-x_i=y_i$.

The probability of n heads in tossing all the coins once is $x_1x_2\cdots x_n$, because the events are independent. Hence $1-x_1x_2\cdots x_n$ is the probability of at least one tail. Consequently, the probability of *at least one tail in each of m consecutive tosses af all the coins* equals

$$(1-x_1x_2\cdots x_n)^m$$

With probability y_i^m, each of m consecutive tosses of the i-th coin is a tail; with probability $1-y_i^m$, we have at least one head. Therefore the probability that *after m tosses of all coins each coin has been a head at least once* equals

$$(1-y_1^m)(1-y_2^m)\cdots(1-y_n^m)$$

Denote the events given above in italics by A and B, respectively. It is easy to observe that *at leat one of them must occur* as a result of m tosses. Indeed, suppose A has not occurred. This means that the outcome of some toss has been n heads, which implies that B has occurred. Now we need a line more to finish the proof. Since one of the events A and B occurs as a result of m tosses, the sum of their probabilities is greater than or equal to 1, that is

$$(1 - x_1 x_2 \cdots x_n)^m + (1 - y_1^m)(1 - y_2^m)\cdots(1 - y_n^m) \geq 1$$

$\square$

Remark. Murray Klamkin - Problem 68-1 (SIAM Review 11(1969)402-406).

60. (**Macedonia 2000**) $(x, y, z > 0)$

$$x^2 + y^2 + z^2 \geq \sqrt{2}\,(xy + yz)$$

Solution. (*Ercole Suppa*) By AM-GM inequality we have

$$\begin{aligned} x^2 + y^2 + z^2 &= x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 + z^2 \geq \\ &\geq 2x\frac{y}{\sqrt{2}} + 2\frac{y}{\sqrt{2}}z = \\ &= \sqrt{2}xy + \sqrt{2}yz = \\ &= \sqrt{2}\,(xy + yz) \end{aligned}$$

$\square$

61. (**Russia 1999**) $(a, b, c > 0)$

$$\frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3$$

First Solution. (*Anh Cuong - ML Forum*)
First let $f(a, b, c) = \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ac}{a^2+c^2} + \frac{c^2+2ab}{a^2+b^2}$. We will prove that:

$$f(a, b, c) \geq \frac{2bc}{b^2 + c^2} + \frac{b}{c} + \frac{c}{b}$$

Suppose that: $b \geq c \geq a$. Since

$$\frac{a^2 + 2bc}{b^2 + c^2} \geq \frac{2bc}{b^2 + c^2}$$

we just need to prove that:

$$\frac{b^2 + 2ac}{a^2 + c^2} + \frac{c^2 + 2ab}{a^2 + b^2} \geq \frac{b}{c} + \frac{c}{b}$$

We have:

$$\begin{aligned}
&\frac{b^2+2ac}{a^2+c^2}+\frac{c^2+2ab}{a^2+b^2}-\frac{b}{c}-\frac{c}{b}=\\
&=\frac{b^3+2abc-c^3-ca^2}{b(c^2+a^2)}+\frac{c^3+2abc-b^3-ba^2}{c(b^2+a^2)}\geq\\
&\geq\frac{b^3-c^3}{b(a^2+c^2)}+\frac{c^3-b^3}{c(a^2+b^2)}=\\
&=\frac{\left(bc-a^2\right)(b-c)^2\left(b^2+bc+c^2\right)}{bc\left(a^2+b^2\right)\left(a^2+c^2\right)}\geq 0
\end{aligned}$$

Hence:

$$f(a,b,c)\geq\frac{2bc}{b^2+c^2}+\frac{b}{c}+\frac{c}{b}$$

But

$$\frac{2bc}{b^2+c^2}+\frac{b}{c}+\frac{c}{b}\geq 3\Leftrightarrow(b-c)^2(b^2+c^2-bc)\geq 0.$$

So we have done now. $\square$

Second Solution. (*Charlie- ML Forum*)
Brute force proof: Denote $T(x,y,z)=\sum_{\text{sym}}a^xb^yc^z$. Expanding and simplifying yields

$$\frac{1}{2}\cdot T(6,0,0)+T(4,1,1)+2\cdot T(3,2,1)+T(3,3,0)\geq 2\cdot T(4,2,0)+\frac{1}{2}\cdot T(2,2,2)$$

which is true since

$$\frac{1}{2}\cdot T(6,0,0)+\frac{1}{2}\cdot T(4,1,1)\geq T(5,1,0)$$

by Schur's inequality, and

$$T(5,1,0)+T(3,3,0)\geq 2\cdot T(4,2,0)$$

by AM-GM ($a^5b+a^3b^3\geq 2a^4b^2$), and

$$2\cdot T(3,2,1)\geq 2\cdot T(2,2,2)\geq\frac{1}{2}\cdot T(2,2,2)$$

by bunching. $\square$

Third Solution. (*Darij Grinberg - ML Forum*)
Using the $\sum_{\text{cyc}}$ notation for cyclic sums, the inequality in question rewrites as

$$\sum_{\text{cyc}}\frac{a^2+2bc}{b^2+c^2}>3$$

But

$$\sum_{\text{cyc}} \frac{a^2 + 2bc}{b^2 + c^2} - 3 = \sum_{\text{cyc}} \left(\frac{a^2 + 2bc}{b^2 + c^2} - 1 \right) =$$

$$= \sum_{\text{cyc}} \frac{a^2}{b^2 + c^2} - \sum_{\text{cyc}} \frac{(b - c)^2}{b^2 + c^2}$$

Thus, we have to show that

$$\sum_{\text{cyc}} \frac{a^2}{b^2 + c^2} > \sum_{\text{cyc}} \frac{(b - c)^2}{b^2 + c^2}$$

Now, by the Cauchy-Schwarz inequality in the Engel form, we have

$$\sum_{\text{cyc}} \frac{a^2}{b^2 + c^2} = \sum_{\text{cyc}} \frac{\left(a^2\right)^2}{a^2 b^2 + c^2 a^2} \geq$$

$$\geq \frac{\left(a^2 + b^2 + c^2\right)^2}{(a^2 b^2 + c^2 a^2) + (b^2 c^2 + a^2 b^2) + (c^2 a^2 + b^2 c^2)} =$$

$$= \frac{\left(a^2 + b^2 + c^2\right)^2}{2\left(b^2 c^2 + c^2 a^2 + a^2 b^2\right)}$$

Hence, it remains to prove that

$$\frac{\left(a^2 + b^2 + c^2\right)^2}{2\left(b^2 c^2 + c^2 a^2 + a^2 b^2\right)} > \sum_{\text{cyc}} \frac{(b - c)^2}{b^2 + c^2}$$

i. e. that

$$\left(a^2 + b^2 + c^2\right)^2 > 2\left(b^2 c^2 + c^2 a^2 + a^2 b^2\right) \sum_{\text{cyc}} \frac{(b - c)^2}{b^2 + c^2}$$

Now,

$$2\left(b^2 c^2 + c^2 a^2 + a^2 b^2\right) \sum_{\text{cyc}} \frac{(b - c)^2}{b^2 + c^2} = \sum_{\text{cyc}} \frac{2\left(b^2 c^2 + c^2 a^2 + a^2 b^2\right)}{b^2 + c^2} (b - c)^2 =$$

$$= \sum_{\text{cyc}} \left(\frac{2b^2 c^2}{b^2 + c^2} + 2a^2 \right) (b - c)^2$$

The HM-GM inequality, applied to the numbers b^2 and c^2, yields

$$\frac{2b^2 c^2}{b^2 + c^2} \leq \sqrt{b^2 c^2} = bc$$

thus,

$$2\left(b^2c^2+c^2a^2+a^2b^2\right)\sum_{\text{cyc}}\frac{(b-c)^2}{b^2+c^2}=\sum_{\text{cyc}}\left(\frac{2b^2c^2}{b^2+c^2}+2a^2\right)(b-c)^2\leq$$
$$\leq\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$$

Hence, instead of proving

$$\left(a^2+b^2+c^2\right)^2>2\left(b^2c^2+c^2a^2+a^2b^2\right)\sum_{\text{cyc}}\frac{(b-c)^2}{b^2+c^2}$$

it will be enough to show the stronger inequality

$$\left(a^2+b^2+c^2\right)^2>\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$$

With a bit of calculation, this is straightforward; here is a longer way to show it without great algebra:

$$\begin{aligned}
&\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2=\\
&=\sum_{\text{cyc}}\left(a(a+b+c)-(c-a)(a-b)\right)(b-c)^2=\\
&=\sum_{\text{cyc}}a(a+b+c)(b-c)^2-\sum_{\text{cyc}}(c-a)(a-b)(b-c)^2=\\
&=(a+b+c)\sum_{\text{cyc}}a(b-c)^2-(b-c)(c-a)(a-b)\underbrace{\sum_{\text{cyc}}(b-c)}_{=0}=\\
&=(a+b+c)\sum_{\text{cyc}}a(b-c)^2=\\
&=(a+b+c)\sum_{\text{cyc}}a\left((b-c)(b-a)+(c-a)(c-b)\right)=\\
&=(a+b+c)\left(\sum_{\text{cyc}}a(b-c)(b-a)+\sum_{\text{cyc}}a(c-a)(c-b)\right)=\\
&=(a+b+c)\left(\sum_{\text{cyc}}c(a-b)(a-c)+\sum_{\text{cyc}}b(a-b)(a-c)\right)=\\
&=(a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c)
\end{aligned}$$

Thus, in order to prove that $\left(a^2+b^2+c^2\right)^2>\sum_{\text{cyc}}\left(bc+2a^2\right)(b-c)^2$, we will show the equivalent inequality

$$\left(a^2+b^2+c^2\right)^2>(a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c)$$

In fact, we will even show the stronger inequality

$$\left(a^2+b^2+c^2\right)^2 > \sum_{\text{cyc}} a^2(a-b)(a-c) + (a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c)$$

which is indeed stronger since $\sum_{\text{cyc}} a^2(a-b)(a-c) \geq 0$ by the Schur inequality.
Now, this stronger inequality can be established as follows:

$$\sum_{\text{cyc}} a^2(a-b)(a-c) + (a+b+c)\sum_{\text{cyc}}(b+c)(a-b)(a-c) =$$

$$= \sum_{\text{cyc}}\left(a^2+(a+b+c)(b+c)\right)(a-b)(a-c) =$$

$$= \sum_{\text{cyc}}\left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)+bc\right)(a-b)(a-c) =$$

$$= \left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)\right)\underbrace{\sum_{\text{cyc}}(a-b)(a-c)}_{=(a^2+b^2+c^2)-(bc+ca+ab)} + \sum_{\text{cyc}} bc\underbrace{(a-b)(a-c)}_{=a^2+bc-ca-ab<2a^2+bc} <$$

$$< \left(\left(a^2+b^2+c^2\right)+(bc+ca+ab)\right)\left(\left(a^2+b^2+c^2\right)-(bc+ca+ab)\right) + \sum_{\text{cyc}} bc\left(2a^2+bc\right) =$$

$$= \left(\left(a^2+b^2+c^2\right)^2-(bc+ca+ab)^2\right)+(bc+ca+ab)^2 = \left(a^2+b^2+c^2\right)^2$$

and the inequality is proven.

□

62. (**Belarus 1999**) $(a^2+b^2+c^2=3,\ a,b,c>0)$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca} \geq \frac{3}{2}$$

Solution. (*Ercole Suppa*) From Cauchy-Schwartz inequality we have

$$9 = \left(a^2+b^2+c^2\right)^2 \leq \left(\sum_{\text{cyc}}\frac{1}{1+bc}\right)\left(\sum_{\text{cyc}} a^2(1+bc)\right) \qquad (1)$$

From GM-AM-QM inequality we have:

$$\begin{aligned}\left(\sum_{\text{cyc}} a^2(1+bc)\right) &= a^2+b^2+c^2+abc(a+b+c) \leq \\ &\leq 3+\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)^3}\cdot 3\sqrt{\frac{a^2+b^2+c^2}{3}} = \qquad (2)\\ &= 3+\frac{1}{3}\left(a^2+b^2+c^2\right)^2 = 3+3 = 6\end{aligned}$$

The required inequality follows from (1) and (2). □

63. (**Czech-Slovak Match 1999**) $(a, b, c > 0)$

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

Solution. (*Ercole Suppa*) Using Cauchy-Schwartz inequality and the well-know

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

we have

$$\begin{aligned}(a+b+c)^2 &\leq \sum_{cyc} \frac{a}{b+2c} \cdot \sum_{cyc} a(b+2c) = \qquad\qquad \text{(Cauchy-Schwarz)}\\ &= \sum_{cyc} \frac{a}{b+2c} \cdot 3(ab+bc+ca) \leq \\ &\leq \sum_{cyc} \frac{a}{b+2c} \cdot (a+b+c)^2\end{aligned}$$

Dividing for $(a+b+c)^2$ we get the result. □

64. (**Moldova 1999**) $(a, b, c > 0)$

$$\frac{ab}{c(c+a)} + \frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} \geq \frac{a}{c+a} + \frac{b}{b+a} + \frac{c}{c+b}$$

First Solution. (*Ghang Hwan, Bodom - ML Forum*)
After the substitution $x = c/a$, $y = a/b$, $z = b/c$ we get $xyz = 1$ and the inequality becomes

$$\frac{z}{x+1} + \frac{x}{y+1} + \frac{y}{z+1} \geq \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$$

Taking into account that $xyz = 1$, this inequality can be rewritten as

$$\frac{z-1}{x+1} + \frac{x-1}{y+1} + \frac{y-1}{z+1} \geq 0 \quad \Longleftrightarrow$$

$$yz^2 + zx^2 + xy^2 + x^2 + y^2 + z^2 \geq x + y + z + 3 \qquad (*)$$

The inequality $(*)$ is obtained summing the well-know inequality

$$x^2 + y^2 + z^2 \geq x + y + z$$

and

$$yz^2 + zx^2 + xy^2 \geq 3\sqrt[3]{x^3y^3z^3} = 3xyz = 3$$

which follows from the AM-GM inequality. □

Second Solution. (*Gibbenergy - ML Forum*) We have

$$L-R=\frac{abc\left[\left(\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ac}{b^2}-3\right)+\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}\right)-\left(\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)\right]}{(a+b)(b+c)(c+a)}\geq 0$$

because

$$\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ac}{b^2}-3\geq 0$$

by AM-GM inequality and

$$\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}\right)-\left(\frac{b}{c}+\frac{c}{a}+\frac{a}{b}\right)\geq 0$$

by the well-know inequality $x^2+y^2+z^2\geq x+y+z$. □

65. (**United Kingdom 1999**) $(p+q+r=1,\ p,q,r>0)$

$$7(pq+qr+rp)\leq 2+9pqr$$

First Solution. (*Ercole Suppa*) From Schur inequality we have

$$(p+q+r)^3+9pqr\geq 4(p+q+r)(pq+qr+rp)$$

Therefore, since $p+q+r=1$, we obtain

$$1+9pqr\geq 4(pq+qr+rp)$$

Hence

$$\begin{aligned}2+9pqr-7(pq+qr+rp)&\geq 2+4(pq+qr+rp)-1-7(pq+qr+rp)=\\&=1-3(pq+qr+rp)=\\&=(p+q+r)^2-3(pq+qr+rp)=\\&=\frac{1}{2}\left[(p-q)^2+(q-r)^2+(r-p)^2\right]\geq 0\end{aligned}$$

and the inequality is proven. □

Second Solution. (*See* [8] *pag.* 189)
Because $p+q+r=1$ the inequality is equivalent to

$$\begin{aligned}&7(pq+qr+rp)(p+q+r)\leq 2(p+q+r)^3+9pqr &&\iff\\&7\sum_{\text{cyc}}\left(p^2q+pq^2+pqr\right)\leq 9pqr+\sum_{\text{cyc}}\left(2p^3+6p^2q+6pq^2+4pqr\right) &&\iff\\&\sum_{\text{cyc}}p^2q+\sum_{\text{cyc}}pq^2\leq\sum_{\text{cyc}}2p^3=\sum_{\text{cyc}}\frac{2p^3+q^3}{3}+\sum_{\text{cyc}}\frac{p^3+2q^3}{3}\end{aligned}$$

This last inequality is true by weighted AM-GM inequality. □

66. (**Canada 1999**) $(x+y+z=1,\ x,y,z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

First Solution. (*See* [8] *pag.* 42)
Assume WLOG that $x=max(x,y,z)$. If $x\geq y\geq z$, then

$$x^2y+y^2z+z^2x\leq x^2y+y^2z+z^2x+z\left[xy+(x-y)(y-z)\right]=$$
$$=(x+y)^2y=4\left(\frac{1}{2}-\frac{1}{2}y\right)\left(\frac{1}{2}-\frac{1}{2}y\right)y\leq\frac{4}{27}$$

where the last inequality follows from AM-GM inequality. Equality occurs if and only if $z=0$ (from the first inequality) and $y=\frac{1}{3}$, in which case $(x,y,z)=\left(\frac{2}{3},\frac{1}{3},0\right)$.
If If $x\geq y\geq z$, then

$$x^2y+y^2z+z^2x\leq x^2z+z^2y+y^2x-(x-z)(z-y)(x-y)\leq$$
$$\leq x^2z+z^2y+y^2x\leq\frac{4}{27}$$

where the second inequality is true from the result we proved for $x\geq y\geq z$ (except with y and z reversed. Equality holds in the first inequality only when two of x, y, z are equal, and in the second inequality only when $(x,z,y)=\left(\frac{2}{3},\frac{1}{3},0\right)$. Because these conditions can't both be true, the inequality is actually strict in this case.
Therefore the inequality is indeed true, and the equality olds when (x,y,z) equals $\left(\frac{2}{3},\frac{1}{3},0\right)$, $\left(\frac{1}{3},0,\frac{2}{3}\right)$ or $\left(0,\frac{2}{3},\frac{1}{3}\right)$. $\square$

Second Solution. (*CMO Committee - Crux Mathematicorum 1999, pag. 400*)
Let $f(x,y,z)=x^2y+y^2z+z^2x$. We wish to determine where f is maximal. Since f is cyclic WLOG we may assume that $x=max(x,y,z)$. Since

$$f(x,y,z)-f(x,z,y)=x^2y+y^2z+z^2x-x^2z-z^2y-y^2x=$$
$$=(y-z)(x-y)(x-z)$$

we may also assume $y\geq z$. Then

$$f(x+z,y,0)-f(x,y,z)=(x+z)^2y-x^2y-y^2z-z^2x=$$
$$=z^2y+yz(x-y)+xz(y-z)\geq 0$$

so we may now assume $z=0$. The rest follows from AM-GM inequality

$$f(x,y,0)=\frac{2x^2y}{2}\leq\frac{1}{2}\left(\frac{x+x+2y}{3}\right)^3=\frac{4}{27}$$

Equality occurs when $x=2y$, hence when (x,y,z) equals $\left(\frac{2}{3},\frac{1}{3},0\right)$, $\left(\frac{1}{3},0,\frac{2}{3}\right)$ or $\left(0,\frac{2}{3},\frac{1}{3}\right)$. $\square$

Third Solution. (*CMO Committee - Crux Mathematicorum 1999, pag. 400*)
With f as above, and $x = max(x, y, z)$ we have

$$f\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) - f(x, y, z) = yz(x - y) + \frac{xz}{2}(x - z) + \frac{z^2 y}{4} + \frac{z^3}{8}$$

so we may assume that $z = 0$. The rest follows as for second solution. $\square$

Fourth Solution. (*See* [4] *pag.* 46, *problem* 32)
Assume WLOG that $x = max(x, y, z)$. We have

$$x^2 y + y^2 z + z^2 x \leq \left(x + \frac{z}{2}\right)^2 \left(y + \frac{z}{2}\right) \tag{1}$$

because $xyz \geq y^2 z$ and $\frac{x^2 z}{2} \geq \frac{xz^2}{2}$.
Then by AM-GM inequality and (1) we have

$$\begin{aligned} 1 = \frac{x + \frac{z}{2}}{2} + \frac{x + \frac{z}{2}}{2} + \left(y + \frac{z}{2}\right) \geq \\ \geq 3\sqrt[3]{\frac{\left(y + \frac{z}{2}\right)\left(x + \frac{z}{2}\right)^2}{4}} \geq \\ \geq 3\sqrt[3]{\frac{x^2 y + y^2 z + z^2 x}{4}} \end{aligned}$$

from which follows the desidered inequality $x^2 y + y^2 z + z^2 x \leq \frac{4}{27}$. $\square$

67. (**Proposed for 1999 USAMO**, [AB, pp.25]) $(x, y, z > 1)$

$$x^{x^2+2yz} y^{y^2+2zx} z^{z^2+2xy} \geq (xyz)^{xy+yz+zx}$$

First Solution. (*See* [15] *pag.* 67)
The required inequality is equivalent to

$$\begin{aligned} \left(x^2 + 2yz\right)\log x + \left(y^2 + 2xz\right)\log y + \left(z^2 + 2xy\right)\log z \geq \\ \geq (xy + yz + x + zx)(\log x + \log y + \log z) \end{aligned}$$

that is

$$(x - y)(x - z)\log x + (y - z)(y - x)\log y + (z - x)(z - y)\log z \geq 0$$

We observe that $\log x, \log y, \log z > 0$ because $x, y, z > 1$. Furthermore, since the last inequality is symmetric, we can assume WLOG that $x \geq y \geq z$. Thus

$$(z - x)(z - y)\log z \geq 0 \tag{1}$$

and, since the function $\log x$ is increasing on $x > 0$, we get

$$(x-y)(x-z)\log x \geq (y-z)(x-y)\log y \tag{2}$$

because each factor of LHS is greater or equal of a different factor of RHS. The required inequality follows from (1) and (2). $\square$

Second Solution. (*Soarer - ML Forum*)
The required equality is equivalent to

$$x^{x^2+yz-xy-xz}y^{y^2+xz-xy-yz}z^{z^2+xy-xz-yz} \geq 1$$
$$x^{(x-y)(x-z)}y^{(y-x)(y-z)}z^{(z-x)(z-y)} \geq 1$$
$$\left(\frac{x}{y}\right)^{x-y} \cdot \left(\frac{y}{z}\right)^{y-z} \cdot \left(\frac{x}{z}\right)^{x-z} \geq 1 \tag{$\star$}$$

By symmetry we can assume WLOG tath $x \geq y \geq z$. Therefore ($\star$) is verifyied. $\square$

68. (**Turkey, 1999**) ($c \geq b \geq a \geq 0$)

$$(a+3b)(b+4c)(c+2a) \geq 60abc$$

First Solution. (*ML Forum*) By AM-GM inequality we have

$$\begin{aligned}(a+3b)(b+4c)(c+2a) &\geq 4\sqrt[4]{ab^3}\cdot 5\sqrt[5]{bc^4}\cdot 3\sqrt[3]{ca^2} = \\ &= 60\left(a^{\frac{1}{4}}a^{\frac{2}{3}}\right)\left(b^{\frac{3}{4}}b^{\frac{1}{5}}\right)\left(c^{\frac{4}{5}}c^{\frac{1}{3}}\right) = \\ &= 60a^{\frac{11}{12}}b^{\frac{19}{20}}c^{\frac{17}{15}} = \\ &= 60abc\cdot a^{-\frac{1}{12}}b^{-\frac{1}{20}}c^{\frac{2}{15}} \geq \\ &\geq 60abc\cdot c^{-\frac{1}{12}}c^{-\frac{1}{20}}c^{\frac{2}{15}} = \\ &= 60abc\end{aligned}$$

where the last inequality is true because $c \geq b \geq a \geq 0$ and the function $f(x) = x^\alpha$ (with $\alpha < 0$) is decreasing. $\square$

Second Solution. (*See* [8] *pag.* 176) By the AM-GM inequality we have $a + b + b \geq 3\sqrt[3]{ab^2}$. Multiplying this inequality and the analogous inequalities for $b + 2c$ and $c + 2a$ yields $(a+2b)(b+2c)(c+2a) \geq 27abc$. Then

$$\begin{aligned}&(a+2b)(b+2c)(c+2a) \geq \\ &\geq \left(a+\frac{1}{3}a+\frac{8}{3}b\right)\left(b+\frac{2}{3}b+\frac{10}{3}c\right)(c+2a) = \\ &= \frac{20}{9}(a+2b)(b+2c)(c+2a) \geq 60abc\end{aligned}$$

$\square$

69. (**Macedonia 1999**) ($a^2+b^2+c^2=1,\ a,b,c>0$)

$$a+b+c+\frac{1}{abc}\geq 4\sqrt{3}$$

First Solution. (*Frengo, Leepakhin - ML Forum*) By AM-GM, we have

$$1=a^2+b^2+c^2\geq 3\sqrt[3]{(abc)^2}\Rightarrow (abc)^2\leq\frac{1}{27}$$

Thus, by AM-GM

$$\begin{aligned}a+b+c+\frac{1}{abc}&=a+b+c+\frac{1}{9abc}+\frac{1}{9abc}+\cdots+\frac{1}{9abc}\geq\\&\geq 12\sqrt[12]{\frac{1}{9^9(abc)^8}}\geq\\&\geq 12\sqrt[12]{\frac{1}{9^9(\frac{1}{27})^4}}=4\sqrt{3}\end{aligned}$$

Equality holds if and only if $a=b=c=\frac{1}{9abc}$ or $a=b=c=\frac{1}{\sqrt{3}}$. □

Second Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$abc\left(a+b+c-4\sqrt{3}\right)+1\geq 0$$

From Schur inequality we have

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca)$$

Since

$$ab+bc+ca=\frac{1}{2}\left[(a+b+c)^2-\left(a^2+b^2+c^2\right)\right]=\frac{1}{2}\left[(a+b+c)^2-1\right]$$

we get

$$abc\geq\frac{1}{9}\left[(a+b+c)^3-2(a+b+c)\right]$$

After setting $S=a+b+c$ from Cauchy-Schwarz inequality follows that

$$S=a+b+c\leq\sqrt{1+1+1}\sqrt{a^2+b^2+c^2}=\sqrt{3}$$

and, consequently

$$\begin{aligned}abc\left(a+b+c-4\sqrt{3}\right)+1&\geq\frac{1}{9}\left(S^3-2S\right)\left(S-4\sqrt{3}\right)+1=\\&=\frac{1}{9}\left[\left(S^3-2S\right)\left(S-4\sqrt{3}\right)+9\right]=\\&=\frac{1}{9}\left[\left(\sqrt{3}-S\right)^4+20S\left(\sqrt{3}-S\right)\right]\geq 0\end{aligned}$$

□

Third Solution. (*Ercole Suppa*)
After setting $S = a+b+c$, $Q = ab+bc+ca$, from the constraint $a^2+b^2+c^2 = 1$ we have $S^2 = 1 + 2Q \geq 1$. Then $S \geq 1$ and, by Cauchy-Schwarz inequality we get

$$S = a + b + c \leq \sqrt{1+1+1}\sqrt{a^2 + b^2 + c^2} = \sqrt{3}$$

From the well-know inequality $(ab + bc + ca)^2 \geq 3abc(a + b + c)$ follows that $\frac{1}{abc} \geq \frac{3S}{Q^2}$. Thus, to establish the required inequality is enough to show that

$$S + \frac{3S}{Q^2} \geq 4\sqrt{3} \quad \Longleftrightarrow \quad 4\left(S - 4\sqrt{3}Q^2\right)Q^2 + 12S \geq 0$$

Sine $1 < S \leq \sqrt{3}$ we have

$$\begin{aligned} 4\left(S - 4\sqrt{3}\right)Q^2 + 12S &= \left(S - 4\sqrt{3}\right)\left(S^2 - 1\right)^2 + 12S = \\ &= \left(\sqrt{3} - S\right)\left(-S^4 + 3\sqrt{3}S^3 + 11S^2 + 3\sqrt{3}S - 4\right) \geq \\ &\geq \left(\sqrt{3} - S\right)\left(-S^4 + 3S^4 + 11S^2 + 3S^2 - 4\right) \geq \\ &\geq \left(\sqrt{3} - S\right)\left(2S^4 + 14S^2 - 4\right) \geq 12\left(\sqrt{3} - S\right) \geq 0 \end{aligned}$$

□

Fourth Solution. (*Ercole Suppa*) Since $a^2 + b^2 + c^2 = 1$, the inequality

$$abc\left(a + b + c - 4\sqrt{3}\right) + 1 \geq 0$$

can be tranformed into a homogeneous one in the following way

$$abc\,(a + b + c) - 4\sqrt{3}abc\sqrt{a^2 + b^2 + c^2} + \left(a^2 + b^2 + c^2\right)^2 \geq 0$$

Squaring and expanding the expression we get

$$\begin{aligned} &\frac{1}{2}\sum_{\text{sym}} a^8 + 4\sum_{\text{sym}} a^6b^2 + \sum_{\text{sym}} a^6bc + 2\sum_{\text{sym}} a^5b^2c + 3\sum_{\text{sym}} a^4b^4 + \\ &+ \sum_{\text{sym}} a^4b^3c + \frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \geq 24\sum_{\text{sym}} a^4b^2c^2 \end{aligned}$$

The last inequality can be obtained addind the following inequalities which are true by Muirhead theorem:

$$\frac{1}{2}\sum_{\text{sym}} a^8 \geq \frac{1}{2}\sum_{\text{sym}} a^4b^2c^2 \tag{1}$$

$$4\sum_{\text{sym}} a^6b^2 \geq 4\sum_{\text{sym}} a^4b^2c^2 \tag{2}$$

$$\sum_{\text{sym}} a^6bc \geq \sum_{\text{sym}} a^4b^2c^2 \tag{3}$$

$$2\sum_{\text{sym}} a^5b^2c \geq 2\sum_{\text{sym}} a^4b^2c^2 \tag{4}$$

$$3\sum_{\text{sym}} a^4b^4 \geq 3\sum_{\text{sym}} a^4b^2c^2 \tag{5}$$

$$\sum_{\text{sym}} a^4b^3c \geq 4\sum_{\text{sym}} a^4b^2c^2 \tag{6}$$

$$\frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \geq \frac{13}{2}\sum_{\text{sym}} a^4b^2c^2 \tag{7}$$

□

Fifth Solution. (*Tiks - ML Forum*)

$$\begin{aligned}
&a+b+c+\frac{1}{abc} \geq 4\sqrt{3} \\
&\iff a+b+c+\frac{(a^2+b^2+c^2)^2}{abc} \geq 4\sqrt{3(a^2+b^2+c^2)} \\
&\iff \frac{(a^2+b^2+c^2)^2-3abc(a+b+c)}{abc} \geq 4(\sqrt{3(a^2+b^2+c^2)}-(a+b+c)) \\
&\iff \frac{(a^2+b^2+c^2)^2-3abc(a+b+c)}{abc} \geq 4\frac{3(a^2+b^2+c^2)-(a+b+c)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c} \\
&\iff \sum(a-b)^2\frac{(a+b)^2+3c^2}{2abc} \geq \sum(a-b)^2\frac{4}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}
\end{aligned}$$

but we have that $\sqrt{3(a^2+b^2+c^2)}+a+b+c \geq 2(a+b+c)$ so we have to prove that

$$\begin{aligned}
&\sum(a-b)^2\frac{(a+b)^2+3c^2}{2abc} \geq \sum(a-b)^2\frac{2}{a+b+c} \\
&\iff \sum(a-b)^2\left[\frac{(a+b)^2+3c^2}{2abc}-\frac{2}{a+b+c}\right] \geq 0
\end{aligned}$$

We have that

$$(a+b+c)\left[(a+b)^2+3c^2\right] \geq c(a+b)^2 \geq 4abc$$

hence

$$\frac{(a+b)^2+3c^2}{2abc}-\frac{2}{a+b+c}\geq 0$$

So the inequality is done. $\square$

70. (**Poland 1999**) $(a+b+c=1,\ a,b,c>0)$

$$a^2+b^2+c^2+2\sqrt{3abc}\leq 1$$

Solution. (*Ercole Suppa*) From the well-know inequality

$$(x+y+z)^2\geq 3(xy+yz+xz)$$

by putting $x=ab$, $y=bc$ e $z=ca$ we have

$$(ab+bc+ca)^2\geq 3abc(a+b+c) \quad\Longrightarrow\quad ab+bc+ca\geq\sqrt{3abc} \qquad (1)$$

From the constraint $a+b+c=1$ follows that

$$1-a^2-b^2-c^2=(a+b+c)^2-a^2-b^2-c^2=2ab+2ac+2ca \qquad (2)$$

(1) and (2) implies

$$1-a^2-b^2-c^2-2\sqrt{3abc}=2ab+2bc+2ca-2\sqrt{3abc}\geq 0$$

$\square$

71. (**Canada 1999**) $(x+y+z=1, x,y,z\geq 0)$

$$x^2y+y^2z+z^2x\leq\frac{4}{27}$$

Solution. (*Ercole Suppa*) See: problem n.66. $\square$

72. (**Iran 1998**) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2,\ x,y,z>1\right)$

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*)

$$\begin{aligned}\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1} &= \left(\sum_{\text{cyc}} \frac{\sqrt{x-1}}{\sqrt{x}}\sqrt{x}\right) \leq \\ &\leq \left(\sum_{\text{cyc}} \frac{x-1}{x}\right)^{\frac{1}{2}} (x+y+z)^{\frac{1}{2}} = \qquad \text{(Cauchy-Schwarz)} \\ &= \left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^{\frac{1}{2}} \sqrt{x+y+z} = \\ &= \sqrt{x+y+z}\end{aligned}$$

□

73. (**Belarus 1998**, I. Gorodnin) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{b+c}+\frac{b+c}{a+b}+1$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$a^2b^3+ab^4+a^3c^2+b^3c^2+b^2c^3 \geq a^2b^2c+2ab^3c+2ab^2c^2 \qquad (\star)$$

From AM-GM inequality we have

$$a^2b^3+b^3c^2 \geq 2\sqrt{a^2b^6c^2} = 2ab^3c \qquad (1)$$

$$\frac{1}{2}ab^4+\frac{1}{2}a^3c^2 \geq 2\sqrt{\frac{a^4b^4c^2}{4}} = a^2b^2c \qquad (2)$$

$$\frac{1}{2}ab^4+\frac{1}{2}a^3c^2+b^2c^3 \geq a^2b^2c+b^2c^3 = 2\sqrt{a^2b^4c^4} = 2ab^2c^2 \qquad (3)$$

The $(\star)$ is obtained adding (1),(2) and (3). □

74. (**APMO 1998**) $(a, b, c > 0)$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Solution. (*See* [7] *pag.* 174) We have

$$\begin{aligned}
&\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)= \\
&\qquad =2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}= \\
&\qquad =\left(\frac{a}{b}+\frac{a}{c}+\frac{a}{a}\right)+\left(\frac{b}{c}+\frac{b}{a}+\frac{b}{b}\right)+\left(\frac{c}{a}+\frac{c}{b}+\frac{c}{c}\right)-1\geq \\
&\qquad \geq 3\left(\frac{a}{\sqrt[3]{abc}}+\frac{b}{\sqrt[3]{abc}}+\frac{c}{\sqrt[3]{abc}}\right)-1= \\
&\qquad =2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)+\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)-1\geq \\
&\qquad \geq 2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right)+3-1= \\
&\qquad =2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)
\end{aligned}$$

by two applications of AM-GM inequality. □

75. (**Poland 1998**) $\left(a+b+c+d+e+f=1,\ ace+bdf\geq\frac{1}{108}\ a,b,c,d,e,f>0\right)$

$$abc+bcd+cde+def+efa+fab\leq\frac{1}{36}$$

Solution. (*Manlio - ML Forum*) Put $A=ace+bdf$ and $B=abc+bcd+cde+def+efa+fab$.By AM-GM inequality we have

$$A+B=(a+d)(b+e)(c+f)\leq(((a+d)+(b+e)+(c+f))/3)^3=1/27$$

so

$$B\leq 1/27-A\leq 1/27-1/108=1/36$$

□

Remark. (Arqady) This is a private case of Walther Janous's inequality: If $x_1+x_2+...+x_n=1$ where x_i are non-negative real numbers and $2\leq k<n, k\in\mathbb{N}$, then

$$x_1x_2...x_k+x_2x_3...x_{k+1}+...+x_nx_1...x_{k-1}\leq max\{\frac{1}{k^k},\frac{1}{n^{k-1}}\}$$

76. (**Korea 1998**) $(x+y+z=xyz,\ x,y,z>0)$

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}$$

First Solution. (*See* [32], *pag.* 14)
We can write $x=\tan A,\ y=\tan B,\ z=\tan C$, where $A,B,C\in\left(0,\frac{\pi}{2}\right)$. Using the fact that $1+\tan^2\theta=\left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of $A,\ B,\ C$:

$$\cos A+\cos B+\cos C\leq\frac{3}{2}\qquad(\star)$$

It follows from $\tan(\pi-C)=-z=\frac{x+y}{1-xy}=\tan(A+B)$ and from $\pi-C, A+B\in(0,\pi)$ that $\pi-C=A+B$ or $A+B+C=\pi$.
Since $\cos x$ is concave on $\left(0,\frac{\pi}{2}\right)$, $(\star)$ a direct consequence of Jensen's inequality and we are done. $\square$

Second Solution. (*See* [32], *pag.* 17)
The starting point is letting $a=\frac{1}{x},\ b=\frac{1}{y},\ c=\frac{1}{z}$. We find that $a+b+c=abc$ is equivalent to $1=xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\begin{aligned}\frac{x}{\sqrt{(x+y)(x+z)}}&=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\leq\\&\leq\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\\&=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right)\end{aligned}$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\leq\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)$$

and

$$\frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right)$$

Adding these three yields the required result. □

77. (**Hong Kong 1998**) $(a, b, c \geq 1)$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{c(ab+1)}$$

First Solution. (*Ercole Suppa*) After setting $x = \sqrt{a-1}$, $y = \sqrt{b-1}$, $z = \sqrt{c-1}$, with $x, y, z \geq 0$, by easy calculations the required inequality in transformed in

$$\begin{aligned} x+y+z \leq \sqrt{\left(1+z^2\right)\left[\left(1+x^2\right)\left(1+y^2\right)+1\right]} \qquad &\Longleftrightarrow \\ (x+y+z)^2 \leq \left(1+z^2\right)\left(x^2y^2+x^2+y^2+2\right) \qquad &\Longleftrightarrow \\ \left(x^2y^2+x^2+y^2+1\right)z^2-2(x+y)z+x^2y^2-2xy+2 \geq 0 & \qquad (\star) \end{aligned}$$

The $(\star)$ is true for all $x, y, z \in \mathbb{R}$ because:

$$\begin{aligned} \frac{\Delta}{4} &= (x+y)^2-\left(x^2y^2+x^2+y^2+1\right)\left(x^2y^2-2xy+2\right) = \\ &= (x+y)^2-\left[x^2y^2+(x+y)^2-2xy+1\right]\left[(xy-1)^2+1\right] = \\ &= (x+y)^2-\left[(xy-1)^2+(x+y)^2\right]\left[(xy-1)^2+1\right] = \\ &= -(xy-1)^4-(xy-1)^2-(x+y)^2(xy-1)^2 = \\ &= -(xy-1)^2\left(2+x^2+y^2+x^2y^2\right) \leq 0 \end{aligned}$$

□

Second Solution. (*Sung-Yoon Kim - ML Forum*) Use

$$\sqrt{x-1}+\sqrt{y-1} \leq \sqrt{xy} \quad \Longleftrightarrow \quad 2\sqrt{(x-1)(y-1)} \leq (x-1)(y-1)+1$$

Then

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{ab}+\sqrt{c-1} \leq \sqrt{c(ab+1)}$$

□

Remark. The inequality used in the second solution can be generalized in the following way (see [25], pag. 183, n.51): given threee real positive numbers a, b, c con $a > c$, $b > c$ we have

$$\sqrt{c(a-c)}+\sqrt{c(b-c)} \leq \sqrt{ab}$$

The inequality, squaring twice, is transformed in $(ab-ac-bc)^2 \geq 0$. The equality holds if $c = ab/(a+b)$.

78. (**IMO Short List 1998**) ($xyz = 1,\ x, y, z > 0$)

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

First Solution. (*IMO Short List Project Group - ML Forum*)
The inequality is equivalent to the following one:

$$x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{3}{4}(x+1)(y+1)(z+1).$$

In fact, a stronger inequality holds true, namely

$$x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{1}{4}[(x+1)^3 + (y+1)^3 + (z+1)^3].$$

(It is indeed stronger, since $u^3 + v^3 + w^3 \geq 3uvw$ for any positive numbers u, v and w.) To represent the difference between the left- and the right-hand sides, put

$$f(t) = t^4 + t^3 - \frac{1}{4}(t+1)^3, \qquad g(t) = (t+1)(4t^2 + 3t + 1).$$

We have $f(t) = \frac{1}{4}(t-1)g(t)$. Also, g is a strictly increasing function on $(0, \infty)$, taking on positive values for $t > 0$. Since

$$\begin{aligned}
& x^4 + x^3 + y^4 + y^3 + z^4 + z^3 - \frac{1}{4}[(x+1)^3 + (y+1)^3 + (z+1)^3] \\
=& f(x) + f(y) + f(z) \\
=& \frac{1}{4}(x-1)g(x) + \frac{1}{4}(y-1)g(y) + \frac{1}{4}(z-1)g(z),
\end{aligned}$$

it suffices to show that the last expression is nonnegative.
Assume that $x \geq y \geq z$; then $g(x) \geq g(y) \geq g(z) > 0$. Since $xyz = 1$, we have $x \geq 1$ and $z \leq 1$. Hence $(x-1)g(x) \geq (x-1)g(y)$ and $(z-1)g(y) \leq (z-1)g(z)$. So,

$$\begin{aligned}
& \frac{1}{4}(x-1)g(x) + \frac{1}{4}(y-1)g(y) + \frac{1}{4}(z-1)g(z) \\
\geq & \frac{1}{4}[(x-1) + (y-1) + (z-1)]g(y) \\
=& \frac{1}{4}(x + y + z - 3)g(y) \\
\geq & \frac{1}{4}(3\sqrt[3]{xyz} - 3)g(y) = 0,
\end{aligned}$$

because $xyz = 1$. This completes the proof. Clearly, the equality occurs if and only if $x = y = z = 1$. $\square$

Second Solution. (*IMO Short List Project Group - ML Forum*)
Assume $x \leq y \leq z$ so that

$$\frac{1}{(1+y)(1+z)} \leq \frac{1}{(1+z)(1+x)} \leq \frac{1}{(1+x)(1+y)}.$$

Then Chebyshev's inequality gives that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)}$$

$$\geq \frac{1}{3}(x^3+y^3+z^3)\left[\frac{1}{(1+y)(1+z)} + \frac{1}{(1+z)(1+x)} + \frac{1}{(1+x)(1+y)}\right]$$

$$= \frac{1}{3}(x^3+y^3+z^3)\frac{3+(x+y+z)}{(1+x)(1+y)(1+z)}.$$

Now, setting $(x+y+z)/3 = a$ for convenience, we have by the AM-GM inequality

$$\frac{1}{3}(x^3+y^3+z^3) \geq a^3,$$

$$x+y+z \geq 3\sqrt[3]{xyz} = 3,$$

$$(1+x)(1+y)(1+z) \leq \left[\frac{(1+x)+(1+y)+(1+z)}{3}\right]^3 = (1+a)^3.$$

It follows that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq a^3 \cdot \frac{3+3}{(1+a)^3}.$$

So, it suffices to show that

$$\frac{6a^3}{(1+a)^3} \geq \frac{3}{4};$$

or, $8a^3 \geq (1+a)^3$. This is true, because $a \geq 1$. Clearly, the equality occurs if and only if $x = y = z = 1$. The proof is complete. $\square$

Third Solution. (*Grobber - ML Forum*)
Amplify the first, second and third fraction by x, y, z respectively. The LHS becomes

$$\sum \frac{x^4}{x(1+y)(1+z)} \geq \frac{\left(x^2+y^2+z^2\right)^2}{x+y+z+2(xy+yz+zx)+3} \geq \frac{\left(x^2+y^2+z^2\right)^2}{4\left(x^2+y^2+z^2\right)} \geq \frac{3}{4}$$

I used the inequalities

$$x^2+y^2+z^2 \geq xy+yz+zx$$

$$x^2+y^2+z^2 \geq 3$$

$$x^2+y^2+z^2 \geq \frac{(x+y+z)^2}{3} \geq x+y+z$$

$\square$

Fourth Solution. (*MysticTerminator - ML Forum*) First, note that

$$\sum_{\text{cyc}} \frac{x^3}{(1+y)(1+z)} \geq \frac{\left(x^2+y^2+z^2\right)^2}{x(1+y)(1+z)+(1+x)y(1+z)+(1+x)(1+y)z}$$

by Cauchy, so we need to prove:

$$4\left(x^2+y^2+z^2\right)^2 \geq 3\left(x(1+y)(1+z)+(1+x)y(1+z)+(1+x)(1+y)z\right)$$

Well, let $x=a^3$, $y=b^3$, $z=c^3$ (with $abc=1$), and homogenize it to find that we have to prove:

$$\sum_{\text{cyc}} \left(4a^{12}+8a^6b^6\right) \geq \sum_{\text{cyc}} \left(3a^6b^6c^3+6a^5b^5c^2+3a^4b^4c^4\right)$$

which is perfectly Muirhead. $\square$

Remark. None of the solutions 1 and 2 above actually uses the condition $xyz=1$. They both work, provided that $x+y+z \geq 3$. Moreover, the alternative solution also shows that the inequality still holds if the exponent 3 is replaced by any number greater than or equal to 3.

79. (**Belarus 1997**) $(a,x,y,z>0)$

$$\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z \geq x+y+z \geq \frac{a+z}{a+z}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z$$

First Solution. (*Soarer - ML Forum*) First one

$$\begin{aligned}\sum x\frac{a+z}{a+x} &= \sum a+z-a\left(\sum \frac{a+z}{a+x}\right) = \\ &= x+y+z-a\left(\sum \frac{a+z}{a+x}-3\right) \leq \\ &\leq x+y+z\end{aligned}$$

Second one

$$\begin{aligned} & \sum x\frac{a+y}{a+z} \geq x+y+z \\ \Leftrightarrow \quad & \sum \frac{y-z}{a+z}x \geq 0 \\ \Leftrightarrow \quad & \sum \frac{xy}{a+z} \geq \sum \frac{xz}{a+z} \\ \Leftrightarrow \quad & \sum \frac{1}{z(a+z)} \geq \sum \frac{1}{y\,(a+z)}\end{aligned}$$

which is rearrangement. $\square$

Second Solution. (*Darij Grinberg- ML Forum*)
Let's start with the first inequality:

$$\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\leq x+y+z$$

It is clearly equivalent to

$$\left(\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\right)-(x+y+z)\leq 0$$

But

$$\begin{aligned}
&\left(\frac{a+z}{a+x}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z\right)-(x+y+z)=\\
&=\left(\frac{a+z}{a+x}-1\right)x+\left(\frac{a+x}{a+y}-1\right)y+\left(\frac{a+y}{a+z}-1\right)z=\\
&=\frac{z-x}{a+x}x+\frac{x-y}{a+y}y+\frac{y-z}{a+z}z=(z-x)\frac{x}{a+x}+(x-y)\frac{y}{a+y}+(y-z)\frac{z}{a+z}=\\
&=\left(z\frac{x}{a+x}-x\frac{x}{a+x}\right)+\left(x\frac{y}{a+y}-y\frac{y}{a+y}\right)+\left(y\frac{z}{a+z}-z\frac{z}{a+z}\right)=\\
&=\left(z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\right)-\left(x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}\right)
\end{aligned}$$

thus, it is enough to prove the inequality

$$\left(z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\right)-\left(x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}\right)\leq 0$$

This inequality is clearly equivalent to

$$z\frac{x}{a+x}+x\frac{y}{a+y}+y\frac{z}{a+z}\leq x\frac{x}{a+x}+y\frac{y}{a+y}+z\frac{z}{a+z}$$

And this follows from the rearrangement inequality, applied to the equally sorted number arrays $(x;\ y;\ z)$ and $\left(\frac{x}{a+x};\ \frac{y}{a+y};\ \frac{z}{a+z}\right)$ (proving that these arrays are equally sorted is very easy: if, for instance, $x\leq y$, then $\frac{a}{x}\geq\frac{a}{y}$, so that $\frac{a+x}{x}=\frac{a}{x}+1\geq\frac{a}{y}+1=\frac{a+y}{y}$, so that $\frac{x}{a+x}\leq\frac{y}{a+y}$).

Now we will show the second inequality:

$$x+y+z\leq\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z$$

It is equivalent to

$$0\leq\left(\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z\right)-(x+y+z)$$

Since

$$\left(\frac{a+y}{a+z}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z\right)-(x+y+z)=$$

$$=\left(\frac{a+y}{a+z}-1\right)x+\left(\frac{a+z}{a+x}-1\right)y+\left(\frac{a+x}{a+y}-1\right)z=$$

$$=\frac{y-z}{a+z}x+\frac{z-x}{a+x}y+\frac{x-y}{a+y}z=(xy-zx)\frac{1}{a+z}+(yz-xy)\frac{1}{a+x}+(zx-yz)\frac{1}{a+y}=$$

$$=\left(xy\frac{1}{a+z}-zx\frac{1}{a+z}\right)+\left(yz\frac{1}{a+x}-xy\frac{1}{a+x}\right)+\left(zx\frac{1}{a+y}-yz\frac{1}{a+y}\right)=$$

$$=\left(xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}\right)-\left(zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\right)$$

thus, it is enough to verify the inequality

$$0\le\left(xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}\right)-\left(zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\right)$$

This inequality is equivalent to

$$zx\frac{1}{a+z}+xy\frac{1}{a+x}+yz\frac{1}{a+y}\le xy\frac{1}{a+z}+yz\frac{1}{a+x}+zx\frac{1}{a+y}$$

But this follows from the rearrangement inequality, applied to the equally sorted number arrays $(yz;\ zx;\ xy)$ and $\left(\frac{1}{a+x};\ \frac{1}{a+y};\ \frac{1}{a+z}\right)$ (proving that these arrays are equally sorted is almost trivial: if, for instance, $x\le y$, then $y\ge x$ and $yz\ge zx$, while on the other hand $a+x\le a+y$ and thus $\frac{1}{a+x}\ge\frac{1}{a+y}$).
This completes the proof of your two inequalities.

□

80. (**Ireland 1997**) $(a+b+c\ge abc,\ a,b,c\ge 0)$

$$a^2+b^2+c^2\ge abc$$

Solution. (*Ercole Suppa*) See problem 51. □

81. (**Iran 1997**) $(x_1x_2x_3x_4=1, x_1,x_2,x_3,x_4>0)$

$$x_1^3+x_2^3+x_3^3+x_4^3\ge max\left(x_1+x_2+x_3+x_4,\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

Solution. (*See* [15] *pag.* 69)
After setting $A = \sum_{i=1}^{4} x_i^3$, $A_i = A - x_i^3$, from AM-GM inequality we have

$$\frac{1}{3}A_1 \geq \sqrt[3]{x_2^3x_3^3x_4^3} = x_2x_3x_4 = \frac{1}{x_1}$$

Similarly can be proved that $\frac{1}{3}A_i \geq \frac{1}{x_i}$ for all $i = 2, 3, 4$. Therefore

$$A = \frac{1}{3}\sum_{i=1}^{4} A_i \geq \sum_{i=1}^{4} \frac{1}{x_i}$$

On the other hand by Power Mean inequality we have

$$\begin{aligned}\frac{1}{4}A = \frac{1}{4}\sum_{i=1}^{4} x_i^3 \geq \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)^3 &= \\ = \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)\left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)^2 &\geq \\ \geq \left(\frac{1}{4}\sum_{i=1}^{4} x_i\right)&\end{aligned}$$

(in the last step we used AM-GM inequality: $\sum_{i=1}^{4} x_i \geq \sqrt[4]{x_1x_2x_3x_4} = 1$). Thus

$$A \geq \sum_{i=1}^{4} x_i$$

and the inequality is proven. □

82. (**Hong Kong 1997**) $(x, y, z > 0)$

$$\frac{3+\sqrt{3}}{9} \geq \frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)}$$

Solution. (*Ercole Suppa*) From QM-AM-GM inequality we have

$$x + y + z \geq \sqrt{3}\sqrt{x^2+y^2+z^2} \tag{1}$$
$$xy + yz + zx \geq 3\sqrt[3]{(xyz)^2} \tag{2}$$
$$\sqrt{x^2+y^2+z^2} \geq \sqrt{3}\sqrt[3]{xyz} \tag{3}$$

Therefore

$$\begin{aligned}
\frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)} &\le \frac{xyz\sqrt{x^2+y^2+z^2}\left(\sqrt{3}+1\right)}{(x^2+y^2+z^2)3\sqrt[3]{(xyz)^2}} \le \\
&\le \frac{xyz\left(\sqrt{3}+1\right)}{3\sqrt{x^2+y^2+z^2}\sqrt[3]{(xyz)^2}} \le \\
&\le \frac{xyz\left(\sqrt{3}+1\right)}{3\sqrt{3}\sqrt[3]{xyz}\sqrt[3]{(xyz)^2}} = \\
&= \frac{\sqrt{3}+1}{3\sqrt{3}} = \frac{3+\sqrt{3}}{9}
\end{aligned}$$

□

Remark. See: Crux Mathematicorum 1988, pag. 203, problem 1067.

83. (**Belarus 1997**) $(a, b, c > 0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

First Solution. (*Ghang Hwan - ML Forum, Siutz - ML Contest 1st Ed. 1R*) The inequality is equivalent with

$$\frac{1+\frac{b}{a}}{1+\frac{c}{a}}+\frac{1+\frac{c}{b}}{1+\frac{a}{b}}+\frac{1+\frac{a}{c}}{1+\frac{b}{c}} \le \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Let $x = a/b$, $y = c/a$, $z = b/c$ and note that $xyz = 1$. After some boring calculation we see that the inequality become

$$\left(x^2+y^2+z^2-x-y-z\right)+\left(x^2z+y^2x+z^2y-3\right) \ge 0$$

This inequality is true. In fact the first and second term are not negative because

$$x^2+y^2+z^2 \ge (x+y+z)\frac{x+y+z}{3} \ge x+y+z \qquad \text{(by CS and AM-GM)}$$

and

$$x^2z+y^2x+z^2y \ge 3\sqrt[3]{x^3y^3z^3} = 3 \qquad \text{(by AM-GM)}$$

□

Second Solution. (*See* [4], *pag.* 43, *problem* 29) Let us take $x = a/b$, $y = c/a$, $z = b/c$ and note that $xyz = 1$. Observe that

$$\frac{a+c}{b+c} = \frac{1+xy}{1+y} = x + \frac{1-x}{1+y}$$

Using similar relations, the problem reduces to proving that if $xyz = 1$, then

$$\frac{x-1}{y+1}\frac{y-1}{z+1} + \frac{z-1}{x+1} \geq 0 \quad \Longleftrightarrow$$
$$\left(x^2-1\right)(z+1) + \left(y^2-1\right)(x+1) + \left(z^2-1\right)(y+1) \geq 0 \quad \Longleftrightarrow$$
$$\sum x^2 z + \sum x^2 \geq \sum x + 3$$

But this inequality is very easy. Indeed, using the AM-GM inequality we have $\sum x^2 z \geq 3$ and so it remains to prove that $\sum x^2 \geq \sum x$, which follows from the inequalities

$$\sum x^2 \geq \frac{\left(\sum x\right)^2}{3} \sum x$$

□

Third Solution. (*Darij Grinberg, ML Forum*) We first prove a lemma:

Lemma. Let a, b, c be three reals; let x, y, z, u, v, w be six nonnegative reals. Assume that the number arrays $(a;b;c)$ and $(x;y;z)$ are equally sorted, and

$$u(a-b) + v(b-c) + w(c-a) \geq 0$$

Then,

$$xu(a-b) + yv(b-c) + zw(c-a) \geq 0$$

Proof Since the statement of Lemma is invariant under cyclic permutations (of course, when these are performed for the number arrays $(a;b;c)$, $(x;y;z)$ and $(u;v;w)$ simultaneously), we can WLOG assume that b is the "medium one" among the numbers a, b, c; in other words, we have either $a \geq b \geq c$, or $a \leq b \leq c$. Then, since the number arrays $(a;b;c)$ and $(x;y;z)$ are equally sorted, we get either $x \geq y \geq z$, or $x \leq y \leq z$, respectively. What is important is that $(x-z)(a-b) \geq 0$ (since the numbers $x-z$ and $a-b$ have the same sign: either both ≥ 0, or both ≤ 0), and that $(y-z)(b-c) \geq 0$ (since the numbers $y-z$ and $b-c$ have the same sign: either both ≥ 0, or both ≤ 0). Now,

$$\begin{aligned} &xu(a-b) + yv(b-c) + zw(c-a) = \\ &= u\underbrace{(x-z)(a-b)}_{\geq 0} + v\underbrace{(y-z)(b-c)}_{\geq 0} + z\underbrace{(u(a-b) + v(b-c) + w(c-a))}_{\geq 0} \geq 0 \end{aligned}$$

and the Lemma is proven. $\square$

Proof of inequality. The inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

can be written as

$$\sum\frac{c+a}{c+b}\leq\sum\frac{a}{b}\quad\Longleftrightarrow\quad\sum\frac{a}{b}-\sum\frac{c+a}{c+b}\geq 0$$

But

$$\sum\frac{a}{b}-\sum\frac{c+a}{c+b}=\sum\left(\frac{a}{b}-\frac{c+a}{c+b}\right)=\sum\frac{1}{b+c}\cdot\frac{c}{b}\cdot(a-b)$$

So it remains to prove that

$$\sum\frac{1}{b+c}\cdot\frac{c}{b}\cdot(a-b)\geq 0$$

In fact, denote $u=c/b$; $v=a/c$; $w=b/a$. Then,

$$\sum u\,(a-b)=\sum\frac{c}{b}\,(a-b)=\sum\left(\frac{ca}{b}-c\right)=\sum\frac{ca}{b}-\sum c=$$
$$=\frac{\sum c^2a^2-\sum c^2ab}{abc}=\frac{\frac{1}{2}\sum(ca-ab)^2}{abc}\geq 0$$

Now, denote

$$x=\frac{1}{b+c}\quad;\quad y=\frac{1}{c+a}\quad;\quad z=\frac{1}{a+b}$$

Then, the number arrays $(a;b;c)$ and $(x;y;z)$ are equally sorted (in fact, e. g., if $a\geq b$, then $c+a\geq b+c$, so that $\frac{1}{b+c}\geq\frac{1}{c+a}$, or, equivalently, $x\geq y$); thus, according to the Lemma, the inequality

$$\sum u\,(a-b)\geq 0$$

implies

$$\sum xu\,(a-b)\geq 0$$

In other words, $\sum\frac{1}{b+c}\cdot\frac{c}{b}\cdot(a-b)\geq 0$. And the problem is solved. $\square$

84. (**Bulgaria 1997**) $(abc=1,\ a,b,c>0)$

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$$

Solution. (*Official solution*) Let $x = a + b + c$ and $y = ab + bc + ca$. It follows from CS inequality that $x \geq 3$ and $y \geq 3$. Since both sides of the given inequality are symmetric functions of a, b and c, we transform the expression as a function of x, y. Taking into account that $abc = 1$, after simple calculations we get

$$\frac{3 + 4x + y + x^2}{2x + y + x^2 + xy} \leq \frac{12 + 4x + y}{9 + 4x + 2y}$$

which is equivalent to

$$3x^2y + xy^2 + 6xy - 5x^2 - y^2 - 24x - 3y - 27 \geq 0$$

Write the last inequality in the form

$$\left(\frac{5}{3}x^2y - 5x^2\right) + \left(\frac{xy^2}{3} - y^2\right) + \left(\frac{xy^2}{3} - 3y\right) + \left(\frac{4}{3}x^2y - 12x\right) + \\ + \left(\frac{xy^3}{3} - 3x\right) + (3xy - 9x) + (3xy - 27) \geq 0$$

When $x \geq 3$, $y \geq 3$, all terms in the left hand side are nonnegative and the inequality is true. Equality holds when $x = 3, y = 3$, which implies $a = b = c = 1$. □

Remark. 1 The inequality can be proved by the general result: if $\prod x_i = 1$ then

$$\sum \frac{1}{n - 1 + x_i} \leq 1$$

PROOF. $f(x_1, x_2, ..., x_n) = \sum \frac{1}{n-1+x_i}$.As $\prod x_i = 1$ we may assume $x_1 \geq 1, x_2 \leq 1$. We shall prove that $f(x_1, x_2, ..., x_n) \leq f(1, x_1x_2, ..., x_n)$. And this is true because after a little computation we obtain $(1-x_1)(x_2-1)(x_1x_2+(n-1)^2) \geq 0$ which is obviously true. So we have $f(x_1, x_2, ..., x_n) \leq f(1, x_1x_2, ..., x_n) \leq ... \leq f(1, ..., 1) = 1$. □

Remark 2. (Darij Grinberg) I want to mention the appearance of the inequality with solution in two sources:

1. Titu Andreescu, Vasile Cîrtoaje, Gabriel Dospinescu, Mircea Lascu, Old and New Inequalities, Zalau: GIL 2004, problem 99.
2. American Mathematics Competitions: Mathematical Olympiads 1997-1998: Olympiad Problems from Around the World, Bulgaria 21, p. 23.

Both solutions are almost the same: Brute force. The inequality doesn't seem to have a better proof.

85. (**Romania 1997**) ($xyz = 1,\ x, y, z > 0$)

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}\ge 2$$

Solution. (*Ercole Suppa*) By setting $a = x^3$, $b = y^3$, $c = z^3$ we have $abc = 1$. From the know inequality $a^3 + b^3 \ge ab(a + b)$ follows that

$$\begin{aligned}\frac{a^3+b^3}{a^2+ab+b^2} &= \frac{a^3+b^3+2\left(a^3+b^3\right)}{3\left(a^2+ab+b^2\right)} \ge \\ &\ge \frac{a^3+b^3+2ab(a+b)}{3\left(a^2+ab+b^2\right)} = \\ &= \frac{(a+b)\left(a^2+ab+b^2\right)}{3\left(a^2+ab+b^2\right)} = \\ &= \frac{a+b}{3}\end{aligned}$$

Similarly can be proved the following inequalities:

$$\frac{b^3+c^3}{b^2+bc+c^2}\ge\frac{b+c}{3}\quad ; \quad \frac{c^3+a^3}{c^2+ca+a^2}\ge\frac{c+a}{3}$$

Then, by AM-GM inequality we have:

$$\begin{aligned}&\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6} = \\ &= \frac{a^3+b^3}{a^2+ab+b^2}+\frac{b^3+c^3}{b^2+bc+c^2}+\frac{c^3+a^3}{c^2+ca+a^2} = \\ &= \frac{a+b}{3}+\frac{b+c}{3}+\frac{c+a}{3} = \\ &= \frac{2(a+b+c)}{3} \ge \\ &\ge \frac{2\cdot 3\sqrt[3]{abc}}{3} = 2\end{aligned}$$

□

86. (**Romania 1997**) ($a, b, c > 0$)

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\ge 1\ge\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}$$

Solution. (*Pipi - ML Forum*) Let

$$I = \frac{a^2}{a^2+2bc} + \frac{b^2}{b^2+2ca} + \frac{c^2}{c^2+2ab} \quad , \quad J = \frac{bc}{a^2+2bc} + \frac{ca}{b^2+2ca} + \frac{ab}{c^2+2ab}$$

We wish to show that $I \geq 1 \geq J$. Since $x^2+y^2 \geq 2xy$ we have

$$\frac{a^2}{a^2+2bc} \geq \frac{a^2}{a^2+b^2+c^2}$$

Similarly,

$$\frac{b^2}{b^2+2ca} \geq \frac{b^2}{a^2+b^2+c^2} \quad , \quad \frac{c^2}{c^2+2ab} \geq \frac{c^2}{a^2+b^2+c^2}$$

Then it is clear that $I \geq 1$. Next, note that $I+2J = 3$ or $I = 3-2J$. By $I \geq 1$, it is easy to see that $J \leq 1$. $\square$

87. (**USA 1997**) $(a, b, c > 0)$

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Solution. (*ML Forum*) By Muirhead (or by factoring) we have

$$a^3+b^3 \geq ab^2+a^2b$$

so we get that:

$$\sum_{\text{cyc}} \frac{abc}{a^3+b^3+abc} \leq \sum_{\text{cyc}} \frac{abc}{ab(a+b+c)} = \sum_{\text{cyc}} \frac{c}{a+b+c} = 1$$

$\square$

88. (**Japan 1997**) $(a, b, c > 0)$

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

Solution. (*See* [40])
WLOG we can assume that $a+b+c = 1$. Then the first term on the left become

$$\frac{(1-2a)^2}{(1-a)^2+a^2} = 2 - \frac{2}{1+(1-2a)^2}$$

Next, let $x_1 = 1 - 2a$, $x_2 = 1 - 2b$, $x_3 = 1 - 2c$, then $x_1 + x_2 + x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desidered inequality is

$$\frac{1}{1+x_1^2} + \frac{1}{1+x_2^2} + \frac{1}{1+x_3^2} \leq \frac{27}{10}$$

We consider the equation of the tangent line to $f(x) = \frac{1}{1+x^2}$ at $x = 1/3$ which is $y = \frac{27}{50}(-x+2)$. We have $f(x) \leq \frac{27}{50}(-x+2)$ for $-1 < x < 1$ because

$$\frac{27}{50}(-x+2) - \frac{1}{1+x^2} = \frac{(3x-1)^2(4-3x)}{50(x^2+1)} \geq 0$$

Then

$$f(x_1) + f(x_2) + f(x_3) \leq \frac{27}{10}$$

and the desidered inequality follows. □

89. (**Estonia 1997**) $(x, y \in \mathbb{R})$

$$x^2 + y^2 + 1 > x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

Solution. (*Ercole Suppa*) We have:

$$\left(x - \sqrt{y^2+1}\right) + \left(y - \sqrt{x^2+1}\right) \geq 0$$

and, consequently,

$$x^2 + y^2 + 1 \geq x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

The equality holds if and only if $x = \sqrt{y^2+1}$ and $y = \sqrt{x^2+1}$, i.e.

$$x^2 + y^2 = x^2 + y^2 + 2$$

Since this last equality is impossible, the result is proven. □

90. (**APMC 1996**) $(x + y + z + t = 0, x^2 + y^2 + z^2 + t^2 = 1, x, y, z, t \in \mathbb{R})$

$$-1 \leq xy + yz + zt + tx \leq 0$$

Solution. (*Ercole Suppa*) After setting $A = xy + yz + zt + tx$ we have

$$0 = (x + y + z + t)^2 = 1 + 2A + 2(xz + yt) \quad \Longrightarrow \quad A = -\frac{1}{2} - xz - yt$$

The required inequality is equivalent to

$$-1 \leq -\frac{1}{2} - xz - yt \leq 0 \quad \Longleftrightarrow \quad -\frac{1}{2} \leq xz + yt \leq \frac{1}{2} \Longleftrightarrow \quad |xz + yt| \leq \frac{1}{2}$$

and can be proved by means of Cauchy-Schwarz and AM-GM inequalities

$$\begin{aligned} |xz + yt| &\leq \sqrt{x^2 + y^2} \cdot \sqrt{t^2 + z^2} = && \text{(CS)} \\ &= \sqrt{(x^2 + y^2)(t^2 + z^2)} \leq && \text{(AM-GM)} \\ &\leq \frac{x^2 + y^2 + t^2 + z^2}{2} = \frac{1}{2} \end{aligned}$$

□

91. (**Spain 1996**) $(a, b, c > 0)$

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a - b)(b - c)$$

Solution. (*Ercole Suppa*) We have:

$$\begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca - 3(a - b)(b - c) &= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac = \\ &= (a - 2b + c)^2 \geq 0 \end{aligned}$$

□

92. (**IMO Short List 1996**) $(abc = 1,\ a, b, c > 0)$

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

Solution. (*by IMO Shortlist Project Group - ML Forum*)
We have

$$\begin{aligned} a^5 + b^5 &= (a + b)\left(a^4 - a^3 b + a^2 b^2 - ab^3 + b^4\right) = \\ &= (a + b)\left[(a - b)^2\left(a^2 + ab + b^2\right) + a^2 b^2\right] \geq \\ &\geq a^2 b^2 (a + b) \end{aligned}$$

with equality if and only if $a = b$. Hence

$$\begin{aligned}\frac{ab}{a^5+b^5+ab} &\le \frac{ab}{ab(a+b)+1} = \\ &= \frac{1}{ab(a+b+c)} = \\ &= \frac{c}{a+b+c}\end{aligned}$$

Taking into account the other two analogous inequalities we have

$$\sum \frac{ab}{a^5+b^5+ab} \le \frac{c}{a+b+c} + \frac{a}{a+b+c} + \frac{b}{a+b+c} = 1$$

and the required inequality is established. Equality holds if and only if $a = b = c = 1$. $\square$

93. (**Poland 1996**) $\left(a+b+c=1,\ a,b,c \ge -\frac{3}{4}\right)$

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \le \frac{9}{10}$$

Solution. (*Ercole Suppa*) The equality holds if $a = b = c = 1/3$. The line tangent to the graph of $f(x) = \frac{x}{x^2+1}$ in the point with abscissa $x = 1/3$ has equation $y = \frac{18}{25}x + \frac{30}{50}$ and the graph of $f(x)$, per $x > -3/4$, if entirely below that line, i.e.

$$\frac{x}{x^2+1} \le \frac{18}{25}x + \frac{30}{50} \quad , \quad \forall x > -\frac{3}{4}$$

because

$$\frac{18}{25}x + \frac{30}{50} - \frac{x}{x^2+1} = (3x-1)^2(4x+3) \ge 0 \quad , \quad \forall x > -\frac{3}{4}$$

Therefore

$$\sum_{\text{cyc}} \frac{a}{a^2+1} \le f(a) + f(b) + f(c) = \frac{9}{10}$$

$\square$

Remark. It is possible to show that the inequality

$$\sum_{\text{cyc}} \frac{a}{a^2+1} \le \frac{9}{10}$$

is true for all $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. See ML Forum.

94. (**Hungary 1996**) $(a+b=1,\ a,b>0)$

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}$$

Solution. (*See* [32], *pag.* 30) Using the condition $a+b=1$, we can reduce the given inequality to homogeneous one, i. e.,

$$\frac{1}{3}\leq\frac{a^2}{(a+b)(a+(a+b))}+\frac{b^2}{(a+b)(b+(a+b))}\ \text{ or }\ a^2b+ab^2\leq a^3+b^3,$$

which follows from $(a^3+b^3)-(a^2b+ab^2)=(a-b)^2(a+b)\geq 0$. The equality holds if and only if $a=b=\frac{1}{2}$.

$\square$

95. (**Vietnam 1996**) $(a,b,c\in\mathbb{R})$

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}\left(a^4+b^4+c^4\right)$$

First Solution. (*Namdung - ML Forum*) Let

$$f(a,b,c)=(a+b)^4+(b+c)^4+(c+a)^4-\frac{4}{7}\left(a^4+b^4+c^4\right).$$

We will show that $f(a,b,c)\geq 0$ for all a, b, c. Among a, b, c, there exist at least one number which has the same sign as $a+b+c$, say a. By long, but easy computation, we have

$$f(a,b,c)-f(a,\frac{b+c}{2},\frac{b+c}{2})=3a(a+b+c)(b-c)^2+\frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2\geq 0$$

So, it sufficient (and necessary) to show that $f(a,t,t)>=0$ for all a, t. Is equivalent to $f(0,t,t)\geq 0$ and $f(1,t,t)\geq 0$ (due homogeneousness). The first is trivial, the second because

$$\begin{aligned}f(1,t,t)&=59t^4+28t^3+42t^2+28t+5=\\&=\frac{6}{59}(20t+7)^2+\left(\sqrt{59}t^2+\frac{14t}{\sqrt{59}}-\frac{1}{\sqrt{59}}\right)^2>0\end{aligned}$$

$\square$

Remark. To find the identity

$$f(a,b,c)-f(a,\frac{b+c}{2},\frac{b+c}{2})=3a(a+b+c)(b-c)^2+\frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2\geq 0$$

we can use the following well-known approach. Let

$$h(a,b,c)=f(a,b,c)-f(a,\frac{b+c}{2},\frac{b+c}{2})\geq 0$$

The first thing we must have is $h(0,b,c)\geq 0$. $h(0,b,c)$ is symmetric homogenus polynomial of b, c and it's easily to find that

$$h(0,b,c)=\frac{3}{56}\left(7b^2+10bc+7c^2\right)(b-c)^2$$

Now, take $h(a,b,c)-h(0,b,c)$ and factor, we will get

$$h(a,b,c)-h(0,b,c)=3a(a+b+c)(b-c)^2$$

Second Solution. (*Iandrei - ML Forum*)
Let $f(a,b,c)=(a+b)^4+(b+c)^4+(c+a)^4-\frac{4}{7}\left(a^4+b^4+c^4\right)$. It's clear that $f(0,0,0)=0$. We prove that $f(a,b,c)\geq 0$. We have

$$f(a,b,c)=\frac{10}{7}\sum a^4+4\sum ab\left(a^2+b^2\right)+6\sum a^2b^2\geq 0 \iff$$
$$\frac{5}{7}\sum a^4+2\sum ab\left(a^2+b^2\right)+3\sum a^2b^2\geq 0 \iff$$
$$5\sum a^4+14\sum ab\left(a^2+b^2\right)+21\sum a^2b^2\geq 0$$

We prove that

$$\frac{5}{2}\left(a^4+b^4\right)+14ab\left(a^2+b^2\right)+21a^2b^2\geq 0 \qquad (\star)$$

Let $x=ab$, $y=a^2+b^2$. Thus

$$\frac{5}{2}\left(y^2-2x^2\right)+14xy+21x^2\geq 0 \iff 16x^2+14xy+\frac{5}{2}y^2\geq 0$$

If $x\neq 0$, we want prove that

$$32+28\frac{y}{x}+5\left(\frac{y}{x}\right)^2\geq 0.$$

If $y/x=t$ with $|t|>2$, we must prove $32+28t+5t^2\geq 0$. The latter second degree function has roots

$$r_1=\frac{-28+12}{10}=-1,6 \quad , \quad r_2=\frac{-28-12}{10}=-4.$$

It's clear that $|t|>2$ implies $32+28t+5t^2\geq 0$. If $x=0$ then $a=0$ or $b=0$ and $(\star)$ is obviously verified. $\square$

Remark. A different solution is given in [4], pag. 92, problem 98.

96. (**Belarus 1996**) ($x+y+z=\sqrt{xyz},\ x,y,z>0$)

$$xy+yz+zx \geq 9(x+y+z)$$

First Solution. (*Ercole Suppa*)
From the well-know inwquality $(xy+yz+zx)^3 \geq 3xyz(x+y+z)$ and AM-GM inequality we have

$$\begin{aligned}
(xy+yz+zx)^3 &\geq 3xyz(x+y+z) = \\
&= 3(x+y+z)^3 \geq && \text{(AM-GM)}\\
&\geq 3\left(3\sqrt[3]{xyz}\right)^3 = \\
&= 81xyz = 81(x+y+z)^2
\end{aligned}$$

The required inequality follows extracting the square root. $\square$

Second Solution. (*Cezar Lupu - ML Forum*)
We know that $x+y+z=\sqrt{xyz}$ or $(x+y+z)^2=xyz$. The inequality is equivalent with this one:

$$xyz(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9(x+y+z),$$

or

$$(x+y+z)^2(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9(x+y+z).$$

Finally, our inequality is equivalent with this well-known one:

$$(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9.$$

$\square$

97. (**Iran 1996**) ($a,b,c>0$)

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right) \geq \frac{9}{4}$$

First Solution. (*Iurie Boreico, see* [4], *pag.* 108, *problem* 114) With the substitution $y+z=a$, $z+x=b$, $x+y=c$ the inequality becomes after some easy computations

$$\sum\left(\frac{2}{ab}-\frac{1}{c^2}\right)(a-b)^2$$

Assume LOG that $a \geq b \geq c$. If $2c^2 \geq ab$, each term in the above expression is positive and we are done. So, suppose $2c^2 < ab$. First, we prove that $2b^2 \geq ac$, $2a^2 \geq bc$. Suppose that $2b^2 < ac$. Then $(b+c)^2 \leq 2(b^2+c^2) < a(b+c)$ and so $b+c < a$, false. Clearly, we can write the inequality like that

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)(a-c)^2+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2 \geq \left(\frac{1}{c^2}-\frac{2}{ab}\right)(a-b)^2$$

We can immediately see that the inequality $(a-c)^2 \geq (a-b)^2+(b-c)^2$ holds and thus it suffices to prove that

$$\left(\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\right)(b-c)^2 \geq \left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}\right)(a-b)^2$$

But is clear that

$$\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}\right) < \left(\frac{1}{b}-\frac{1}{c}\right)^2$$

and so the right hand side is at most

$$\frac{(a-b)^2(b-c)^2}{b^2c^2}$$

Also, it is easy to see that

$$\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2} \geq \frac{1}{ac}+\frac{1}{bc} > \frac{(a-b)^2}{b^2c^2}$$

which show that the left hand side is at least

$$\frac{(a-b)^2(b-c)^2}{b^2c^2}$$

and this ends the solution. $\square$

Second Solution. (*Cezar Lupu - ML Forum*) We take $x=p-a, y=p-b, z=p-c$ so the inequality becomes:

$$(p-a)(p-b)+(p-b)(p-c)+(p-c)(p-a)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq \frac{9}{4} \iff$$

$$(p^2-16Rr+5r^2)\left[(4R+r)(p^2-16Rr+5r^2)+4r\left(3R(5R-r)+r(R-2r)\right)\right]+$$
$$+4R^3(R-2r)^2 \geq 0$$

But using Gerrestein's inequality $p^2 \geq 16Rr-5r^2$ and Euler's inequality $R \geq 2r$ we are done. Hope I did not make any stupid mistakes in my calculations. $\square$

Remark. Gerrestein's inequality.
In the triangle ABC we have $p^2 + 5r^2 \geq 16Rr$.

Put $a = x + y, b = y + z, c = z + x, x, y, z > 0$. The inital inequality becomes

$$(x + y + z)^3 \geq 4(x + y)(y + z)(z + x) - 5xyz$$

This one is homogenous so consider $x + y + z = 1$. So we only must prove that

$$1 \geq 4(1 - x)(1 - y)(1 - z) - 5xyz \Leftrightarrow 1 + 9xyz \geq 4(xy + yz + zx)$$

which is true by Shur . Anyway this one is weak , it also follows from

$$s^2 \geq 2R^2 + 8Rr + 3r^2$$

which is little bit stronger.

Third Solution. (*payman_pm - ML Forum*)

$$(xy + yz + zx)(\frac{1}{(x + y)^2} + \frac{1}{(y + z)^2} + \frac{1}{(z + x)^2} =$$
$$= (xy + yz + xz)(\frac{(x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2}{(x + y)^2(y + z)^2(z + x)^2}$$

but we have

$$(xy + yz + zx)((x + y)^2(y + z)^2 + (y + z)^2(z + x)^2 + (z + x)^2(x + y)^2) =$$
$$\sum(x^5y + 2x^4y^2 + \frac{5}{2}x^4yz + 13x^3y^2z + 4x^2y^2z^2)$$

and

$$(x + y)^2(y + z)^2(z + x)^2 = \sum(x^4y^2 + x^4yz + x^3y^3 + 6x^3y^2z + \frac{5}{3}x^2y^2z^2)$$

and by some algebra

$$\sum(4x^5y - x^4y^2 - 3x^3y^3 + x^4yz - 2x^3y^2z + x^2y^2z^2) \geq 0$$

and by using Sschur inequality we have $\sum(x^3 - 2x^2y + xyz) \geq 0$ and if multiply this inequality to xyz :

$$\sum(x^4yz - 2x^3y^2z + x^2y^2z^2) \geq 0 \qquad (1)$$

and by using$AM - GM$ inequality we have

$$\sum((x^5y - x^4y^2) + 3(x^5y - x^3y^3)) \geq 0 \qquad (2)$$

and by using (1), (2) the problem is solved. $\square$

Fourth Solution. (*Darij Grinberg - ML Forum*) I have just found another proof of the inequality which seems to be a bit less ugly than the familiar ones. We first prove a lemma:

Lemma 1. If a, b, c, x, y, z are six nonnegative reals such that $a \geq b \geq c$ and $x \leq y \leq z$, then

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+ \\ +z(a-b)^2\left(3ab+bc+ca-c^2\right) \geq 0.$$

Proof of Lemma 1. Since $a \geq b$, we have $ab \geq b^2$, and since $b \geq c$, we have $bc \geq c^2$. Thus, the terms $3ca+ab+bc-b^2$ and $3ab+bc+ca-c^2$ must be nonnegative. The important question is whether the term $3bc+ca+ab-a^2$ is nonnegative or not. If it is, then we have nothing to prove, since the whole sum

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+ \\ +z(a-b)^2\left(3ab+bc+ca-c^2\right)$$

is trivially nonnegative, as a sum of nonnegative expressions. So we will only consider the case when it is not; i. e., we will consider the case when $3bc+ca+ab-a^2<0$. Then, since $(b-c)^2 \geq 0$, we get $(b-c)^2\left(3bc+ca+ab-a^2\right) \leq 0$, and this, together with $x \leq y$, implies that

$$x(b-c)^2\left(3bc+ca+ab-a^2\right) \geq y(b-c)^2\left(3bc+ca+ab-a^2\right)$$

On the other hand, since $3ab+bc+ca-c^2 \geq 0$ and $(a-b)^2 \geq 0$, we have $(a-b)^2\left(3ab+bc+ca-c^2\right) \geq 0$, which combined with $y \leq z$, yields

$$z(a-b)^2\left(3ab+bc+ca-c^2\right) \geq y(a-b)^2\left(3ab+bc+ca-c^2\right)$$

Hence,

$$x(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+z(a-b)^2\left(3ab+bc+ca-c^2\right)$$
$$\geq y(b-c)^2\left(3bc+ca+ab-a^2\right)+y(c-a)^2\left(3ca+ab+bc-b^2\right)+y(a-b)^2\left(3ab+bc+ca-c^2\right)$$
$$=y\left((b-c)^2\left(3bc+ca+ab-a^2\right)+(c-a)^2\left(3ca+ab+bc-b^2\right)+(a-b)^2\left(3ab+bc+ca-c^2\right)\right)$$

But

$$(b-c)^2\left(3bc+ca+ab-a^2\right)+(c-a)^2\left(3ca+ab+bc-b^2\right)+(a-b)^2\left(3ab+bc+ca-c^2\right)$$
$$\geq (b-c)^2\left(-bc+ca+ab-a^2\right)+(c-a)^2\left(-ca+ab+bc-b^2\right)+(a-b)^2\left(-ab+bc+ca-c^2\right)$$

(since squares of real numbers are always nonnegative)

$$=(b-c)^2(c-a)(a-b)+(c-a)^2(a-b)(b-c)+(a-b)^2(b-c)(c-a)$$

$$=(b-c)(c-a)(a-b)\left(\underbrace{(b-c)+(c-a)+(a-b)}_{=0}\right)=0$$

thus,

$$x\left(b-c\right)^{2}\left(3bc+ca+ab-a^{2}\right)+y\left(c-a\right)^{2}\left(3ca+ab+bc-b^{2}\right)+$$
$$+z\left(a-b\right)^{2}\left(3ab+bc+ca-c^{2}\right)\geq 0$$

and Lemma 1 is proven.

Now to the proof of the Iran 1996 inequality:

We first rewrite the inequality using the $\sum$ notation as follows:

$$\sum\frac{1}{\left(b+c\right)^{2}}\geq\frac{9}{4\left(bc+ca+ab\right)}$$

Upon multiplication with $4(bc+ca+ab)$, this becomes

$$\sum\frac{4\left(bc+ca+ab\right)}{\left(b+c\right)^{2}}\geq 9$$

Subtracting 9, we get

$$\sum\frac{4\left(bc+ca+ab\right)}{\left(b+c\right)^{2}}-9\geq 0$$

which is equivalent to

$$\sum\left(\frac{4\left(bc+ca+ab\right)}{\left(b+c\right)^{2}}-3\right)\geq 0$$

But

$$\frac{4\left(bc+ca+ab\right)}{\left(b+c\right)^{2}}-3=\frac{\left(3b+c\right)\left(a-b\right)}{\left(b+c\right)^{2}}-\frac{\left(3c+b\right)\left(c-a\right)}{\left(b+c\right)^{2}}$$

Hence, it remains to prove

$$\sum\left(\frac{\left(3b+c\right)\left(a-b\right)}{\left(b+c\right)^{2}}-\frac{\left(3c+b\right)\left(c-a\right)}{\left(b+c\right)^{2}}\right)\geq 0$$

But

$$\sum \left(\frac{(3b+c)(a-b)}{(b+c)^2} - \frac{(3c+b)(c-a)}{(b+c)^2} \right) =$$

$$= \sum \frac{(3b+c)(a-b)}{(b+c)^2} - \sum \frac{(3c+b)(c-a)}{(b+c)^2} =$$

$$= \sum \frac{(3b+c)(a-b)}{(b+c)^2} - \sum \frac{(3a+c)(a-b)}{(c+a)^2} =$$

$$= \sum \left(\frac{(3b+c)(a-b)}{(b+c)^2} - \frac{(3a+c)(a-b)}{(c+a)^2} \right) =$$

$$= \sum \frac{(a-b)^2 \left(3ab+bc+ca-c^2\right)}{(b+c)^2 (c+a)^2}$$

Thus, the inequality in question is equivalent to

$$\sum \frac{(a-b)^2 \left(3ab+bc+ca-c^2\right)}{(b+c)^2 (c+a)^2} \geq 0$$

Upon multiplication with $(b+c)^2 (c+a)^2 (a+b)^2$, this becomes

$$\sum (a+b)^2 (a-b)^2 \left(3ab+bc+ca-c^2\right) \geq 0$$

In other words, we have to prove the inequality

$$(b+c)^2 (b-c)^2 \left(3bc+ca+ab-a^2\right) + (c+a)^2 (c-a)^2 \left(3ca+ab+bc-b^2\right) +$$
$$+ (a+b)^2 (a-b)^2 \left(3ab+bc+ca-c^2\right) \geq 0$$

But now it's clear how we prove this - we WLOG assume that $a \geq b \geq c$, and define $x = (b+c)^2$, $y = (c+a)^2$ and $z = (a+b)^2$; then, the required inequality follows from Lemma 1 after showing that $x \leq y \leq z$ (what is really easy: since $a \geq b \geq c$, we have $(a+b+c) - a \leq (a+b+c) - b \leq (a+b+c) - c$, what rewrites as $b+c \leq c+a \leq a+b$, and thus $(b+c)^2 \leq (c+a)^2 \leq (a+b)^2$, or, in other words, $x \leq y \leq z$).
This completes the proof of the Iran 1996 inequality. Feel free to comment or to look for mistakes (you know, chances are not too low that applying a new method one can make a number of mistakes). $\square$

Remark. For different solutions proof see: [19], pag.306; [65], pag.163; Crux Mathematicorum [1994:108; 1995:205; 1996:321; 1997:170,367].

98. (**Vietnam 1996**) $(2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab = 16, a,b,c,d \geq 0)$

$$a+b+c+d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$$

Solution. (*Mohammed Aassila - Crux Mathematicorum 2000, pag.332*)
We first prove a lemma:

LEMMA If x, y, z are real positive numbers such that $x+y+z+xyz=4$, then

$$x+y+z \geq xy+yz+zx$$

PROOF. Suppose that $x+y+z < xy+yz+zx$. From Schur inequality we have

$$\begin{aligned} 9xyz &\geq 4(x+y+z)(xy+yz+zx)-(x+y+z)^3 \geq \\ &\geq 4(x+y+z)^2-(x+y+z)^3 = \\ &= (x+y+z)^2\left[4-(x+y+z)\right] = \\ &= xyz(x+y+z)^2 \end{aligned}$$

Thus

$$(x+y+z)^2 < 9 \quad \Longrightarrow \quad x+y+z < 3$$

and AM-GM implies

$$xyz < \left(\frac{x+y+z}{3}\right)^3 = 1$$

Hence $x+y+z+xyz < 4$ and this is impossible. Therefore we have $x+y+z \geq xy+yz+zx$ and the lemma is proved. □

Now, the given inequality can be proved in the following way. Put

$$S_1 = \sum a \quad , \quad S_2 = \sum ab \quad , \quad S_3 = \sum abc.$$

Let

$$P(x) = (x-a)(x-b)(x-c)(x-d) = x^4 - S_1x^3 + S_2x^2 - S_3x + abcd.$$

Rolle's theorem says that $P'(x)$ has 3 positive roots u, v, w. Thus

$$P'(x) = 4(x-u)(x-v)(x-w) = 4x^3-4(u+v+w)x^2+4(uv+vw+wu)x-4uvw$$

Since $P'(x) = 4x^3 - 3S_1x^2 + 2S_2x - S_3$ we have:

$$S_1 = \frac{4}{3}(u+v+w) \quad , \quad S_2 = 2(uv+vw+wu) \quad , \quad S_3 = 4uvw \qquad (1)$$

From (1) we have

$$\begin{gathered} 2(ab+ac+ad+bc+bd+cd)+abc+bcd+cda+dab=16 \quad \Longleftrightarrow \\ 2S_2+S_3=16 \quad \Longleftrightarrow \quad 4(uv+vw+wu)+4uvw=16 \quad \Longleftrightarrow \\ uv+vw+wu+uvw=4 \qquad (2) \end{gathered}$$

From the Lemma and (1) follows that

$$u+v+w \geq uv+vw+wu \quad \Longleftrightarrow \quad \frac{3}{4}S_1 \geq \frac{1}{2}S_2 \quad \Longleftrightarrow \quad S_1 \geq \frac{2}{3}S_2$$

i.e. $a+b+c+d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$ and we are done. □

Remark. A different solution is given in [6] pag. 98.

3 Years 1990 ∼ 1995

99. (**Baltic Way 1995**) $(a,b,c,d>0)$

$$\frac{a+c}{a+b}+\frac{b+d}{b+c}+\frac{c+a}{c+d}+\frac{d+b}{d+a}\geq 4$$

Solution. (*Ercole Suppa*) From HM-AM inequality we have

$$\frac{1}{x}+\frac{1}{y}\geq\frac{4}{x+y}\quad;\quad\forall x,y\in\mathbb{R}_0^+.$$

Therefore:

$$\begin{aligned}LHS &= (a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)=\\ &=(a+b+c+d)(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)\geq \qquad\text{(HM-AM)}\\ &\geq(a+b+c+d)\frac{4}{a+b+c+d}=4\end{aligned}$$

□

100. (**Canada 1995**) $(a,b,c>0)$

$$a^ab^bc^c\geq abc^{\frac{a+b+c}{3}}$$

First Solution. (*Ercole Suppa*) From Weighted AM-GM inequality applied to the numbers $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ with weights $p_1=\frac{a}{a+b+c}$, $p_2=\frac{b}{a+b+c}$, $p_3=\frac{c}{a+b+c}$ we have

$$\begin{aligned}p_1\cdot\frac{1}{a}+p_2\cdot\frac{1}{b}+p_3\cdot\frac{1}{c}\geq\left(\frac{1}{a}\right)^{p_1}\cdot\left(\frac{1}{b}\right)^{p_2}\cdot\left(\frac{1}{c}\right)^{p_3}\qquad\Longleftrightarrow\\ \frac{3}{a+b+c}\geq\frac{1}{\sqrt[a+b+c]{a^ab^bc^c}}\end{aligned}$$

Thus, the AM-GM inequality yields:

$$\sqrt[a+b+c]{a^ab^bc^c}\geq\frac{a+b+c}{3}\geq\sqrt[3]{abc}\quad\Longrightarrow\quad a^ab^bc^c\geq abc^{\frac{a+b+c}{3}}$$

□

Second Solution. (See [56]*pag.* 15)
We can assume WLOG that $a \leq b \leq c$. Then

$$\log a \leq \log b \leq \log c$$

and, from Chebyshev inequality we get

$$\frac{a+b+c}{3} \cdot \frac{\log a + \log b + \log c}{3} \leq \frac{a \log a + b \log b + c \log c}{3}$$

Therefore

$$a \log a + b \log b + c \log c \geq \frac{a+b+c}{3} (\log a + \log b + \log c) \qquad \Longrightarrow$$
$$a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

$\square$

Third Solution. (*Official solution - Crux Mathematicorum 1995, pag. 224*)
We prove equivalently that

$$a^{3a} b^{3b} c^{3c} \geq (abc)^{a+b+c}$$

Due to complete symmetry in a, b and c, we may assume WLOG that $a \geq b \geq c > 0$. Then $a - b \geq 0$, $b - c \geq 0$, $a - c \geq 0$ and $a/b \geq 1$, $b/c \geq 1$, $a/c \geq 1$. Therefore

$$\frac{a^{3a} b^{3b} c^{3c}}{(abc)^{a+b+c}} = \left(\frac{a}{b}\right)^{a-b} \left(\frac{b}{c}\right)^{b-c} \left(\frac{a}{c}\right)^{a-c} \geq 1$$

$\square$

101. (**IMO 1995, Nazar Agakhanov**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

First Solution. (*See* [32], *pag.* 17)
After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

It follows from the Cauchy-Schwarz inequality that

$$[(y+z) + (z+x) + (x+y)] \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq (x+y+z)^2$$

so that, by the AM-GM inequality,

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$

$\square$

Second Solution. (*See* [32], *pag.* 36)
It's equivalent to

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2(abc)^{4/3}}.$$

Set $a=x^3,b=y^3,c=z^3$ with $x,y,z>0$. Then, it becomes

$$\sum_{\text{cyc}}\frac{1}{x^9(y^3+z^3)}\geq\frac{3}{2x^4y^4z^4}.$$

Clearing denominators, this becomes

$$\sum_{\text{sym}}x^{12}y^{12}+2\sum_{\text{sym}}x^{12}y^9z^3+\sum_{\text{sym}}x^9y^9z^6\geq 3\sum_{\text{sym}}x^{11}y^8z^5+6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}}x^{12}y^{12}-\sum_{\text{sym}}x^{11}y^8z^5\right)+2\left(\sum_{\text{sym}}x^{12}y^9z^3-\sum_{\text{sym}}x^{11}y^8z^5\right)+$$
$$+\left(\sum_{\text{sym}}x^9y^9z^6-\sum_{\text{sym}}x^8y^8z^8\right)\geq 0$$

and every term on the left hand side is nonnegative by Muirhead's theorem. □

102. (**Russia 1995**) $(x,y>0)$

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}$$

Solution. (*Ercole Suppa*) From AM-GM inequality we have:

$$\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}\leq\frac{x}{2\sqrt{x^4y^2}}+\frac{y}{2\sqrt{x^2y^4}}=$$
$$=\frac{1}{2xy}+\frac{1}{2xy}=\frac{1}{xy}$$

□

103. (**Macedonia 1995**) $(a,b,c>0)$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2$$

Solution. (*Manlio Marangelli - ML Forum*)
After setting $A = 1$, $B = \frac{a}{b+c}$ the HM-AM yields

$$\sqrt{AB} \geq \frac{2}{\frac{1}{A} + \frac{1}{B}}$$

so

$$\sqrt{\frac{a}{b+c}} \geq \frac{2}{1 + \frac{b+c}{a}} = \frac{2a}{a+b+c}$$

Similar inequalities are true also for the other two radicals. Therefore

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \geq \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} = 2$$

□

104. (**APMC 1995**) $(m, n \in \mathbb{N},\ x, y > 0)$

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n) \geq nm(x^{n+m-1}y+xy^{n+m-1})$$

Solution. (*See* [5] *pag.* 147)
We rewrite the given inequality in the form

$$mn(x-y)\left(x^{n+m-1}+y^{n+m-1}\right) \geq (n+m-1)\left(x^n-y^n\right)\left(x^m-y^m\right)$$

and divide both sides by $(x-y)^2$ to get the equivalent form

$$nm\left(x^{n+m-2}+x^{n+m-3}y+\cdots+y^{n+m-2}\right) \geq$$
$$\geq (n+m-1)\left(x^{n-1}+\cdots+y^{n-1}\right)\left(x^{m-1}+\cdots+y^{m-2}\right)$$

We now will prove a more general result. Suppose

$$P(x, y) = a_d x^d + \cdots + a_{-d} y^d$$

is a homogeneous polynomial of degree d with the following properties

(a) For $i = 1, \ldots, d$, $a_i = a_{-i}$ (equivalently $P(x, y) = P(y, x)$)

(b) $\sum\limits_{i=-d}^{d} a_d = 0$, (equivalently $P(x, x) = 0$)

(c) For $i = 0, \ldots, d-1$, $a_d + \cdots + a_{d-i} \geq 0$.

Then $P(x,y) \geq 0$ for all $x, y > 0$. (The properties are easily verified for $P(x,y)$ equal to the difference of the two sides in our desidered inequality. The third property follows from the fact that in this case, $a_d \geq a_{d-1} \geq \cdots \geq a_0$). We prove the general result by induction on d, as it is obvious for $d = 0$. Suppose P has the desidered properties, and let

$$\begin{aligned} Q(x,y) = &(a_d + a_{d-1})\, x^{d-1} + a_{d-2}x^{d-2}y + \cdots \\ &+ a_{-d+2}xy^{d-2} + (a_{-d} + a_{-d+1})\, y^{d-1}. \end{aligned}$$

Then Q has smaller degree and satisfies the required properties, so by the induction hypotesis $Q(x,y) \geq 0$. Moreover,

$$\begin{aligned} P(x,y) - Q(x,y) &= a_d \left(x^d - x^{d-1}y - xy^{d-1} + y^d\right) = \\ &= a_d(x-y)\left(x^d - y^d\right) \geq 0 \end{aligned}$$

since $a_d \geq 0$ and the sign of $x - y$ is the same as the sign of $x^d - y^d$. Adding these two inequalities give $P(x,y) \geq 0$, as desidered. $\square$

105. (**Hong Kong 1994**) $(xy + yz + zx = 1,\ x, y, z > 0)$

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}$$

First Solution. (*Grobber - ML Forum*)
What we must prove is

$$x + y + z + xyz(xy + yz + zx) \leq \frac{4\sqrt{3}}{9} + xy(x+y) + yz(y+z) + zx(z+x).$$

By adding $3xyz$ to both sides we get (we can eliminate $xy + yz + zx$ since it's equal to 1)

$$x + y + z + 4xyz \leq \frac{4\sqrt{3}}{9} + x + y + z.$$

By subtracting $x + y + z$ from both sides and dividing by 4 we are left with $xyz \leq \frac{\sqrt{3}}{9}$, which is true by AM-GM applied to xy, yz, zx. $\square$

Second Solution. (*Murray Klamkin - Crux Mathematicorum 1998, pag.394*)
We first convert the inequality to the following equivalent homogeneous one:

$$\begin{aligned} x\left(T_2 - y^2\right)\left(T_2 - z^2\right) + yx\left(T_2 - z^2\right)\left(T_2 - x^2\right) + z\left(T_2 - x^2\right)\left(T_2 - y^2\right) \leq \\ \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}} \end{aligned}$$

where $T_2 = xy + yz + zx$, and for subsequent use $T_1 = x + y + z$, $T_3 = xyz$. Expanding out, we get

$$T_1T_2^2 - T_2 \sum x\left(y^2 + z^2\right) + T_2T_3 \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}}$$

or

$$T_1T_2^2 - T_2\left(T_1T_2 - 3T_3\right) + T_2T_3 = 4T_2T_3 \leq \frac{4\sqrt{3}}{9}\left(T_2\right)^{\frac{5}{2}}$$

Squaring, we get one of the know Maclaurin inequalities for symmetric functions:

$$\sqrt[3]{T_3} \leq \sqrt[3]{\frac{T_2}{3}}$$

There is equality if and only if $x = y = z$. $\square$

106. (**IMO Short List 1993**) $(a, b, c, d > 0)$

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \geq \frac{2}{3}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) From Cauchy-Schwarz inequality we have:

$$\begin{aligned}
(a+b+c+d)^2 &= \left(\sum_{\text{cyc}} \frac{\sqrt{a}}{\sqrt{b+2c+3d}}\sqrt{a}\sqrt{b+2c+3d}\right)^2 \leq \\
&\leq \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\left(\sum_{\text{cyc}} ab + 2ac + 3ad\right) = \\
&= \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\left(\sum_{\text{sym}} ab\right) \leq \\
&\leq \left(\sum_{\text{cyc}} \frac{a}{b+2c+3d}\right)\frac{3}{2}\left(a+b+c+d\right)^2 \qquad (1)
\end{aligned}$$

The inequality of the last step can be proved by BUNCHING principle (Muirhead Theorem) in the following way:

$$\sum_{\text{sym}} ab \leq \frac{3}{2}\left(a+b+c+d\right)^2$$

$$\sum_{\text{sym}} ab \leq \frac{3}{2}\sum_{\text{cyc}} a^2 + \frac{3}{4}\sum_{\text{sym}} ab$$

$$\frac{1}{4}\sum_{\text{sym}} ab \leq \frac{3}{2}\sum_{\text{cyc}} a^2$$

$$\sum_{\text{sym}} ab \leq 6\sum_{\text{cyc}} a^2$$

$$\sum_{\text{sym}} ab \leq \sum_{\text{sym}} a^2$$

From (1) follows that

$$\sum_{\text{cyc}} \frac{a}{b+2c+3d} \geq \frac{2}{3}$$

$\square$

107. (**APMC 1993**) $(a, b \geq 0)$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \leq \frac{a+\sqrt{ab}+b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

Solution. (*Tsaossoglou - Crux Mathematicorum 1997, pag. 73*)
Let $A=\sqrt[6]{a}$, $B=\sqrt[6]{b}$. The first inequality

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4}$$

is equivalent to

$$\begin{aligned} &\left(\sqrt{a}+\sqrt{b}\right)^2 \leq \left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \\ \iff \quad &\left(A^3+B^3\right)^2 \leq \left(A^4+B^4\right)\left(A^2+B^2\right) \end{aligned}$$

which holds by the Cauchy inequality.
The second inequality

$$\frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \leq \frac{a+\sqrt{ab}+b}{3}$$

is equivalent to

$$\begin{aligned}
&3(a+b)+3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\le 4\left(a+\sqrt{ab}+b\right)\\
\Longleftrightarrow\quad &a+3\sqrt[3]{a^2b}+3\sqrt[3]{ab^2}+b\le 2\left(a+\sqrt{ab}+b\right)\\
\Longleftrightarrow\quad &\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3\le 2\left(\sqrt{a}+\sqrt{b}\right)^2\\
\Longleftrightarrow\quad &\left(\frac{A^2+B^2}{2}\right)^3\le\left(\frac{A^3+B^3}{2}\right)^2
\end{aligned}$$

which holds by the power mean inequality.
The third inequality

$$\frac{a+\sqrt{ab}+b}{3}\le\sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

is equivalent to

$$\left(\frac{A^6+A^3B^3+B^6}{3}\right)^2\le\left(\frac{A^4+B^4}{2}\right)^3$$

For this it is enough to prove that

$$\left(\frac{A^4+B^4}{2}\right)^3-\left(\frac{A^6+A^3B^3+B^6}{3}\right)^2\ge 0$$

or

$$\begin{aligned}
&9\left(A^4+B^4\right)^3-8\left(A^6+A^3B^3+B^6\right)^2=\\
=&(A-B)^4(A^8+4A^7B+10A^6B^2+4A^5B^3-2A^4B^4+\\
&+4A^3B^5+10A^2B^6+4AB^7+B^8)\ge\\
\ge&(A-B)^4\left(A^8-2A^4B^4+B^8\right)=\\
=&(A-B)^4\left(A^4-B^4\right)^2\ge 0
\end{aligned}$$

□

108. (**Poland 1993**) $(x,y,u,v>0)$

$$\frac{xy+xv+uy+uv}{x+y+u+v}\ge\frac{xy}{x+y}+\frac{uv}{u+v}$$

Solution. (*Ercole Suppa*)
Is enough to note that

$$\begin{aligned}
&\frac{xy+xv+uy+uv}{x+y+u+v}-\frac{xy}{x+y}+\frac{uv}{u+v}=\\
=&\frac{(x+u)(y+v)}{x+y+u+v}-\frac{xy}{x+y}+\frac{uv}{u+v}=\\
=&\frac{(vx-uy)^2}{(x+y)(u+v)(x+y+u+v)}\geq 0
\end{aligned}$$

□

109. (**IMO Short List 1993**) $(a+b+c+d=1,\ a,b,c,d>0)$

$$abc+bcd+cda+dab\leq\frac{1}{27}+\frac{176}{27}abcd$$

Solution. (*See* [23] *pag.* 580)
Put $f(a,b,c,d)=abc+bcd+cda+dab-\frac{176}{27}abcd$ and note that f is symmetric with respect to the four variables a, b, c, d. We can write

$$f(a,b,c,d)=ab(c+d)+cd(a+b-\frac{176}{27}ab)$$

If $a+b-\frac{176}{27}ab\leq 0$, then using AM-GM for a, b, $c+d$, we have

$$f(a,b,c,d)\leq ab(c+d)=\frac{1}{27}$$

If $a+b-\frac{176}{27}ab>0$ by AM-GM inequality applied to c, d we get

$$f(a,b,c,d)\leq ab(c+d)+\frac{1}{4}(c+d)^2\left(a+b-\frac{176}{27}ab\right)=f\left(a,b,\frac{c+d}{2},\frac{c+d}{2}\right)$$

Consider now the following fourtplets:

$$P_0(a,b,c,d)\quad,\quad P_1\left(a,b,\frac{c+d}{2},\frac{c+d}{2}\right)\quad,\quad P_2\left(\frac{a+b}{2},\frac{a+b}{2},\frac{c+d}{2},\frac{c+d}{2}\right)$$
$$P_3\left(\frac{1}{4},\frac{a+b}{2},\frac{c+d}{2},\frac{1}{4}\right)\quad,\quad P_4\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$$

From the above considerations we deduce that for $i=0,1,2,3$ either $f(P_i)=1/27$ or $f(P_i)\leq f(P_{i+1})$. Since $f(P_4)=1/27$, in every case we are led to

$$f(a,b,c,d)=f(P_0)=\frac{1}{27}$$

Equality occurs only in the cases $\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (with permutations) and $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$.
□

110. (**Italy 1993**) $(0 \le a, b, c \le 1)$

$$a^2 + b^2 + c^2 \le a^2b + b^2c + c^2a + 1$$

First Solution. (*Ercole Suppa*) The given inequality is equivalent to

$$a^2(1-b) + b^2(1-c) + c^2(1-a) \le 1$$

The function

$$f(a, b, c) = a^2(1-b) + b^2(1-c) + c^2(1-a)$$

after setting b, c is convex with respect to the variable a so take its maximum value in $a = 0$ or in $a = 1$. A similar reasoning is true if we fix a, c or a, b. Thus is enough to compute $f(a, b, c)$ when $a, b, c \in \{0, 1\}$. Since f is symmetric (with respect to a, b, c) and:

$$f(0,0,0) = 0, \quad f(0,0,1) = 1, \quad f(0,1,1) = 1, \quad f(1,1,1) = 0$$

the result is proven. $\square$

Second Solution. (*Ercole Suppa*) We have

$$\begin{aligned} a^2(1-b) + b^2(1-c) + c^2(1-a) &\le a(1-b) + b(1-c) + c(1-a) = \\ &= a + b + c - (ab + bc + ca) = \\ &= 1 - (1-a)(1-b)(1-c) - abc \le 1 \end{aligned}$$

$\square$

111. (**Poland 1992**) $(a, b, c \in \mathbb{R})$

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \ge (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

Solution. (*Harazi - ML Forum*) It can be proved observing that

$$(a+b-c)^2(c+a-b)^2 \ge (a^2+b^2-c^2)(c^2+a^2-b^2)$$

which is true because:

$$(a+c-b)^2(b+a-c)^2 = (a^2-(b-c)^2)^2 = a^4 - 2a^2(b-c)^2 + (b-c)^4$$

But $(a^2+c^2-b^2)(b^2+a^2-c^2) = a^4 - (b^2-c^2)^2$. So, it is enough to prove that

$$(b^2-c^2)^2 + (b-c)^4 \ge 2a^2(b-c)^2 \iff (b+c)^2 + (b-c)^2 \ge 2a^2 \iff b^2 + c^2 - a^2 \ge 0$$

We can assume that $b^2 + c^2 - a^2 \ge 0$, $c^2 + a^2 - b^2 \ge 0$, $a^2 + b^2 - c^2 \ge 0$ (only one of them could be negative and then it's trivial), so these inequalities hold. Multiply them and the required inequality is proved. $\square$

112. (**Vietnam 1991**) ($x \geq y \geq z > 0$)

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

First Solution. (*Gabriel - ML Forum*) Since $x \geq y \geq z \geq 0$ we have that,

$$\begin{aligned}\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} = \frac{x^3y^2 + y^3z^2 + z^3x^2}{xyz} &\geq \\ \geq \frac{(x^3 + y^3 + z^3)(x^2 + y^2 + z^2)}{3(xyz)} &\geq \\ \geq \frac{3xyz(x^2 + y^2 + z^2)}{3(xyz)} &= \\ = x^2 + y^2 + z^2\end{aligned}$$

by Chebyshev's inequality $\square$

Second Solution. (*Murray Klamkin - Crux Mathematicorum 1996, pag.111*) Let $z = a$, $y = a + b$, $x = a + b + c$ where $a > 0$ and $b, c \geq 0$. Substituting back in the inequality, multiplying by the least common denominator, we get

$$\begin{aligned}&\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} - x^2 - y^2 - z^2 = \\ =&\frac{1}{a(a+b)(a+b+c)}(a^3b^2 + 3a^2b^3 + 3ab^4 + b^5 + a^3bc + 6a^2b^2c + 8ab^3c + \\ &+ 3b^4c + a^3c^2 + 3a^2bc^2 + 6ab^2c^2 + 3b^3c^2 + abc^3 + b^2c^3 \geq 0\end{aligned}$$

and the inequality is proved. $\square$

Third Solution. (*ductrung - ML Forum*) First, note that

$$\sum \frac{ab(a-b)}{c} = \frac{(a-b)(a-c)(b-c)(ab+bc+ca)}{abc} \geq 0$$

Hence

$$\sum \frac{a^2b}{c} \geq \sum \frac{ab^2}{c}$$

and so

$$2\sum \frac{a^2b}{c} \geq \sum \frac{ab(a+b)}{c}$$

It remains to show that

$$\sum \frac{ab(a+b)}{c} \geq 2(a^2 + b^2 + c^2)$$

or equivalently

$$\sum a^2b^2(a+b) \geq 2abc(a^2+b^2+c^2)$$

But

$$\sum a^2b^2(a+b) - 2abc(a^2+b^2+c^2) = \sum c^3(a-b)^2 \geq 0$$

Remark. Different solutions are given in *Crux Mathematicorum* 1994, pag.43.

113. (**Poland 1991**) $(x^2+y^2+z^2=2,\ x,y,z \in \mathbb{R})$

$$x+y+z \leq 2+xyz$$

First Solution. (*See* [4], *pag.* 57, *problem* 50)
Using the Cauchy-Schwarz inequality, we find that

$$x+y+z-xyz = x(1-yz)+(y+z) \leq \sqrt{[x^2+(y+z)^2]\,[1+(1-yz)^2]}$$

So, it is enough to prove that this last quantity is at most 2, which is equivalent to

$$(2+2yz)\left[2-2yz+(yz)^2\right] \leq 4 \quad \Longleftrightarrow \quad (2yz)^3 \leq (2yz)^2$$

which is clearly true because $2yz \leq y^2+z^2 \leq 2$. □

Second Solution. (*Crux Mathematicorum 1989, pag. 106*)
Put $S=x+y+z$, $P=xyz$. It is enough to show that

$$E = 4-(S-P)^2 \geq 0$$

Now using $x^2+y^2+z^2=2$ we have

$$\begin{aligned}4E &= 2^3-2^2\left(S^2-2\right)+2(4SP)-4P^2 = \\ &= 2^3-2^2(2xy+2yz+2zx)+2\left(4x^2yz+4xy^2z+4xyz^2\right)-8x^2y^2z^2+4P^2 = \\ &= (2-2xy)(2-2yz)(2-2zx)+4P^2\end{aligned}$$

Since

$$\begin{aligned}2-2xy &= z^2+(x-y)^2 \\ 2-2yz &= x^2+(x-z)^2 \\ 2-2zx &= y^2+(z-x)^2\end{aligned}$$

the above quantities are nonnegative. Thus, so also is E, completing the proof. □

Third Solution. (*Crux Mathematicorum 1989, pag. 106*)
Lagrange multipliers provide a straighforward solution. Here the Lagrangian is

$$\mathcal{L} = x + y + z - xyz - \lambda\left(x^2 + y^2 + z^2 - 2\right)$$

Now setting the partial derivatives equal to zero we obtain

$$\begin{aligned} 1 - yz &= 2\lambda x \\ 1 - xz &= 2\lambda y \\ 1 - xy &= 2\lambda z \end{aligned}$$

On subtraction, we get

$$(x - y)(z - 2\lambda) = 0 = (y - z)(x - 2\lambda)$$

Thus the critical points are $x = y = z$ and $x = y$, $z = \left(1 - x^2\right)/x$ and any cyclic permutation. The maximum value corresponds to the critical point $x = y$, $z = \left(1 - x^2\right)/x$. Since $x^2 + y^2 + z^2 = 2$ this leads to

$$\left(3x^2 - 1\right)\left(x^2 - 1\right) = 0$$

Finally, the critical point $(1, 1, 0)$ and permutations of it give the maximum value of $x + y + z - xyz$ to be 2. □

Fourth Solution. (*See* [1] *pag.* 155)
If one of x, y, z is nagative, for example $z < 0$ then

$$2 + xyz - x - y - z = (2 - x - y) - z(1 - xy) \geq 0$$

since $x + y \leq \sqrt{2\left(x^2 + y^2\right)} \leq 2$ and $2xy \leq x^2 + y^2 \leq 2$. Thus, WLOG, we can suppose $0 < x \leq y \leq z$. If $z \leq 1$ then

$$2 + xyz - x - y - z = (1 - x)(1 - y) + (1 - z)(1 - xy) \geq 0$$

If, on the contrary $z > 1$ then by Cauchy-Schwartz inequality we have

$$x + y + z \leq \sqrt{2\left[(x + y)^2 + z^2\right]} = 2\sqrt{xy + 1} \leq xy + 2 \leq xyz + 2$$

□

Remark 1. This inequality was proposted in IMO shortlist 1987 by United Kingdom.

Remark 2. The inequality admit the following generalization: Given real numbers x, y, z such that $x^2 + y^2 + z^2 = k$, $k > 0$, prove the inequality

$$\frac{2}{k}xyz - \sqrt{2k} \leq x + y + z \leq \frac{2}{k}xyz + \sqrt{2k}$$

When $k = 2$, see problem 113.

114. (**Mongolia 1991**) $(a^2+b^2+c^2=2,\ a,b,c\in\mathbb{R})$

$$|a^3+b^3+c^3-abc|\le 2\sqrt{2}$$

Solution. (*ThAzN1 - ML Forum*) It suffices to prove

$$(a^3+b^3+c^3-abc)^2\le 8=(a^2+b^2+c^2)^3.$$

This is equivalent to

$$(a^2+b^2+c^2)^3-(a^3+b^3+c^3)^2+2abc(a^3+b^3+c^3)-a^2b^2c^2\ge 0$$
$$\sum(3a^4b^2+3a^2b^4-2a^3b^3)+2abc(a^3+b^3+c^3)+5a^2b^2c^2\ge 0$$
$$\sum(a^4b^2+a^2b^4+a^2b^2(a-b)^2+a^4(b+c)^2)+5a^2b^2c^2\ge 0$$

$\square$

115. (**IMO Short List 1990**) $(ab+bc+cd+da=1,\ a,b,c,d>0)$

$$\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\ge\frac{1}{3}$$

First Solution. (*See* [23] *pag.* 540)
Let A, B, C, D denote $b+c+d$, $a+c+d$, $a+b+d$, $a+b+c$ respectively. Since $ab+bc+cd+da=1$ the numbers A, B, C, D are all positive. By Cauchy-Schwarz inequality we have

$$a^2+b^2+c^2+d^2\ge ab+bc+cd+da=1$$

We'll prove the required inequality under a weaker condition that A, B, C, D are all positive and $a^2+b^2+c^2+d^2\ge 1$. We may assume, WLOG, that $a\ge b\ge c\ge d\ge 0$. Hence $a^3\ge b^3\ge c^3\ge d^3\ge 0$ and $\frac{1}{A}\ge\frac{1}{B}\ge\frac{1}{C}\ge\frac{1}{D}\ge 0$. Using Chebyshev inequality and Cauchy inequality we obtain

$$\frac{a^3}{A}+\frac{b^3}{B}+\frac{c^3}{C}+\frac{d^3}{D}\ge\frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)\ge$$
$$\ge\frac{1}{16}\left(a^2+b^2+c^2+d^2\right)(a+b+c+d)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)=$$
$$=\frac{1}{48}\left(a^2+b^2+c^2+d^2\right)(A+B+C+D)\left(\frac{1}{A}+\frac{1}{B}\frac{1}{C}+\frac{1}{D}\right)\ge\frac{1}{3}$$

This complete the proof. $\square$

Second Solution. (*Demetres Christofides - J. Sholes WEB site*)
Put

$$\begin{aligned}
&S = a+b+c+d\\
&A = \frac{a^3}{S-a} + \frac{b^3}{S-b} + \frac{c^3}{S-c} + \frac{d^3}{S-d}\\
&B = a^2+b^2+c^2+d^2\\
&C = a(S-a)+b(S-b)+c(S-c)+d(S-d) = 2+2ac+2bd
\end{aligned}$$

By Cauchy-Scwarz we have

$$AC \geq B^2 \tag{1}$$

We also have

$$\begin{aligned}
&(a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2 \geq 0 \quad \Longrightarrow\\
&B \geq ab+bc+cd+da = 1
\end{aligned} \tag{2}$$

and

$$(a-c)^2+(b-d)^2 \geq 0 \quad \Longrightarrow \quad B \geq 2ac+2bd \tag{3}$$

If $2ac+2bd \leq 1$ then $C \leq 3$, so by (1) and (2) we have

$$A \geq \frac{B^2}{C} \geq \frac{1}{C} \geq \frac{1}{3}$$

If $2ac+2bd > 1$ then $C > 3$, so by (1), (2), and (3) we have

$$A \geq \frac{B^2}{C} \geq \frac{B}{C} \geq \frac{2ac+2bd}{C} \geq \frac{C-2}{C} = 1 - \frac{2}{C} > \frac{1}{3}$$

This complete the proof. □

Third Solution. (*Campos - ML Forum*) By Holder we have that

$$\begin{aligned}
&\left(\sum \frac{a^3}{b+c+d}\right)\left(\sum a(b+c+d)\right)\left(\sum 1\right)^2 \geq \left(\sum a\right)^4 \quad \Longrightarrow\\
&\sum \frac{a^3}{b+c+d} \geq \frac{(a+b+c+d)^4}{16(\sum a(b+c+d))}
\end{aligned}$$

but it's easy to verify from the condition that

$$(a+b+c+d)^2 \geq 4(a+c)(b+d) = 4$$

and

$$3(a+b+c+d)^2 \geq 4\sum a(b+c+d)$$

because

$$\begin{aligned}
&3(a+b+c+d)^2-4\sum a(b+c+d)=\\
=&3(a+b+c+d)^2-8(ab+ac+ad+bc+bd+cd)=\\
=&3\left(a^2+b^2+c^2+d^2\right)-2(ab+ac+ad+bc+bd+cd)=\\
=&4\left(a^2+b^2+c^2+d^2\right)-(a+b+c+d)^2\geq 0 \qquad \text{(by Cauchy-Schwarz)}
\end{aligned}$$

This complete the proof. $\square$

4 Supplementary Problems

116. (**Lithuania 1987**) $(x, y, z > 0)$

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\geq\frac{x+y+z}{3}$$

Solution. (*Gibbenergy - ML Forum*) Since $3xy \leq x^2+xy+y^2$ we have

$$\frac{x^3}{x^2+xy+y^2}=x-\frac{xy(x+y)}{x^2+xy+y^2}\geq x-\frac{x+y}{3}$$

Then doing this for all other fractions and summing we obtain the inequality we want to prove. □

Remark. This inequality was proposed in Tournament of the Towns 1998.

117. (**Yugoslavia 1987**) $(a, b > 0)$

$$\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)\geq a\sqrt{b}+b\sqrt{a}$$

First Solution. (*Ercole Suppa*)
From AM-GM inequality follows that $\frac{a+b}{2}\geq\sqrt{ab}$. Therefore

$$\begin{aligned}
&\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-a\sqrt{b}-b\sqrt{a}=\\
&=\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)\geq\\
&\geq\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\frac{a+b}{2}\left(\sqrt{a}+\sqrt{b}\right)=\\
&=\frac{a+b}{2}\left[a+b+\frac{1}{2}-\sqrt{a}-\sqrt{b}\right]=\\
&=\frac{a+b}{2}\left[\left(\sqrt{a}-\frac{1}{2}\right)^2+\left(\sqrt{b}-\frac{1}{2}\right)^2\right]\geq 0
\end{aligned}$$

□

Second Solution. (*Arne- ML Forum*)
The left-hand side equals

$$\frac{a^2}{2}+\frac{b^2}{2}+ab+\frac{a}{4}+\frac{b}{4}$$

Now note that, by AM-GM inequality

$$\frac{a^2}{2}+\frac{ab}{2}+\frac{a}{8}+\frac{b}{8}\geq 4\sqrt[4]{\frac{a^2}{2}\cdot\frac{ab}{2}\cdot\frac{a}{8}\cdot\frac{b}{8}}=a\sqrt{b}$$

Similarly

$$\frac{b^2}{2}+\frac{ab}{2}+\frac{a}{8}+\frac{b}{8}\geq 4\sqrt[4]{\frac{b^2}{2}\cdot\frac{ab}{2}\cdot\frac{a}{8}\cdot\frac{b}{8}}=b\sqrt{a}$$

Adding these inequalities gives the result. □

118. (**Yugoslavia 1984**) $(a,b,c,d>0)$

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2$$

Solution. (See [65] *pag.* 127)
From Cauchy-Schwarz inequality we have

$$(a+b+c+d)^2\leq\sum_{\text{cyc}}a(b+c)\sum_{\text{cyc}}\frac{a}{b+c}=$$
$$=(ab+2ac+bc+2bd+cd+ad)\cdot\sum_{\text{cyc}}\frac{a}{b+c}$$

Then, to establish the required inequality it will be enough to show that

$$(ab+2ac+bc+2bd+cd+ad)\leq\frac{1}{2}(a+b+c+d)^2$$

This inequality it is true because

$$\frac{1}{2}(a+b+c+d)^2-(ab+2ac+bc+2bd+cd+ad)=$$
$$=\frac{1}{2}(a-c)^2+\frac{1}{2}(b-d)^2\geq 0$$

The equality holds if and only if $a=c$ e $b=d$. □

119. (**IMO 1984**) $(x+y+z=1,\ x,y,z\geq 0)$

$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}$$

First Solution. (*See* [32], *pag.* 23)
Let $f(x,y,z) = xy + yz + zx - 2xyz$. We may assume that $0 \le x \le y \le z \le 1$. Since $x + y + z = 1$, we find that $x \le \frac{1}{3}$. It follows that $f(x,y,z) = (1-3x)yz + xyz + zx + xy \ge 0$. Applying the AM-GM inequality, we obtain $yz \le \left(\frac{y+z}{2}\right)^2 = \left(\frac{1-x}{2}\right)^2$. Since $1 - 2x \ge 0$, this implies that

$$f(x,y,z) = x(y+z)+yz(1-2x) \le x(1-x)+\left(\frac{1-x}{2}\right)^2(1-2x) = \frac{-2x^3+x^2+1}{4}.$$

Our job is now to maximize $F(x) = \frac{1}{4}(-2x^3 + x^2 + 1)$, where $x \in \left[0, \frac{1}{3}\right]$. Since $F'(x) = \frac{3}{2}x\left(\frac{1}{3} - x\right) \ge 0$ on $\left[0, \frac{1}{3}\right]$, we conclude that $F(x) \le F(\frac{1}{3}) = \frac{7}{27}$ for all $x \in \left[0, \frac{1}{3}\right]$. □

Second Solution. (*See* [32], *pag.* 31)
Using the condition $x+y+z = 1$, we reduce the given inequality to homogeneous one, i. e.,

$$0 \le (xy + yz + zx)(x + y + z) - 2xyz \le \frac{7}{27}(x + y + z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \le xyz + \sum_{\text{sym}} x^2y.$$

The right hand side inequality simplifies to

$$7\sum_{\text{cyc}} x^3 + 15xyz - 6\sum_{\text{sym}} x^2y \ge 0.$$

In the view of

$$7\sum_{\text{cyc}} x^3+15xyz-6\sum_{\text{sym}} x^2y = \left(2\sum_{\text{cyc}} x^3 - \sum_{\text{sym}} x^2y\right)+5\left(3xyz + \sum_{\text{cyc}} x^3 - \sum_{\text{sym}} x^2y\right),$$

it's enough to show that

$$2\sum_{\text{cyc}} x^3 \ge \sum_{\text{sym}} x^2y \text{ and } 3xyz + \sum_{\text{cyc}} x^3 \ge \sum_{\text{sym}} x^2y.$$

We note that

$$2\sum_{\text{cyc}} x^3-\sum_{\text{sym}} x^2y = \sum_{\text{cyc}}(x^3+y^3)-\sum_{\text{cyc}}(x^2y+xy^2) = \sum_{\text{cyc}}(x^3+y^3-x^2y-xy^2) \ge 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyc}} x(x-y)(x-z) \ge 0,$$

which is a particular case of Schur's theorem. □

120. (**USA 1980**) ($a, b, c \in [0, 1]$)

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution. (See [43] *pag.* 82)
The function

$$f(a, b, c) = \frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c)$$

is convex in each of the three variables a, b, c, so f takes its maximum value in one of eight vertices of the cube $0 \leq a \leq 1$, $0 \leq b \leq 1$, $0 \leq c \leq 1$. Since $f(a, b, c)$ takes value 1 in each of these points, the required inequality is proven. □

121. (**USA 1979**) ($x + y + z = 1,\ x, y, z > 0$)

$$x^3 + y^3 + z^3 + 6xyz \geq \frac{1}{4}.$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$\begin{gathered} 4\left(x^3 + y^3 + z^3\right) + 24xyz \geq (x+y+z)^3 \quad \Longleftrightarrow \\ 3\left(x^3 + y^3 + z^3\right) + 18xyz \geq 3\sum_{\text{sym}} x^2 y \quad \Longleftrightarrow \\ \sum x^3 + 3\sum xyz \geq \sum_{\text{sym}} x^2 \end{gathered}$$

which is true for all $x, y, z > 0$ by Schur inequality. □

122. (**IMO 1974**) ($a, b, c, d > 0$)

$$1 < \frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < 2$$

Solution. (*Ercole Suppa*) We have

$$\begin{gathered} \frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < \\ < \frac{a}{a+b} + \frac{b}{b+a} + \frac{c}{c+d} + \frac{d}{c+d} = 2 \end{gathered}$$

and

$$\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}>$$
$$>\frac{a}{a+b+c+d}+\frac{b}{a+b+c+d}+\frac{c}{a+b+c+d}+\frac{d}{a+b+c+d}=1$$

□

Remark. In the problem 5 of IMO 1974 is requested to find all possible values of

$$S=\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}$$

for arbitrary positive reals a, b, c, d. A detailed solution is given in [59], pag. 203.

123. (**IMO 1968**) $(x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in \mathbb{R}, x_1y_1 > {z_1}^2, x_2y_2 > {z_2}^2)$

$$\frac{1}{x_1y_1-{z_1}^2}+\frac{1}{x_2y_2-{z_2}^2}\ge\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2}$$

Solution. ([23]*pag.* 369)
Define $u_1=\sqrt{x_1y_1}+z_1$, $u_2=\sqrt{x_2y_2}+z_2$, $v_1=\sqrt{x_1y_1}-z_1$ and $v_2=\sqrt{x_2y_2}-z_2$. By expanding both sides we can easily verify

$$(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2=(u_1+u_2)(v_1+v_2)+\left(\sqrt{x_1y_2}-\sqrt{x_2y_1}\right)^2$$

Thus

$$(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2\ge(u_1+u_2)(v_1+v_2)$$

Since $x_iy_i-z_i^2=u_iv_i$ for $i=1,2$, it suffices to prove

$$\frac{8}{(u_1+u_2)(v_1+v_2)}\le\frac{1}{u_1v_1}+\frac{1}{u_2v_2}$$
$$\iff\quad 8u_1u_2v_1v_2\le(u_1+u_2)(v_1+v_2)(u_1v_1+u_2v_2)$$

which trivially follows from the AM-GM inequalities

$$2\sqrt{u_1u_2}\le u_1+u_2\quad,\quad 2\sqrt{v_1v_2}\le v_1+v_2\quad,\quad 2\sqrt{u_1v_1u_2v_2}\le u_1v_1+u_2v_2$$

Equality holds if and only if $x_1y_2=x_2y_1$, $u_1=u_2$ and $v_1=v_2$, i.e. if and only if $x_1=x_2$, $y_1=y_2$ and $z_1=z_2$. □

124. (**Nesbitt's inequality**) $(a, b, c > 0)$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

Solution. (*See* [32], *pag.* 18) After the substitution $x = b+c,\ y = c+a,\ z = a+b$, it becomes

$$\sum_{\text{cyc}}\frac{y+z-x}{2x}\geq\frac{3}{2}\quad or\quad\sum_{\text{cyc}}\frac{y+z}{x}\geq 6,$$

which follows from the AM-GM inequality as following:

$$\sum_{\text{cyc}}\frac{y+z}{x}=\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6.$$

Remark. In [32], are given many other proofs of this famous inequality.

125. (**Polya's inequality**) $(a \neq b, a, b > 0)$

$$\frac{1}{3}\left(2\sqrt{ab}+\frac{a+b}{2}\right)>\frac{a-b}{\ln a-\ln b}$$

Solution. (*Kee-Wai Lau - Crux Mathematicorum 1999, pag.253*)
We can assume WLOG that $a > b$. The required inequality is equivalent to

$$\frac{1}{3}\left(2\sqrt{ab}+\frac{a+b}{2}\right)\geq\frac{a-b}{\ln\frac{a}{b}}$$

or, dividing both members by b :

$$\frac{1}{3}\left(2\sqrt{\frac{a}{b}}+\frac{\frac{a}{b}+1}{2}\right)\geq\frac{\frac{a}{b}-1}{\ln\frac{a}{b}}$$

After setting $x = \sqrt{\frac{a}{b}}$ we must show that

$$\ln x-\frac{3\left(x^2-1\right)}{x^2+4x+1}\geq 0\quad,\quad\forall x\geq 1$$

By putting

$$f(x)=\ln x-\frac{3\left(x^2-1\right)}{x^2+4x+1}$$

a direct calculation show that

$$f'(x)=\frac{(x-1)^4}{x\left(x^2+4x+1\right)^2}$$

Thus $f'(x) > 0$ for all $x > 1$ (i.e. $f(x)$ is increasing for all $x > 1$). Since $f(1) = 0$, we have $f(x) > 0$ for all $x > 1$ and the the result is proven. $\square$

126. (**Klamkin's inequality**) $(-1 < x, y, z < 1)$

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq 2$$

Solution. (*Ercole Suppa*)
From AM-GM inequality we have

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \geq \frac{2}{\sqrt{(1-x^2)(1-y^2)(1-z^2)}} \geq 2$$

$\square$

127. (**Carlson's inequality**) $(a, b, c > 0)$

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}$$

First Solution. (*P. E. Tsaoussoglou - Crux Mathematicorum 1995, pag. 336*)
It is enough to prove that for all positive real numbers a, b, c the following inequality holds

$$64(ab+bc+ca)^3 \leq 27(a+b)^2(b+c)^2(c+a)^2$$

or

$$64 \cdot 3(ab+bc+ca)(ab+bc+ca)^2 \leq 81\left[(a+b)(b+c)(c+a)\right]^2$$

It is know that $3(ab+bc+ca) \leq (a+b+c)^2$. Thus, it is enough to prove one of the following equivalent inequalities

$$8(a+b+c)(ab+bc+ca) \leq 9(a+b)(b+c)(c+a)$$
$$8(a+b)(b+c)(c+a) + 8abc \leq 9(a+b)(b+c)(c+a)$$
$$8abc \leq (a+b)(b+c)(c+a)$$

The last inequality is well-know and this complete the proof. $\square$

Second Solution. (See [49] *pag.* 141)
It is enough to prove that for all positive real numbers a, b, c the following inequality holds

$$64(ab+bc+ca)^3 \leq 27(a+b)^2(b+c)^2(c+a)^2$$

Write $s = a+b+c$, $u = ab+bc+ca$, $v = abc$. Since $a^2+b^2+c^2 > ab+bc+ca = u$ we have

$$s = \sqrt{a^2 + b^2 + c^2 + 2u} \geq \sqrt{3u}$$

By AM-GM inequality

$$s \geq 3\sqrt[3]{abc} \quad , \quad u \geq 3\sqrt[3]{(ab)(bc)(ca)} = 3\sqrt[3]{v^2}$$

and hence $su \geq 9v$. Consequently,

$$\begin{aligned}(a+b)(b+c)(c+a) &= (s-a)(s-b)(s-c) = s^3 + su - v \geq \\ &\geq su - \frac{1}{9}su = \frac{8}{9}su \geq \frac{8}{9}u\sqrt{3u} = \\ &= \frac{8\sqrt{3}}{9}\sqrt{(ab+bc+ca)^3}\end{aligned}$$

and raising both sides to the second power we obtain the asserted inequality. Equality holds if and only if $a = b = c$. $\square$

Remark. The problem was proposed in Austrian-Polish Competition 1992, problem 6.

128. (*See* [4], *Vasile Cirtoaje*) $(a, b, c > 0)$

$$\left(a + \frac{1}{b} - 1\right)\left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right)\left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right)\left(a + \frac{1}{b} - 1\right) \geq 3$$

Solution. (*See* [4], *pag.* 89, *problem* 94) Assume WLOG that $x = \max\{x, y, z\}$. Then

$$x \geq \frac{1}{3}(x+y+z) = \frac{1}{3}\left(a + \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c} - 3\right) \geq \frac{1}{3}(2+2+2-3) = 1$$

On the other hand,

$$\begin{aligned}(x+1)(y+1)(z+1) &= abc + \frac{1}{abc} + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \\ &\geq 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 5 + x + y + z\end{aligned}$$

and hence

$$xyz + xy + yz + zx \geq 4$$

Since

$$y + z = \frac{1}{a} + b + \frac{(c-1)^2}{c} > 0$$

two cases are possible.

(a) Case $yz \leq 0$. We have $xyz \leq 0$, and from $xy+yz+zx \geq 4$ it follows that $xy+yz+zx \geq 4 > 3$.

(b) Case $y, z > 0$. Let $d = \sqrt{\frac{xy+yz+zx}{3}}$. We have to show that $d \geq 1$. By AM-GM we have $xyz \leq d^3$. Thus $xyz+xy+yz+zx \geq 4$ implies $d^3+3d^2 \geq 4$, $(d-1)(d+2)^2 \geq 0$, $d \geq 1$. Equality occurs for $a=b=c=1$.

□

129. (**[ONI], Vasile Cirtoaje**) $(a, b, c, d > 0)$

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0$$

Solution. (*See* [4], *pag.* 60, *n.* 54)
By AM-HM inequality we have

$$\begin{aligned}
&\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}= \\
=&\frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{c+a}{d+a}+\frac{d+b}{a+b}-4= \\
=&(a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)+(b+d)\left(\frac{1}{c+d}+\frac{1}{a+b}\right)-4 \geq \\
\geq&\frac{4(a+c)}{(b+c)+(d+a)}+\frac{4(b+d)}{(c+d)+(a+b)}-4=0
\end{aligned}$$

□

130. (**Elemente der Mathematik, Problem 1207, Šefket Arslanagić**) $(x, y, z > 0)$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$x^2z+y^2x+z^2y \geq (x+y+z)\sqrt[3]{(xyz)^2}$$

The above inequality is obtained by adding the following

$$\begin{aligned}
\frac{1}{3}x^2z+\frac{1}{3}x^2z+\frac{1}{3}xy^2 &\geq x\sqrt[3]{(xyz)^2} \\
\frac{1}{3}xy^2+\frac{1}{3}xy^2+\frac{1}{3}yz^2 &\geq y\sqrt[3]{(xyz)^2} \\
\frac{1}{3}yz^2+\frac{1}{3}yz^2+\frac{1}{3}x^2z &\geq z\sqrt[3]{(xyz)^2}
\end{aligned}$$

which follows from AM-GM inequality. □

131. ($\sqrt{WURZEL}$, **Walther Janous**) $(x+y+z=1,\ x,y,z>0)$

$$(1+x)(1+y)(1+z) \geq (1-x^2)^2+(1-y^2)^2+(1-z^2)^2$$

First Solution. (*Ercole Suppa*) By setting $A = xy+yz+zx$, $B = xyz$, since $x+y+z=1$, we get

$$x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = 1-2A$$

and

$$\begin{aligned} x^4+y^4+z^4 &= \left(x^2+y^2+z^2\right)^2 - 2(x^2y^2+y^2z^2+z^2x^2) = \\ &= (1-2A)^2 - 2\left[(xy+yz+zx)^2 - 2\left(x^2yz+xy^2z+xyz^2\right)\right] = \\ &= (1-2A)^2 - 2\left[A^2 - 2B(x+y+z)\right] = \\ &= (1-2A)^2 - 2A^2 + 4B = \\ &= 2A^2 - 4A + 4B + 1 \end{aligned}$$

The required inequality is equivalent to

$$\begin{aligned} 1+x+y+z+xy+yz+zx+xyz \geq x^4+y^4+z^4-2\left(x^2+y^2+z^2\right)+3 \quad &\iff \\ 2+A+B \geq 2A^2-4A+4B+1-2(1-2A)+3 \quad &\iff \\ A \geq 2A^2+3B \quad &\iff \\ xy+yz+zx \geq 2(xy+yz+zx)^2+3xyz \quad &\iff \\ (x+y+z)^2(xy+yz+zx) \geq 2(xy+yz+zx)^2+3xyz(x+y+z) \quad &\iff \\ \left(x^2+y^2+z^2\right)(xy+yz+zx) \geq 3xyz(x+y+z) \quad &(\star) \end{aligned}$$

The inequality $(\star)$ follows from Muirhead theorem since

$$\left(x^2+y^2+z^2\right)(xy+yz+zx) \geq 3xyz(x+y+z) \quad \iff$$

$$x^3y+x^3z+xy^3+y^3z+xz^3+yz^3 \geq 2x^2yz+2xy^2z+2xyz^2 \quad \iff$$

$$\sum_{\text{sym}} x^3y \geq \sum_{\text{sym}} x^2yz$$

Alternatively is enough to observe that for all $x,y,z \geq 0$ we get

$$\begin{aligned} &\left(x^2+y^2+z^2\right)(xy+yz+zx) - 3xyz(x+y+z) \geq \\ \geq &(xy+yz+zx)^2 - 3xyz(x+y+z) = \\ = &\frac{1}{2}\left[x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2\right] \geq 0 \end{aligned}$$

$\square$

Second Solution. (*Yimin Ge - ML Forum*) Homogenizing gives

$$(x+y+z)(2x+y+z)(x+2y+z)(x+y+2z) \geq \sum((y+z)(2x+y+z))^2$$

By using the Ravi-substitution, we obtain

$$(a+b+c)(a+b)(b+c)(c+a) \geq 2\sum(a(b+c))^2$$

which is equivalent to

$$\sum_{\text{sym}} a^3b \geq \sum_{\text{sym}} a^2b^2$$

which is true. $\square$

Remark. This inequality was proposed in Austrian-Polish Competition 2000, problem 6.

132. ($\sqrt{WURZEL}$**, Heinz-Jürgen Seiffert**) ($xy > 0, x, y \in \mathbb{R}$)

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2}$$

Solution. (*Campos - ML Forum*)
We have

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2} \quad \Leftrightarrow$$

$$\sqrt{\frac{x^2+y^2}{2}} - \sqrt{xy} \geq \frac{x+y}{2} - \frac{2xy}{x+y} \quad \Leftrightarrow$$

$$\frac{(x-y)^2}{2} \geq \frac{(x-y)^2}{2(x+y)} \cdot \left(\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}\right) \quad \Leftrightarrow$$

$$x+y \geq \sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy} \qquad \Leftrightarrow$$

$$\frac{(x+y)^2}{2} \geq 2\sqrt{\frac{x^2+y^2}{2}}\sqrt{xy}$$

and this is AM-GM. $\square$

133. ($\sqrt{WURZEL}$**, Šefket Arslanagić**) $(a,b,c>0)$

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3\,(x+y+z)}$$

Solution. (*Ercole Suppa*) First we prove the following lemma:

Lemma. If $a_1,\cdots,a_n,b_1,\cdots,b_n,c_1,\cdots,c_n$ are real positive numbers, the following inequality holds

$$\left(\sum_{i=1}^{n}a_ib_ic_i\right)^3\leq\left(\sum_{i=1}^{n}{a_i}^3\right)\left(\sum_{i=1}^{n}{b_i}^3\right)\left(\sum_{i=1}^{n}{c_i}^3\right)$$

Proof. By Holder and Cauchy-Schwarz inequalities we have

$$\begin{aligned}\sum a_ib_ic_i&\leq\left(\sum a_i^3\right)^{\frac{1}{3}}\left(\sum(b_ic_i)^{\frac{3}{2}}\right)^{\frac{2}{3}}\leq\\&\leq\left(\sum a_i^3\right)^{\frac{1}{3}}\left[\left(\sum b_i^3\right)^{\frac{1}{2}}\left(\sum c_i^3\right)^{\frac{1}{2}}\right]^{\frac{2}{3}}=\\&=\left(\sum a_i^3\right)^{\frac{1}{3}}\cdot\left(\sum b_i^3\right)^{\frac{1}{3}}\cdot\left(\sum c_i^3\right)^{\frac{1}{3}}\end{aligned}$$

□

In order to show the required inequality we put

$$(a_1,a_2,a_3)=\left(\frac{a}{\sqrt[3]{x}},\frac{b}{\sqrt[3]{y}},\frac{c}{\sqrt[3]{z}}\right)$$

$$(b_1,b_2,b_3)=\left(\sqrt[3]{x},\sqrt[3]{y},\sqrt[3]{z}\right)$$

$$(c_1,c_2,c_3)=(1,1,1)$$

and we use the Lemma:

$$\begin{aligned}(a+b+c)^3&=\left(\sum a_ib_ic_i\right)^3\leq\\&\leq\sum a_i^3\sum b_i^3\sum c_i^3=\\&=\left(\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\right)(x+y+z)(1+1+1)\end{aligned}$$

Finally, dividing by $(x+y+z)$ we have

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3\,(x+y+z)}$$

□

134. ($\sqrt{WURZEL}$**, Šefket Arslanagić**) ($abc = 1, a, b, c > 0$)

$$\frac{1}{a^2(b+c)} + \frac{1}{b^2(c+a)} + \frac{1}{c^2(a+b)} \geq \frac{3}{2}.$$

Solution. (*Ercole Suppa*) After setting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$ we have $xyz = 1$ and the required inequality is equivalent to

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$$

which is the well-know Nesbitt inequality (see Problem 124). □

135. ($\sqrt{WURZEL}$**, Peter Starek, Donauwörth**) ($abc = 1, a, b, c > 0$)

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{2}(a+b)(c+a)(b+c) - 1.$$

Solution. (*Ercole Suppa*) After setting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$ we have $xyz = 1$ and the required inequality is equivalent to

$$x^3 + y^3 + z^3 \geq \frac{1}{2} \cdot \frac{x+y}{xy} \cdot \frac{y+z}{yz} \cdot \frac{z+x}{zx} - 1 \quad \Longleftrightarrow$$

$$2\left(x^3 + y^3 + z^3\right) \geq x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 \quad \Longleftrightarrow$$

$$\sum_{\text{sym}} x^3 \geq \sum_{\text{sym}} x^2y$$

The above inequality follows from Muirhead theorem or can be obtained adding the three inequalities

$$x^3 + y^3 \geq x^2y + xy^2, \qquad y^3 + z^3 \geq y^2z + yz^2, \qquad z^3 + x^3 \geq z^2x + zx^2$$

□

136. ($\sqrt{WURZEL}$**, Peter Starek, Donauwörth**) ($x+y+z = 3, x^2+y^2+z^2 = 7, x, y, z > 0$)

$$1 + \frac{6}{xyz} \geq \frac{1}{3}\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)$$

Solution. (*Ercole Suppa*)
From the constraints $x+y+z=3$, $x^2+y^2+z^2=7$ follows that

$$9=(x+y+z)^2=7+2(xy+yz+zx) \quad \Longrightarrow \quad xy+yz+zx=1$$

The required inequality is equivalent to

$$\begin{aligned}
&3xyz+18 \geq x^2y+y^2z+z^2x && \Longleftrightarrow \\
&3xyz+6(x+y+x)(xy+yz+zx) \geq x^2y+y^2z+z^2x && \Longleftrightarrow \\
&21xyz+5\left(x^2y+y^2z+z^2x\right)+6\left(x^2z+xy^2+yz^2\right) \geq 0
\end{aligned}$$

which is true for all $x,y,z>0$. □

137. ($\sqrt{WURZEL}$**, Šefket Arslanagić**) $(a,b,c>0)$

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} \geq \frac{3\,(a+b+c)}{a+b+c+3}.$$

Solution. (*Ercole Suppa*)
We can assume WLOG that $a+b+c=1$. The required inequality is equivalent to

$$\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} \geq \frac{3}{4} \tag{$\star$}$$

From Cauchy-Scwarz inequality we have

$$1=(a+b+c)^2 \leq \left(\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\right)[a(b+1)+b(c+1)+c(a+1)]$$

Thus, by using the well-know inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get

$$\begin{aligned}
\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1} &\geq \frac{1}{ab+bc+ca+a+b+c}= \\
&=\frac{1}{ab+bc+ca+1} \geq \\
&\geq \frac{1}{\frac{1}{3}+1}=\frac{3}{4}
\end{aligned}$$

and $(\star)$ is proven. □

138. (**[ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva**) $(a,b,c>0)$

$$a^2+b^2+c^2+2abc+3 \geq (1+a)(1+b)(1+c)$$

Solution. (*See* [4], *pag.* 75, *problem* 74)
Let $f(a,b,c) = a^2+b^2+c^2+2abc+3-(1+a)(1+b)(1+c)$. We have to prove that all values of f are nonnegative. If $a,b,c>3$, then we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1 \quad \Longrightarrow \quad ab+bc+ca<abc$$

hence

$$\begin{aligned} f(a,b,c) &= a^2+b^2+c^2+abc+2-a-b-c-ab-bc-ca > \\ &> a^2+b^2+c^2+2-a-b-c>0 \end{aligned}$$

So, we may assume that $a \le 3$ and let $m = \frac{b+c}{2}$. Easy computations show that

$$f(a,b,c)-f(a,m,m) = \frac{(3-a)(b-c)^2}{4} \ge 0$$

and so it remains to prove that

$$f(a,m,m) \ge 0 \quad \Longleftrightarrow \quad (a+1)m^2-2(a+1)m+a^2-a+2 \ge 0$$

This is cleary true, because the discriminant of the quadratic equation is

$$\Delta = -4(a+1)(a-1)^2 \le 0$$

$\square$

139. (**Gazeta Matematică**) $(a,b,c>0)$

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \ge a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}$$

Solution. (*See* [32], *pag.* 43) We obtain the chain of equalities and inequalities

$$\begin{aligned} \sum_{cyc}\sqrt{a^4+a^2b^2+b^4} &= \sum_{cyc}\sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)} \ge \\ &\ge \frac{1}{\sqrt{2}}\sum_{cyc}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) = && \text{(Cauchy-Schwarz)} \\ &= \frac{1}{\sqrt{2}}\sum_{cyc}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right) \ge \\ &\ge \sqrt{2}\sum_{cyc}\sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} \ge && \text{(AM-GM)} \\ &\ge \sqrt{2}\sum_{cyc}\sqrt{a^4+\frac{a^2bc}{2}} = && \text{(Cauchy-Schwarz)} \\ &= \sum_{cyc}\sqrt{2a^4+a^2bc} \end{aligned}$$

$\square$

140. (**C**[2]**2362, Mohammed Aassila**) $(a, b, c > 0)$

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

Solution. (*Crux Mathematicorum 1999, pag. 375, n.2362*) We use the well-know inequality $t + 1/t \geq 2$ for $t > 0$. Equality occurs if and only if $t = 1$. Note that

$$\begin{aligned}
\frac{1+abc}{a(1+b)} &= \frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} - 1 \\
\frac{1+abc}{b(1+c)} &= \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} - 1 \\
\frac{1+abc}{c(1+a)} &= \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a} - 1
\end{aligned}$$

Then

$$\frac{1+abc}{a(1+b)} + \frac{1+abc}{b(1+c)} + \frac{1+abc}{c(1+a)} \geq 2 + 2 + 2 - 3 = 3$$

by the above inequality. Equality holds when

$$\frac{1+a}{a(1+b)} = \frac{1+b}{b(1+c)} = \frac{1+c}{c(1+a)} = 1$$

that is, when $a = b = c = 1$. □

141. (**C2580**) $(a, b, c > 0)$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab}$$

Solution. (*Crux Mathematicorum 2001, pag. 541, n.2580*)
Let $D = abc\left(a^2 + bc\right)\left(b^2 + ac\right)\left(c^2 + ab\right)$. Clearly $D > 0$ and

$$\begin{aligned}
&\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{b+c}{a^2+bc} - \frac{c+a}{b^2+ac} + \frac{a+b}{c^2+ab} = \\
=&\frac{a^4b^4 + b^4c^4 + c^4a^4 - a^4b^2c^2 - b^4c^2a^2 - c^4a^2b^2}{D} = \\
=&\frac{\left(a^2b^2 - b^2c^2\right)^2 + \left(b^2c^2 - c^2a^2\right)^2 + \left(c^2a^2 - a^2b^2\right)^2}{2D} \geq 0
\end{aligned}$$

which shows that the given inequality is true. Equality holds if and only if $a = b = c$. □

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142. (**C2581**) $(a, b, c > 0)$

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c$$

Solution. (*Crux Mathematicorum 2001, pag. 541, n.2581*)
Let $D = (a+b)(b+c)(c+a)$. Clearly $D > 0$. We show that the difference between the left-hand side and the right-hand side of the inequality is nonnegative.

$$\begin{aligned}
&\frac{a^2+bc}{b+c}-a+\frac{b^2+ca}{c+a}-b+\frac{c^2+ab}{a+b}-c=\\
=&\frac{a^2+bc-ab-ac}{b+c}+\frac{b^2+ac+ab-bc}{a+c}+\frac{c^2+ab-ac-bc}{a+b}=\\
=&\frac{(a-b)(a-c)}{b+c}+\frac{(b-a)(b-c)}{a+c}+\frac{(c-a)(c-b)}{a+b}=\\
=&\frac{\left(a^2-b^2\right)\left(a^2-c^2\right)+\left(b^2-a^2\right)\left(b^2-c^2\right)+\left(c^2-a^2\right)\left(c^2-b^2\right)}{D}=\\
=&\frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{D}=\\
=&\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{2D}\geq 0
\end{aligned}$$

Equality holds if and only if $a = b = c$. $\square$

143. (**C2532**) $(a^2 + b^2 + c^2 = 1,\ a, b, c > 0)$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 3+\frac{2(a^3+b^3+c^3)}{abc}$$

Solution. (*Crux Mathematicorum 2001, pag. 221, n.2532*)
We have

$$\begin{aligned}
&\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-3-\frac{2(a^3+b^3+c^3)}{abc}=\\
=&\frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2}-3-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)=\\
=&a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+b^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-2\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)=\\
=&a^2\left(\frac{1}{b}-\frac{1}{c}\right)^2+b^2\left(\frac{1}{c}-\frac{1}{a}\right)^2+c^2\left(\frac{1}{a}-\frac{1}{b}\right)^2\geq 0
\end{aligned}$$

Equality holds if and only if $a = b = c$. $\square$

144. (**C3032, Vasile Cirtoaje**) ($a^2+b^2+c^2=1,\ a,b,c>0$)

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\leq\frac{9}{2}$$

Solution. (*Crux Mathematicorum 2006, pag. 190, problem 3032*)
Note first that the given inequality is equivalent to

$$3-5(ab+bc+ca)+7abc(a+b+c)-9a^2b^2c^2\geq 0$$
$$3-5(ab+bc+ca)+6abc(a+b+c)+abc(a+b+c-9abc)\geq 0 \qquad (1)$$

By the AM-GM inequality we have

$$\begin{aligned} a+b+c-9abc &= (a+b+c)\left(a^2+b^2+c^2\right)-9abc\geq \\ &\geq 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}-9abc=0 \end{aligned} \qquad (2)$$

On the other hand,

$$\begin{aligned} &3-5(ab+bc+ca)+6abc(a+b+c)= \\ =&3\left(a^2+b^2+c^2\right)^2-5(ab+bc+ca)\left(a^2+b^2+c^2\right)+6abc(a+b+c)= \\ =&3\left(a^4+b^4+c^4\right)+6\left(a^2b^2+b^2c^2+c^2a^2\right)+abc(a+b+c) \\ &-5\left[ab\left(a^2+b^2\right)+bc\left(b^2+c^2\right)+ca\left(c^2+a^2\right)\right]= \\ =&\left[2\sum a^4+6\sum a^2b^2-4\sum ab\left(a^2+b^2\right)\right]+ \\ &+\left[a^4+b^4+c^4+abc(a+b+c)-ab\left(a^2+b^2\right)-bc\left(b^2+c^2\right)-ca\left(c^2+a^2\right)\right]= \\ =&\left[(a-b)^4+(b-c)^4+(c-a)^4\right]+ \\ &+a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\geq 0 \end{aligned} \qquad (3)$$

since

$$(a-b)^4+(b-c)^4+(c-a)^4\geq 0$$

and

$$a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\geq 0$$

is the well-knom Schur's inequality. Now (1) follows from (2) and (3). The equality holds if and only if $a=b=c=\sqrt{3}/3$. $\square$

145. (**C2645**) ($a,b,c>0$)

$$\frac{2(a^3+b^3+c^3)}{abc}+\frac{9(a+b+c)^2}{(a^2+b^2+c^2)}\geq 33$$

First Solution. (*Darij Grinberg - ML Forum*)
Equivalently transform our inequality:

$$\frac{2\left(a^3+b^3+c^3\right)}{abc}+\frac{9\left(a+b+c\right)^2}{a^2+b^2+c^2}\geq 33 \Longleftrightarrow$$
$$\left(\frac{2\left(a^3+b^3+c^3\right)}{abc}-6\right)+\left(\frac{9\left(a+b+c\right)^2}{a^2+b^2+c^2}-27\right)\geq 0 \Longleftrightarrow$$
$$2\frac{a^3+b^3+c^3-3abc}{abc}+9\frac{\left(a+b+c\right)^2-3\left(a^2+b^2+c^2\right)}{a^2+b^2+c^2}\geq 0$$

Now, it is well-known that

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-bc-ca-ab\right)$$

and

$$\left(a+b+c\right)^2-3\left(a^2+b^2+c^2\right)=-2\left(a^2+b^2+c^2-bc-ca-ab\right),$$

so the inequality above becomes

$$2\frac{(a+b+c)\left(a^2+b^2+c^2-bc-ca-ab\right)}{abc}+9\frac{-2\left(a^2+b^2+c^2-bc-ca-ab\right)}{a^2+b^2+c^2}\geq 0$$

Now, according to the well-known inequality $a^2+b^2+c^2\geq bc+ca+ab$, we have $a^2+b^2+c^2-bc-ca-ab\geq 0$, so that we can divide this inequality by $a^2+b^2+c^2-bc-ca-ab$ to obtain

$$\begin{aligned} & 2\frac{a+b+c}{abc}+9\frac{-2}{a^2+b^2+c^2}\geq 0\\ \Longleftrightarrow\quad & 2\frac{a+b+c}{abc}-\frac{2\cdot 9}{a^2+b^2+c^2}\geq 0\\ \Longleftrightarrow\quad & \frac{a+b+c}{abc}\geq\frac{9}{a^2+b^2+c^2}\\ \Longleftrightarrow\quad & (a+b+c)\left(a^2+b^2+c^2\right)\geq 9abc\end{aligned}$$

But this is evident, since AM-GM yields $a+b+c\geq 3\sqrt[3]{abc}$ and $a^2+b^2+c^2\geq 3\sqrt[3]{a^2b^2c^2}$, so that $(a+b+c)\left(a^2+b^2+c^2\right)\geq 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}=9abc$.
Proof complete. $\square$

Second Solution. (*Crux Mathematicorum 2002, pag. 279, n.2645*)
On multiplying by the common denominator and performing the necessary calculations, we have that the given inequality is equivalent to

$$2\left(a^3+b^3+c^3\right)\left(a^2+b^2+c^2\right)+9abc(a+b+c)^2-33abc\left(a^2+b^2+c^2\right)\geq 0$$

The left side of this is the product of

$$a^2+b^2+c^2-ab-bc-ca \tag{1}$$

and

$$2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right) \tag{2}$$

The product of (1) and (2) is nonnegative because

$$a^2+b^2+c^2-ab-bc-ca=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\geq 0$$

and (by AM-GM)

$$\begin{aligned}&2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right)\geq\\ \geq&2\left(9\sqrt[9]{a^9b^9c^9}-9abc\right)=0\end{aligned}$$

Equality holds if and only if $a=b=c$. □

Remark. In order to prove that (2) is positive we can use also the S.O.S method (=sum of squares):

$$\begin{aligned}&2\left(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-9abc\right)=\\ =&(a-b)^2(a+b+3c)+(b-c)^2(b+c+3a)+(c-a)^2(c+a+3b)\geq 0\end{aligned}$$

□

146. $(x,y\in\mathbb{R})$

$$-\frac{1}{2}\leq\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\leq\frac{1}{2}$$

First Solution. (*Ercole Suppa*) The required inequality is equivalent to

$$\begin{aligned}-\left(1+x^2\right)\left(1+y^2\right)\leq 2(x+y)(1-xy)\leq\left(1+x^2\right)\left(1+y^2\right)\quad&\Longleftrightarrow\\ -(x+y)^2-(1-xy)^2\leq 2(x+y)(1-xy)\leq(x+y)^2+(1-xy)^2&\end{aligned}$$

which is true by the well-know inequalitie $a^2+b^2\pm 2ab\geq 0$. □

Second Solution. (*See* [25], pag. 185, n.79)
Let

$$\vec{a}=\left(\frac{2x}{1+x^2},\frac{1-x^2}{1+x^2}\right)\quad,\quad\vec{b}=\left(\frac{1-y^2}{1+y^2},\frac{2y}{1+y^2}\right)$$

Then it is easy to verify that $|\vec{a}|=|\vec{b}|=1$. The Cauchy-Schwarz $|\vec{a}\cdot\vec{b}|\leq|\vec{a}|\cdot|\vec{b}|$ inequality implies that

$$|\vec{a}\cdot\vec{b}|=\left|2\cdot\frac{x\left(1-y^2\right)+y\left(1-x^2\right)}{\left(1+x^2\right)\left(1+y^2\right)}\right|=\left|2\cdot\frac{(x+y)(1-xy)}{\left(1+x^2\right)\left(1+y^2\right)}\right|\leq 1$$

Dividing by 2, we get the result. □

147. $(0 < x, y < 1)$

$$x^y + y^x > 1$$

Solution. (*See* [25], *pag.* 198, *n.* 66) First we prove the following lemma:

LEMMA. If u, x are real numbers such that $u > 0$, $0 < x < 1$, we have

$$(1+u)^x < 1 + ux$$

PROOF. Let $f(u) = 1 + xu - (1+u)^x$. We have $f(0) = 0$ and f is increasing in the interval $]0, 1[$ because

$$f'(u) = x - x(1+u)^{x-1} = x\left[1 - \frac{1}{(1+u)^{1-x}}\right] > x > 0$$

Thus $f(u) > 0$ for all $x \in \mathbb{R}$ and the lemma is proved. $\square$

Now, the given inequality can be proved in the following way:

Let $x = \frac{1}{1+u}$, $y = \frac{1}{1+v}$, $u > 0$, $v > 0$. Then, by the LEMMA, we have

$$\begin{aligned} x^y &= \frac{1}{(1+u)^y} > \frac{1}{1+uy} = \frac{1+v}{1+u+v} \\ y^x &> \frac{1}{(1+v)^x} > \frac{1}{1+vx} = \frac{1+u}{1+u+v} \end{aligned}$$

Thus

$$x^y + y^x > \frac{1+v}{1+u+v} + \frac{1+u}{1+u+v} = 1 + \frac{1}{1+u+v} > 1$$

and the inequality is proven. $\square$

148. $(x, y, z > 0)$

$$\sqrt[3]{xyz} + \frac{|x-y| + |y-z| + |z-x|}{3} \geq \frac{x+y+z}{3}$$

Solution. (*Ercole Suppa*)
We can assume WLOG that $x \leq y \leq z$. Let a, b, c be three real numbers such that $x = a$, $y = a + b$, $z = a + b + c$ con $a > 0$, $b, c \geq 0$. The required inequality

is equivalent to:

$$\sqrt[3]{a(a+b)(a+b+c)} + \frac{b+c+b+c}{3} \geq \frac{3a+2b+c}{3} \quad \Longleftrightarrow$$

$$3\sqrt[3]{a(a+b)(a+b+c)} \geq 3a - c \quad \Longleftrightarrow$$

$$54a^2b + 54a^2c + 27ab^2 + 27abc - 9ac^2 + c^3 \geq 0 \quad \Longleftrightarrow$$

$$54a^2b + 27ab^2 + 27abc + \left(54a^2c + c^3 - 9ac^2\right) \geq 0$$

The above inequality is satisfied for all $a > 0$, $b, c \geq 0$ since AM-GM inequality yields

$$54a^2c + c^3 \geq 2\sqrt{54a^2c^4} = 6\sqrt{6}ac^2 \geq 9ac^2$$

$\square$

149. $(a, b, c, x, y, z > 0)$

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}$$

Solution. (*Massimo Gobbino - Winter Campus 2006*) By generalized Holder inequality we have

$$\sum_{i=1}^{n} a_i b_i c_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} c_i^r\right)^{\frac{1}{r}}$$

which is true for all $p, q, r \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. After setting $a_1 = a^{\frac{1}{3}}, b_1 = b^{\frac{1}{3}}, c_1 = c^{\frac{1}{3}}$ e $a_2 = x^{\frac{1}{3}}, b_2 = y^{\frac{1}{3}}, c_2 = z^{\frac{1}{3}}$ we get:

$$\begin{aligned}\sqrt[3]{abc} + \sqrt[3]{xyz} = \sum a_i b_i c_i \leq& \\ \leq& \left(\sum a_i^3\right)^{\frac{1}{3}} \left(\sum b_i^3\right)^{\frac{1}{3}} \left(\sum c_i^3\right)^{\frac{1}{3}} = \\ =& \sqrt[3]{(a+x)(b+y)(c+z)}\end{aligned}$$

$\square$

150. $(x, y, z > 0)$

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1$$

Solution. (*Walther Janous, see* [4], *pag.* 49, *problem* 37)
We have

$$(x+y)(x+z)=xy+\left(x^2+yz\right)+xz \geq xy+2x\sqrt{yz}+xz=\left(\sqrt{xy}+\sqrt{xz}\right)^2$$

Hence

$$\begin{aligned}\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} &\leq \sum \frac{x}{x+\sqrt{xy}+\sqrt{xz}}= \\ &=\sum \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}=1\end{aligned}$$

and the inequality is proved. □

151. $(x+y+z=1,\ x, y, z>0)$

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}} \geq \sqrt{\frac{3}{2}}$$

First Solution. (*Ercole Suppa*)
The function $f(t)=\frac{t}{\sqrt{1-t}}$ is convex on $]0,1[$ because

$$f''(t)=\frac{4-t}{4(1-t)^{\frac{5}{2}}} \geq 0$$

Then by Jensen inequality

$$\begin{gathered}f(x)+f(y)+f(z) \geq 3f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{1}{3}\right) \quad \Longleftrightarrow \\ \frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}} \geq \sqrt{\frac{3}{2}}\end{gathered}$$

□

Second Solution. (*Ercole Suppa*) After setting $a = \sqrt{1-x}$, $b = \sqrt{1-y}$, $c = \sqrt{1-z}$, we have $0 < a, b, c < 1$, $a^2 + b^2 + c^2 = 2$ and the required inequality is equivalent to:

$$\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-b^2}{c} \geq \sqrt{\frac{3}{2}} \quad \Longleftrightarrow$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{\frac{3}{2}} + a + b + c \tag{$\star$}$$

From Cauchy-Schwarz inequality we have

$$2 = a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \quad \Longrightarrow \quad a + b + c \leq 2\sqrt{\frac{3}{2}} \tag{1}$$

From AM-HM inequality we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c} \geq \frac{9}{2}\sqrt{\frac{2}{3}} = 3\sqrt{\frac{3}{2}} \tag{2}$$

By adding (1) and (2) we get ($\star$) and the result is proven. $\square$

Third Solution. (*Campos - ML Forum*) Assume WLOG that $x \geq y \geq z$. Then

$$\frac{1}{\sqrt{1-x}} \geq \frac{1}{\sqrt{1-y}} \geq \frac{1}{\sqrt{1-z}}$$

and, from Chebyshev and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
\sum \frac{x}{\sqrt{1-x}} &\geq \frac{\sum x \cdot \sum \frac{1}{\sqrt{1-x}}}{3} = \\
&= \frac{1}{3} \cdot \sum \frac{1}{\sqrt{1-x}} = \\
&= \frac{1}{3} \cdot \frac{9}{\sum \sqrt{1-x}} \geq \\
&\geq \frac{3}{\sqrt{3 \cdot \sum (1-x)}} = \\
&= \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}
\end{aligned}$$

$\square$

Remark.
The inequality can be generalized in the following way (India MO 1995):

If $x_1, x_2, ..., x_n$ are n real positive numbers such that $x_1 + x_2 + x_3 + ... + x_n = 1$ the following inequality holds

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + ... + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

152. $(a, b, c \in \mathbb{R})$

$$\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

Solution. (*Ercole Suppa*) After setting $a + b + c = t$, from the Minkowski inequality we have:

$$\begin{aligned} &\sqrt{a^2 + (1-b)^2} + \sqrt{b^2 + (1-c)^2} + \sqrt{c^2 + (1-a)^2} \geq \\ \geq &\sqrt{(a+b+c)^2 + (3-a-b-c)^2} = \\ = &\sqrt{t^2 + (3-t)^3} \geq \frac{3\sqrt{2}}{2} \end{aligned}$$

The last step is true since

$$\sqrt{t^2 + (3-t)^3} \geq \frac{3\sqrt{2}}{2} \quad \Longleftrightarrow \quad t^2 + (3-t)^2 \geq \frac{9}{2} \quad \Longleftrightarrow \quad (2t-3)^2 \geq 0$$

$\square$

153. $(a, b, c > 0)$

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

First Solution. (*Ercole Suppa*)
We have:

$$\begin{gathered} \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2} \quad \Longleftrightarrow \\ a^2 - ab + b^2 + b^2 - bc + c^2 + 2\sqrt{(a^2 - ab + b^2)(b^2 - bc + c^2)} \geq a^2 + ac + c^2 \quad \Longleftrightarrow \\ 2\sqrt{(a^2 - ab + b^2)(b^2 - bc + c^2)} \geq ab + bc + ac - 2b^2 \quad \Longleftrightarrow \\ 4(a^2 - ab + b^2)(b^2 - bc + c^2) \geq \left(ab + bc + ac - 2b^2\right)^2 \quad \Longleftrightarrow \\ 3(ab - ac + bc)^2 \geq 0 \end{gathered}$$

and we are done. $\square$

Second Solution. (*Albanian Eagle - ML Forum*)
This inequality has a nice geometric interpretation:
let O, A, B, C be four points such that $\angle AOB = \angle BOC = 60°$ and $OA = a$, $OB = b$, $OC = c$ then our inequality is just the triangle inequality for $\triangle ABC$.

Remark. The idea of second solution can be used to show the following inequality (given in a Singapore TST competition):

Let a, b, c be real positive numbers. Show that

$$c\sqrt{a^2 - ab + b^2} + a\sqrt{b^2 - bc + c^2} \geq b\sqrt{a^2 + ac + c^2}$$

Proof. By using the same notations of second solution, the required inequality is exactly the Tolomeo inequality applied to the quadrilateral $OABC$. $\square$

Third Solution. (*Lovasz - ML Forum*)
We have

$$\sqrt{a^2 - ab + b^2}+\sqrt{b^2 - bc + c^2} = \sqrt{\left(\frac{a}{2} - b\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}+\sqrt{\left(b - \frac{c}{2}\right)^2 + \left(\frac{c\sqrt{3}}{2}\right)^2}$$

In Cartesian Coordinate, let the two vectors $\left(\frac{a}{2} - b, \frac{b\sqrt{3}}{2}\right)$ and $\left(b - \frac{c}{2}, \frac{c\sqrt{3}}{2}\right)$.
Then

$$\vec{a} + \vec{b} = \left(\frac{a-c}{2}, \frac{(a+c)\sqrt{3}}{2}\right).$$

Now use $\|a\| + \|b\| \geq \|a + b\|$, we get:

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{\frac{(a-c)^2}{4} + \frac{3(a+c)^2}{4}} = \\ = \sqrt{a^2 + ac + c^2}$$

$\square$

154. $(xy + yz + zx = 1,\ x, y, z > 0)$

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2}$$

Solution. (*See* [25], pag.185, n.89)
After setting $x = \tan\alpha/2$, $y = \tan\beta/2$, $z = \tan\gamma/2$, by constraint $xy+yz+zx = 1$ follows that

$$\tan\frac{\gamma}{2} = \frac{1-xy}{x+y} = \frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{1}{\tan\frac{\alpha+\beta}{2}} =$$
$$= \cot\frac{\alpha+\beta}{2} = \tan\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)$$

Thus $\alpha+\beta+\gamma = \pi$, so we can assume that α, β, γ are the angles of a triangle. The required inequality is equivalent to

$$\cos\alpha\,\mathrm{sen}\,\alpha + \cos\beta\,\mathrm{sen}\,\beta + \cos\gamma\,\mathrm{sen}\,\gamma \leq \frac{\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma}{2}$$
$$\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma \leq \mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma \tag{1}$$

By sine law, using the common notations, we have

$$\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma = \frac{a+b+c}{2R} = \frac{2s}{2R} = \frac{sr}{Rr} = \frac{S}{rR} \tag{2}$$

If x, y, z are the distances of circumcenter O fromi BC, CA, AB we have

$$\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma = 2\left(\mathrm{sen}\,\alpha\cos\alpha + \mathrm{sen}\,\beta\cos\beta + \mathrm{sen}\,\gamma\cos\gamma\right) =$$
$$= \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{R} =$$
$$= \frac{a\cdot\frac{x}{R} + b\cdot\frac{y}{R} + c\cdot\frac{z}{R}}{R} = \frac{2S}{R^2} \tag{3}$$

From (2), (3) and Euler inequality $R \geq 2r$ we get

$$\frac{\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta + \mathrm{sen}\,\gamma}{\mathrm{sen}\,2\alpha + \mathrm{sen}\,2\beta + \mathrm{sen}\,2\gamma} = \frac{R}{2r} \geq 1$$

and (1) is proven. □

155. $(x, y, z \geq 0)$

$$xyz \geq (y+z-x)(z+x-y)(x+y-z)$$

Solution. (*See* [32], *pag.* 2)
The inequality follows from Schur's inequality because

$$xyz - (y+z-x)(z+x-y)(x+y-z) =$$
$$= x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0$$

The equality hols if and only if $x = y = z$ or $x = y$ and $z = 0$ and cyclic permutations. □

156. $(a, b, c > 0)$

$$\sqrt{ab(a+b)} + \sqrt{bc(b+c)} + \sqrt{ca(c+a)} \geq \sqrt{4abc + (a+b)(b+c)(c+a)}$$

Solution. (*Ercole Suppa*) Squaring both members with easy computations we get that the required inequality is equivalent to:

$$a\sqrt{bc(a+b)(a+c)} + b\sqrt{ac(b+a)(b+c)} + c\sqrt{ab(c+a)(c+b)} \geq 3abc$$

which is true by AM-GM inequality:

$$\begin{aligned} & a\sqrt{bc(a+b)(a+c)} + b\sqrt{ac(b+a)(b+c)} + c\sqrt{ab(c+a)(c+b)} \geq \\ \geq & 3\sqrt[3]{(abc)^2(a+b)(b+c)(c+a)} \geq \\ \geq & 3\sqrt[3]{8(abc)^3} = 6abc \geq 3abc \end{aligned}$$

$\square$

157. (**Darij Grinberg**) $(x, y, z \geq 0)$

$$\left(\sqrt{x\,(y+z)} + \sqrt{y\,(z+x)} + \sqrt{z\,(x+y)}\right) \cdot \sqrt{x+y+z} \geq 2\sqrt{(y+z)\,(z+x)\,(x+y)}$$

First Solution. (*Darij Grinberg - ML Forum*)
Consider the triangle with sides $a = y+z$, $b = z+x$, $c = x+y$ and semiperimeter $s = \frac{a+b+c}{2} = x+y+z$. Then, our inequality becomes

$$\left(\sqrt{(s-a)\,a} + \sqrt{(s-b)\,b} + \sqrt{(s-c)\,c}\right) \cdot \sqrt{s} \geq 2\sqrt{abc}$$

or

$$\sqrt{\frac{s\,(s-a)}{bc}} + \sqrt{\frac{s\,(s-b)}{ca}} + \sqrt{\frac{s\,(s-c)}{ab}} \geq 2$$

If we call A, B, C the angles of our triangle, then this simplifies to

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq 2$$

i. e.

$$\sin\left(90^\circ - \frac{A}{2}\right) + \sin\left(90^\circ - \frac{B}{2}\right) + \sin\left(90^\circ - \frac{C}{2}\right) \geq 2$$

But $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$ and $90^\circ - \frac{C}{2}$ are the angles of an acute triangle (as one can easily see); hence, we must show that if A, B, C are the angles of an acute triangle, then

$$\sin A + \sin B + \sin C \geq 2$$

(Actually, for any non-degenerate triangle, $sinA + sinB + sinC > 2$, but I don't want to exclude degenerate cases.) Here is an elegant proof of this inequality by Arthur Engel: Since triangle ABC is acute, we have $A - B \leq C$, and $\cos \frac{A-B}{2} \geq \cos \frac{C}{2}$, so that

$$\begin{aligned} \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} = \\ = 2 \cos \frac{C}{2} \cos \frac{A-B}{2} \geq \\ \geq 2 \cos^2 \frac{C}{2} = 1 + \cos C \end{aligned}$$

and

$$\sin A + \sin B + \sin C \geq 1 + \cos C + \sin C \geq 2$$

Hereby, we have used the very simple inequality $\cos C + \sin C \geq 1$ for any acute angle C.
(I admit that I did not find the proof while trying to solve the problem, but I rather constructed the problem while searching for a reasonable application of the $\sin A + \sin B + \sin C \geq 2$ inequality, but this doesn't matter afterwards...)

□

Second Solution. (*Harazi - ML Forum*)
Take $x + y + z = 1$. Square the inequality

$$\sum \sqrt{x(1-x)} \geq 2 \cdot \sqrt{(1-x)(1-y)(1-z)}$$

and reduce it to

$$\sum xy - 2xyz \leq \sum \sqrt{xy(y+z)(z+x)}$$

But

$$\sum xy - 2xyz \leq \sum xy$$

and

$$\sum \sqrt{xy(x+z)(y+z)} \geq \sum xy + \sum z \cdot \sqrt{xy}$$

□

Third Solution. (*Zhaobin, Darij Grinberg - ML Forum*)
We have

$$\sqrt{\frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}} \geq \frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}$$

then we get:

$$\sum \sqrt{\frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)}} \geq \sum \frac{x\left(y+z\right)\left(x+y+z\right)}{\left(y+z\right)\left(z+x\right)\left(x+y\right)} =$$

$$= 2\frac{\left(y+z\right)\left(z+x\right)\left(x+y\right)+xyz}{\left(y+z\right)\left(z+x\right)\left(x+y\right)} \geq 2$$

□

158. (**Darij Grinberg**) $(x, y, z > 0)$

$$\frac{\sqrt{y+z}}{x} + \frac{\sqrt{z+x}}{y} + \frac{\sqrt{x+y}}{z} \geq \frac{4\left(x+y+z\right)}{\sqrt{\left(y+z\right)\left(z+x\right)\left(x+y\right)}}.$$

Solution. (*See* [54], *pag.* 18)
By Cauchy, we have $\sqrt{(a+b)(a+c)} \geq a + \sqrt{bc}$. Now,

$$\sum \frac{\sqrt{b+c}}{a} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}} \quad \Longleftrightarrow$$

$$\sum \frac{b+c}{a}\sqrt{(a+b)(a+c)} \geq 4(a+b+c)$$

Substituting our result from Cauchy, it would suffice to show

$$\sum (b+c)\frac{\sqrt{bc}}{a} \geq 2(a+b+c)$$

Assume WLOG $a \geq b \geq c$, implying $b+c \leq c+a \leq a+b$ and $\frac{\sqrt{bc}}{a} \leq \frac{\sqrt{ca}}{b} \leq \frac{\sqrt{ab}}{c}$.
Hence, by Chebyshev and AM-GM,

$$\sum (b+c)\frac{\sqrt{bc}}{a} \geq \frac{(2(a+b+c))\left(\frac{\sqrt{bc}}{a}+\frac{\sqrt{ca}}{b}+\frac{\sqrt{ab}}{c}\right)}{} \geq 2(a+b+c)$$

as desidered. □

159. (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2\left(b+c\right)}{\left(b^2+c^2\right)\left(2a+b+c\right)} + \frac{b^2\left(c+a\right)}{\left(c^2+a^2\right)\left(2b+c+a\right)} + \frac{c^2\left(a+b\right)}{\left(a^2+b^2\right)\left(2c+a+b\right)} > \frac{2}{3}.$$

Solution. (*Zhaobin - ML Forum*)
Just notice

$$(b+c)(a^2+bc) = ba^2+ca^2+b^2c+bc^2 = b(a^2+c^2)+c(a^2+b^2)$$

then let $x = a\left(b^2+c^2\right)$, $y = \frac{b}{a^2+c^2}$, $z = \frac{c}{a^2+b^2}$. The given inequality is equivalent to the well-know Nesbitt inequality.

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}$$

□

160. (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2}{2a^2+(b+c)^2} + \frac{b^2}{2b^2+(c+a)^2} + \frac{c^2}{2c^2+(a+b)^2} < \frac{2}{3}.$$

Solution. (*Darij Grinberg - ML Forum*) The inequality in question,

$$\sum \frac{a^2}{2a^2+(b+c)^2} < \frac{2}{3}$$

rewrites as

$$\frac{2}{3} - \sum \frac{a^2}{2a^2+(b+c)^2} > 0$$

But

$$\begin{aligned}
&\frac{2}{3} - \sum \frac{a^2}{2a^2+(b+c)^2} = \\
=&\frac{2}{3} \cdot \sum \frac{a}{a+b+c} - \sum \frac{a^2}{2a^2+(b+c)^2} = \\
=&\sum \left(\frac{2}{3} \cdot \frac{a}{a+b+c} - \frac{a^2}{2a^2+(b+c)^2} \right) = \\
=&\sum \frac{a(b+c-a)^2 + a(b+c)(b+c-a)}{3(a+b+c)\left(2a^2+(b+c)^2\right)} = \\
=&\sum \frac{a(b+c-a)^2}{3(a+b+c)\left(2a^2+(b+c)^2\right)} + \sum \frac{a(b+c)(b+c-a)}{3(a+b+c)\left(2a^2+(b+c)^2\right)}
\end{aligned}$$

Now, it is obvious that

$$\sum \frac{a(b+c-a)^2}{3(a+b+c)\left(2a^2+(b+c)^2\right)} \geq 0$$

What remains to be proven is the inequality

$$\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{3\left(a+b+c\right)\left(2a^2+\left(b+c\right)^2\right)} > 0$$

which simplifies to

$$\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} > 0$$

Now,

$$\begin{aligned}
\sum \frac{a\left(b+c\right)\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} &= \sum \frac{ab\left(b+c-a\right)+ca\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} = \\
&= \sum \frac{ab\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} + \sum \frac{ca\left(b+c-a\right)}{2a^2+\left(b+c\right)^2} = \\
&= \sum \frac{bc\left(c+a-b\right)}{2b^2+\left(c+a\right)^2} + \sum \frac{bc\left(a+b-c\right)}{2c^2+\left(a+b\right)^2} = \\
&= \sum bc\left(\frac{c+a-b}{2b^2+\left(c+a\right)^2} + \frac{a+b-c}{2c^2+\left(a+b\right)^2}\right) = \\
&= \sum bc\frac{a\left(\left(a+b+c\right)^2+a^2+2bc\right)+\left(b+c\right)\left(b-c\right)^2}{\left(2b^2+\left(c+a\right)^2\right)\left(2c^2+\left(a+b\right)^2\right)} > 0
\end{aligned}$$

$\square$

161. (**Vasile Cirtoaje**) $(a, b, c \in \mathbb{R})$

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$

Solution. (*Darij Grinberg - ML Forum*)
Vasile Cartoaje established his inequality

$$\left(a^2+b^2+c^2\right)^2 \geq 3\left(a^3b+b^3c+c^3a\right)$$

using the identity

$$\begin{aligned}
&4\left(\left(a^2+b^2+c^2\right)-\left(bc+ca+ab\right)\right)\left(\left(a^2+b^2+c^2\right)^2-3\left(a^3b+b^3c+c^3a\right)\right) = \\
&= \left(\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(b^2a+c^2b+a^2c\right)\right)^2 + \\
&\quad + 3\left(\left(a^3+b^3+c^3\right)-\left(a^2b+b^2c+c^2a\right)-2\left(b^2a+c^2b+a^2c\right)+6abc\right)^2
\end{aligned}$$

Actually, this may look a miracle, but there is a very natural way to find this identity. In fact, we consider the function

$$g(a,b,c) = \left(a^2+b^2+c^2\right)^2 - 3\left(a^3b+b^3c+c^3a\right)$$

over all triples $(a,b,c) \in \mathbb{R}^3$. We want to show that this function satisfies $g(a,b,c) \geq 0$ for any three reals a, b, c. Well, fix a triple (a,b,c) and translate it by some real number d; in other words, consider the triple $(a+d, b+d, c+d)$. For which $d \in \mathbb{R}$ will the value $g(a+d, b+d, c+d)$ be minimal? Well, minimizing $g(a+d, b+d, c+d)$ is equivalent to minimizing $g(a+d, b+d, c+d) - g(a,b,c)$ (since (a,b,c) is fixed), but

$$\begin{aligned} & g(a+d, b+d, c+d) - g(a,b,c) = \\ = & d^2\left(\left(a^2+b^2+c^2\right) - (bc+ca+ab)\right) + \\ & + d\left(\left(a^3+b^3+c^3\right) - 5\left(a^2b+b^2c+c^2a\right) + 4\left(b^2a+c^2b+a^2c\right)\right) \end{aligned}$$

so that we have to minimize a quadratic function, what is canonical, and it comes out that the minimum is achieved for

$$d = -\frac{\left(a^3+b^3+c^3\right) - 5\left(a^2b+b^2c+c^2a\right) + 4\left(b^2a+c^2b+a^2c\right)}{2\left(\left(a^2+b^2+c^2\right) - (bc+ca+ab)\right)}$$

So this is the value of d such that $g(a+d, b+d, c+d)$ is minimal. Hence, for this value of d, we have $g(a,b,c) \geq g(a+d, b+d, c+d)$. Thus, in order to prove that $g(a,b,c) \geq 0$, it will be enough to show that $g(a+d, b+d, c+d) \geq 0$. But, armed with the formula

$$d = -\frac{\left(a^3+b^3+c^3\right) - 5\left(a^2b+b^2c+c^2a\right) + 4\left(b^2a+c^2b+a^2c\right)}{2\left(\left(a^2+b^2+c^2\right) - (bc+ca+ab)\right)}$$

and with a computer algebra system or a sufficient patience, we find that

$$\begin{aligned} & g(a+d, b+d, c+d) = \\ = & \frac{3\left(\left(a^3+b^3+c^3\right) - \left(a^2b+b^2c+c^2a\right) - 2\left(b^2a+c^2b+a^2c\right) + 6abc\right)^2}{4\left(\left(a^2+b^2+c^2\right) - (bc+ca+ab)\right)} \end{aligned}$$

what is incontestably ≥ 0. So we have proven the inequality. Now, writing

$$g(a,b,c) = g(a+d, b+d, c+d) - \left(g(a+d, b+d, c+d) - g(a,b,c)\right)$$

and performing the necessary calculations, we arrive at Vasc's mystic identity. $\square$

A Classical Inequalities

Theorem 1. (**AM-GM inequality**)
Let $a_1, \cdots, a_n$ *be positive real numbers. Then, we have*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 2. (**Weighted AM-GM inequality**)
Let $\lambda_1, \cdots, \lambda_n$ *real positive numbers with* $\lambda_1 + \cdots + \lambda_n = 1$*. For all* $x_1, \cdots, x_n > 0$*, we have*

$$\lambda_1 \cdot x_1 + \cdots + \lambda_n \cdot x_n \geq {x_1}^{\lambda_1} \cdots {x_n}^{\lambda_n}.$$

Theorem 3. (**GM-HM inequality**)
Let $a_1, \cdots, a_n$ *be positive real numbers. Then, we have*

$$\sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Theorem 4. (**QM-AM inequality**)
Let $a_1, \cdots, a_n$ *be positive real numbers. Then, we have*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n}$$

Theorem 5. (**Power Mean inequality**)
Let $x_1, \cdots, x_n > 0$*. The power mean of order* p *is defined by*

$$M_0(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 \cdots x_n},$$

$$M_p(x_1, x_2, \ldots, x_n) = \left(\frac{x_1^p + \cdots + {x_n}^p}{n}\right)^{\frac{1}{p}} \qquad (p \neq 0).$$

Then the function $M_p(x_1, x_2, \ldots, x_n) : \mathbb{R} \to \mathbb{R}$ *is continuous and monotone increasing.*

Theorem 6. (**Rearrangement inequality**)
Let $x_1 \geq \cdots \geq x_n$ *and* $y_1 \geq \cdots \geq y_n$ *be real numbers. For any permutation* σ *of* $\{1, \ldots, n\}$*, we have*

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

Theorem 7. (**The Cauchy[3]-Schwarz[4]-Bunyakovsky[5] inequality**)
Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Remark. This inequality apparently was firstly mentioned in a work of A.L. Cauchy in 1821. The integral form was obtained in 1859 by V.Y. Bunyakovsky. The corresponding version for inner-product spaces obtained by H.A. Schwartz in 1885 is mainly known as Schwarz's inequality. In light of the clear historical precedence of Bunyakovsky's work over that of Schwartz, the common practice of referring to this inequality as CS-*inequality* may seem unfair. Nevertheless in a lot of modern books the inequality is named CSB-*inequality* so that both Bunyakovsky and Schwartz appear in the name of this fundamental inequality.

By setting $a_i = \frac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$ the CSB inequality takes the following form

Theorem 8. (**Cauchy's inequality in Engel's form**)
Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Then,

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}$$

Theorem 9. (**Chebyshev's inequality[6]**)
Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 10. (**Hölder's inequality[7]**)
Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

[3] Louis Augustin Cauchy (1789-1857), french mathematician
[4] Hermann Amandus Schwarz (1843-1921), german mathematician
[5] Viktor Yakovlevich Bunyakovsky (1804-1889), russian mathematician
[6] Pafnuty Lvovich Chebyshev (1821-1894), russian mathematician.
[7] Otto Ludwig Hölder (1859-1937), german mathematician

Theorem 11. (**Minkowski's inequality**[8])
If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ *and* $p > 1$, *then*

$$\left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {y_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}}$$

Definition 1. (**Convex functions.**)
We say that a function $f(x)$ *is convex on a segment* $[a, b]$ *if for all* $x_1, x_2 \in [a, b]$

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

Theorem 12. (**Jensen's inequality**[9])
Let $n \geq 2$ *and* $\lambda_1, \ldots, \lambda_n$ *be nonnegative real numbers such that* $\lambda_1 + \cdots + \lambda_n = 1$. *If* $f(x)$ *is convex on* $[a, b]$ *then*

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n x_n$$

for all $x_1, \ldots, x_n \in [a, b]$.

Definition 2. (**Majorization relation for finite sequences**)
Let $a = (a_1, a_2, \ldots, a_n)$ *and* $b = (b_1, b_2, \ldots, b_n)$ *be two (finite) sequences of real numbers such that* $a_1 \geq a_2 \geq \cdots \geq a_n$ *and* $b_1 \geq b_2 \geq \cdots \geq b_n$. *We say that the sequence* a *majorizes the sequence* b *and we write*

$$a \succ b \qquad or \qquad b \prec a$$

if the following two conditions are satisfyied

(i) $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k$, *for all* k, $1 \leq k \leq n - 1$;

(ii) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$.

Theorem 13. (**Majorization inequality | Karamata's inequality**[10])
Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a convex function. Suppose that* $(x_1, \cdots, x_n)$ *majorizes* $(y_1, \cdots, y_n)$, *where* $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a, b]$. *Then, we obtain*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

[8] Hermann Minkowski (1864-1909), german mathematician.
[9] Johan Ludwig William Valdemar Jensen (1859-1925), danish mathematician.
[10] Jovan Karamata (1902-2967), serbian mathematician.

Theorem 14. (**Muirhead's inequality**[11] | **Bunching Principle**)
If $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are two nonincreasing sequences of nonnegative real numbers such that a majorizes b, then we have

$$\sum_{sym} x_1^{a_1} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} \cdots x_n^{b_n}$$

where the sums are taken over all $n!$ permutations of variables $x_1, x_2, \ldots, x_n$.

Theorem 15. (**Schur's inequality**[12])
Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyc}} x^r (x - y)(x - z) \geq 0.$$

Remark. The case $r = 1$ of Schur's inequality is

$$\sum_{\text{sym}} \left(x^3 - 2x^2 y + xyz\right) \geq 0$$

By espanding both the sides and rearranging terms, each of following inequalities is equivalent to the $r = 1$ case of Schur's inequality

- $x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x)$
- $xyx \geq (x + y - z)(y + z - x)(z + x - y)$
- $(x + y + z)^3 + 9xyz \geq 4(x + y + z)(xy + yz + zx)$

Theorem 16. (**Bernoulli's inequality**[13])
For all $r \geq 1$ and $x \geq -1$, we have

$$(1 + x)^r \geq 1 + rx.$$

Definition 3. (**Symmetric Means**)
For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t + x_1) \cdots (t + x_n)$ is called the i-th elementary symmetric function σ_i. This means that

$$(t + x_1) \cdots (t + x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1} t + \sigma_n.$$

[11] Robert Muirhead (1860-1941), english matematician.

[12] Issai Schur (1875-1941), was Jewish a mathematician who worked in Germany for most of his life. He considered himself German rather than Jewish, even though he had been born in the Russian Empire in what is now Belarus, and brought up partly in Latvia.

[13] Jacob Bernouilli (1654-1705), swiss mathematician founded this inequality in 1689. However the same result was exploited in 1670 by the english mathematician Isaac Barrow.

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 17. (**Newton's inequality**[14])
Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have

$$S_i^2 \geq S_{i-1} \cdot S_{i+1}$$

Theorem 18. (**Maclaurin's inequality**[15])
Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n}$$

[14]Sir Isaac Newton (1643-1727), was the greatest English mathematician of his generation. He laid the foundation for differential and integral calculus. His work on optics and gravitation make him one of the greatest scientists the world has known.
[15]Colin Maclaurin (1698-1746), Scottish mathematican.

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Inequalities from 2008 Mathematical Competition

Editor

- **Manh Dung Nguyen,** Special High School for Gifted Students, Hanoi University of Science, Vietnam
- **Vo Thanh Van,** Special High School for Gifted Students, Hue University of Science, Vietnam

Contact

If you have any question about this ebook, please contact us. Email:

`nguyendunghus@gmail.com`

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Abbreviations

- **IMO** International mathematical Olympiad
- **TST** Team Selection Test
- **MO** Mathematical Olympiad
- **LHS** Left hand side
- **RHS** Right hand side
- **W.L.O.G** Without loss of generality
- $\sum$: $\sum_{cyclic}$

Contents

Chapter 1

Problems

Pro 1. ***(Vietnamese National Olympiad 2008)*** *Let x, y, z be distinct non-negative real numbers. Prove that*

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\geq\frac{4}{xy+yz+zx}.$$

∇

Pro 2. ***(Iranian National Olympiad (3rd Round) 2008)***. *Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:*

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\leq K\sqrt{(x+y)(y+z)(z+x)}$$

∇

Pro 3. ***(Iranian National Olympiad (3rd Round) 2008)***. *Let $x, y, z \in \mathbb{R}^+$ and $x+y+z=3$. Prove that:*

$$\frac{x^3}{y^3+8}+\frac{y^3}{z^3+8}+\frac{z^3}{x^3+8}\geq\frac{1}{9}+\frac{2}{27}(xy+xz+yz)$$

∇

Pro 4. ***(Iran TST 2008.)*** *Let $a, b, c > 0$ and $ab+ac+bc=1$. Prove that:*

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}$$

∇

Pro 5. *(**Macedonian Mathematical Olympiad 2008.**) Positive numbers a, b, c are such that $(a+b)(b+c)(c+a)=8$. Prove the inequality*

$$\frac{a+b+c}{3}\geq\sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

∇

Pro 6. ***(Mongolian TST 2008)*** *Find the maximum number C such that for any non-negative x, y, z the inequality*

$$x^3 + y^3 + z^3 + C(xy^2 + yz^2 + zx^2) \geq (C+1)(x^2y + y^2z + z^2x).$$

holds.

∇

Pro 7. ***(Federation of Bosnia, 1. Grades 2008.)*** *For arbitrary reals x, y and z prove the following inequality:*

$$x^2 + y^2 + z^2 - xy - yz - zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}.$$

∇

Pro 8. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove the inequality:*

$$\frac{a^5 + b^5}{ab(a+b)} + \frac{b^5 + c^5}{bc(b+c)} + \frac{c^5 + a^5}{ca(a+b)} \geq 3(ab + bc + ca) - 2$$

∇

Pro 9. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals prove inequality:*

$$(1 + \frac{4a}{b+c})(1 + \frac{4b}{a+c})(1 + \frac{4c}{a+b}) > 25$$

∇

Pro 10. ***(Croatian Team Selection Test 2008)*** *Let x, y, z be positive numbers. Find the minimum value of:*

$$(a) \quad \frac{x^2 + y^2 + z^2}{xy + yz}$$

$$(b) \quad \frac{x^2 + y^2 + 2z^2}{xy + yz}$$

∇

Pro 11. ***(Moldova 2008 IMO-BMO Second TST Problem 2)*** *Let $a_1, \ldots, a_n$ be positive reals so that $a_1 + a_2 + \ldots + a_n \leq \frac{n}{2}$. Find the minimal value of*

$$A = \sqrt{a_1^2 + \frac{1}{a_2^2}} + \sqrt{a_2^2 + \frac{1}{a_3^2}} + \ldots + \sqrt{a_n^2 + \frac{1}{a_1^2}}$$

∇

Pro 12. ***(RMO 2008, Grade 8, Problem 3)*** *Let $a, b \in [0, 1]$. Prove that*

$$\frac{1}{1+a+b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

∇

Pro 13. ***(Romanian TST 2 2008, Problem 1)*** *Let $n \geq 3$ be an odd integer. Determine the maximum value of*

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \ldots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$

where x_i are positive real numbers from the interval $[0, 1]$

∇

Pro 14. ***(Romania Junior TST Day 3 Problem 2 2008)*** *Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:*

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(b + a)} \leq \frac{1}{abc}.$$

∇

Pro 15. ***(Romanian Junior TST Day 4 Problem 4 2008)*** *Determine the maximum possible real value of the number k, such that*

$$(a + b + c)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - k\right) \geq k$$

for all real numbers $a, b, c \geq 0$ with $a + b + c = ab + bc + ca$.

∇

Pro 16. ***(2008 Romanian Clock-Tower School Junior Competition)*** *For any real numbers $a, b, c > 0$, with $abc = 8$, prove*

$$\frac{a - 2}{a + 1} + \frac{b - 2}{b + 1} + \frac{c - 2}{c + 1} \leq 0$$

∇

Pro 17. ***(Serbian National Olympiad 2008)*** *Let a, b, c be positive real numbers such that $x + y + z = 1$. Prove inequality:*

$$\frac{1}{yz + x + \frac{1}{x}} + \frac{1}{xz + y + \frac{1}{y}} + \frac{1}{xy + z + \frac{1}{z}} \leq \frac{27}{31}.$$

∇

Pro 18. ***(Canadian Mathematical Olympiad 2008)*** *Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that*

$$\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2}.$$

∇

Pro 19. ***(German DEMO 2008)*** *Find the smallest constant C such that for all real x, y*

$$1 + (x+y)^2 \leq C \cdot (1+x^2) \cdot (1+y^2)$$

holds.

∇

Pro 20. ***(Irish Mathematical Olympiad 2008)*** *For positive real numbers a, b, c and d such that $a^2+b^2+c^2+d^2 = 1$ prove that*

$$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq 3/32,$$

and determine the cases of equality.

∇

Pro 21. ***(Greek national mathematical olympiad 2008, P1)*** *For the positive integers $a_1, a_2, ..., a_n$ prove that*

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \prod_{i=1}^n a_i$$

where $k = max\{a_1, a_2, ..., a_n\}$ and $t = min\{a_1, a_2, ..., a_n\}$. When does the equality hold?

∇

Pro 22. ***(Greek national mathematical olympiad 2008, P2)***
If x, y, z are positive real numbers with $x, y, z < 2$ and $x^2+y^2+z^2 = 3$ prove that

$$\frac{3}{2} < \frac{1+y^2}{x+2} + \frac{1+z^2}{y+2} + \frac{1+x^2}{z+2} < 3$$

∇

Pro 23. ***(Moldova National Olympiad 2008)*** *Positive real numbers a, b, c satisfy inequality $a+b+c \leq \frac{3}{2}$. Find the smallest possible value for:*

$$S = abc + \frac{1}{abc}$$

∇

Pro 24. ***(British MO 2008)*** *Find the minimum of $x^2+y^2+z^2$ where $x, y, z \in \mathbb{R}$ and satisfy $x^3+y^3+z^3-3xyz = 1$*

∇

Pro 25. ***(Zhautykov Olympiad, Kazakhstan 2008, Question 6)*** *Let a, b, c be positive integers for which $abc = 1$. Prove that*

$$\sum \frac{1}{b(a+b)} \geq \frac{3}{2}.$$

∇

Pro 26. ***(Ukraine National Olympiad 2008, P1)*** *Let x, y and z are non-negative numbers such that $x^2+y^2+z^2=3$. Prove that:*

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{x+y^2+z}}+\frac{z}{\sqrt{x+y+z^2}}\le\sqrt{3}$$

∇

Pro 27. ***(Ukraine National Olympiad 2008, P2)*** *For positive a,b,c,d prove that*

$$(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4\ge 16abcd(1+a)(1+b)(1+c)(1+d)$$

∇

Pro 28. ***(Polish MO 2008, Pro 5)*** *Show that for all nonnegative real values an inequality occurs:*

$$4(\sqrt{a^3b^3}+\sqrt{b^3c^3}+\sqrt{c^3a^3})\le 4c^3+(a+b)^3.$$

∇

Pro 29. ***(Brazilian Math Olympiad 2008, Problem 3)****. Let x,y,z real numbers such that $x+y+z=xy+yz+zx$. Find the minimum value of*

$$\frac{x}{x^2+1}+\frac{y}{y^2+1}+\frac{z}{z^2+1}$$

∇

Pro 30. ***(Kiev 2008, Problem 1)****. Let $a,b,c\ge 0$. Prove that*

$$\frac{a^2+b^2+c^2}{5}\ge min((a-b)^2,(b-c)^2,(c-a)^2)$$

∇

Pro 31. ***(Kiev 2008, Problem 2)****. Let $x_1,x_2,\cdots,x_n\ge 0, n>3$ and $x_1+x_2+\cdots+x_n=2$ Find the minimum value of*

$$\frac{x_2}{1+x_1^2}+\frac{x_3}{1+x_2^2}+...+\frac{x_1}{1+x_n^2}$$

∇

Pro 32. ***(Hong Kong TST1 2009, Problem 1)****. Let $\theta_1,\theta_2,\ldots,\theta_{2008}$ be real numbers. Find the maximum value of*

$$\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_3+\ldots+\sin\theta_{2007}\cos\theta_{2008}+\sin\theta_{2008}\cos\theta_1$$

∇

Pro 33. ***(Hong Kong TST1 2009, Problem 5).*** *Let a, b, c be the three sides of a triangle. Determine all possible values of*

$$\frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$\nabla$$

Pro 34. ***(Indonesia National Science Olympiad 2008).*** *Prove that for x and y positive reals,*

$$\frac{1}{(1+\sqrt{x})^2}+\frac{1}{(1+\sqrt{y})^2}\geq\frac{2}{x+y+2}.$$

$$\nabla$$

Pro 35. ***(Baltic Way 2008).*** *Prove that if the real numbers a, b and c satisfy $a^2+b^2+c^2=3$ then*

$$\sum\frac{a^2}{2+b+c^2}\geq\frac{(a+b+c)^2}{12}.$$

When does the inequality hold?

$$\nabla$$

Pro 36. ***(Turkey NMO 2008 Problem 3).*** *Let $a.b.c$ be positive reals such that their sum is 1. Prove that*

$$\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{a^2c^2}{b^3(a^2-ac+c^2)}\geq\frac{3}{ab+bc+ac}$$

$$\nabla$$

Pro 37. ***(China Western Mathematical Olympiad 2008).*** *Given $x,y,z\in(0,1)$ satisfying that*

$$\sqrt{\frac{1-x}{yz}}+\sqrt{\frac{1-y}{xz}}+\sqrt{\frac{1-z}{xy}}=2.$$

Find the maximum value of xyz.

$$\nabla$$

Pro 38. ***(Chinese TST 2008 P5).*** *For two given positive integers $m,n>1$, let $a_{ij}(i=1,2,\cdots,n,j=1,2,\cdots,m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where*

$$f=\frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m}a_{ij})^2+m\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2+mn\sum_{i=1}^{n}\sum_{i=j}^{m}a_{ij}^2}$$

$$\nabla$$

Pro 39. ***(Chinese TST 2008 P6)*** *Find the maximal constant M, such that for arbitrary integer $n\geq 3$, there exist two sequences of positive real number $a_1,a_2,\cdots,a_n$, and $b_1,b_2,\cdots,b_n$, satisfying*
(1):$\sum_{k=1}^{n}b_k=1,2b_k\geq b_{k-1}+b_{k+1},k=2,3,\cdots,n-1$;
(2):$a_k^2\leq 1+\sum_{i=1}^{k}a_ib_i,k=1,2,3,\cdots,n,a_n\equiv M$.

$$\nabla$$

Chapter 2

Solutions

Problem 1. ***(Vietnamese National Olympiad 2008)*** *Let x, y, z be distinct non-negative real numbers. Prove that*

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\ge\frac{4}{xy+yz+zx}.$$

*Proof. (Posted by **Vo Thanh Van**).* Assuming $z = min\{x, y, z\}$. We have

$$(x-z)^2+(y-z)^2=(x-y)^2+2(x-z)(y-z)$$

So by the **AM-GM inequality**, we get

$$\begin{aligned}\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}&=\frac{1}{(x-y)^2}+\frac{(x-y)^2}{(y-z)^2(z-x)^2}+\frac{2}{(x-z)(y-z)}\\&\ge\frac{2}{(x-z)(y-z)}+\frac{2}{(x-z)(y-z)}=\frac{4}{(x-z)(y-z)}\\&\ge\frac{4}{xy+yz+zx}\end{aligned}$$

Q.E.D. □

*Proof. (Posted by **Altheman**).* Let $f(x, y, z)$ denote the **LHS** minus the **RHS**. Then $f(x+d, y+d, z+d)$ is increasing in d so we can set the least of $x+d, y+d, z+d$ equal to zero (WLOG $z = 0$). Then we have

$$\frac{1}{(x-y)^2}+\frac{1}{x^2}+\frac{1}{y^2}-\frac{4}{xy}=\frac{(x^2+y^2-3xy)^2}{x^2y^2(x-y)^2}\ge 0$$

□

∇

Problem 2. ***(Iranian National Olympiad (3rd Round) 2008).*** *Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:*

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\le K\sqrt{(x+y)(y+z)(z+x)}$$

*Proof. (Posted by **nayel**).* By the **Cauchy-Schwarz inequality**, we have

$$LHS = \sqrt{x}\sqrt{xy} + \sqrt{y}\sqrt{yz} + \sqrt{z}\sqrt{zx} \le \sqrt{(x+y+z)(xy+yz+zx)}$$
$$\le \frac{3}{2\sqrt{2}}\sqrt{(x+y)(y+z)(z+x)}$$

where the last inequality follows from

$$8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$$

which is well known. □

*Proof. (Posted by **rofler**).* We want to find the **smallest** K. I claim $K = \dfrac{3}{2\sqrt{2}}$. The inequality is equivalent to

$$8(x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 \le 9(x+y)(y+z)(z+x)$$

$$\Longleftrightarrow 8x^2y + 8y^2z + 8z^2x + 16xy\sqrt{yz} + 16yz\sqrt{zx} + 16xz\sqrt{xy} \le 9\sum_{sym} x^2y + 18xyz$$

$$\Longleftrightarrow 16xy\sqrt{yz} + 16yz\sqrt{zx} + 16xz\sqrt{xy} \le x^2y + y^2z + z^2x + 9y^2x + 9z^2y + 9x^2z + 18xyz$$

By the **AM-GM inequality**, we have

$$z^2x + 9y^2x + 6xyz \ge 16\sqrt[16]{z^2x \cdot y^{18}x^9 \cdot x^6y^6z^6} = 16xy\sqrt{xz}$$

Sum up cyclically. We can get equality when $x = y = z = 1$, so we know that K cannot be any smaller. □

*Proof. (Posted by **FelixD**).* We want to find the smallest K such that

$$(x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 \le K^2(x+y)(y+z)(z+x)$$

But

$$(x\sqrt{y} + y\sqrt{z} + z\sqrt{x})^2 = \sum_{cyc} x^2y + 2\Big(\sum_{cyc} xy\sqrt{yz}\Big)$$
$$\le \sum_{cyc} x^2y + 2\Big(\sum_{cyc} \frac{xyz + xy^2}{2}\Big)$$
$$= (x+y)(y+z)(z+x) + xyz$$
$$\le (x+y)(y+z)(z+x) + \frac{1}{8}(x+y)(y+z)(z+x)$$
$$= \frac{9}{8}(x+y)(y+z)(z+x)$$

Therefore,

$$K^2 \ge \frac{9}{8} \to K \ge \frac{3}{2\sqrt{2}}$$

with equality holds if and only if $x = y = z$. □

∇

Problem 3. ***(Iranian National Olympiad (3rd Round) 2008)***. *Let $x, y, z \in \mathbb{R}^+$ and $x + y + z = 3$. Prove that:*

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy + xz + yz)$$

Proof. (Posted by ***rofler****).* By the **AM-GM inequality**, we have

$$\frac{x^3}{(y+2)(y^2-2y+4)} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \geq \frac{x}{3}$$

Summing up cyclically, we have

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} + \frac{x^2+y^2+z^2-(x+y+z)+6*3}{27}$$

$$\geq 1 \geq \frac{1}{3} + \frac{1}{9} - \frac{x^2+y^2+z^2}{27}$$

Hence it suffices to show that

$$\frac{1}{3} - \frac{x^2+y^2+z^2}{27} \geq \frac{2}{27}(xy + xz + yz)$$
$$\iff 9 - (x^2+y^2+z^2) \geq 2(xy + xz + yz)$$
$$\iff 9 \geq (x+y+z)^2 = 9$$

Q.E.D. □

∇

Problem 4. ***(Iran TST 2008.)*** *Let $a, b, c > 0$ and $ab + ac + bc = 1$. Prove that:*

$$\sqrt{a^3+a} + \sqrt{b^3+b} + \sqrt{c^3+c} \geq 2\sqrt{a+b+c}$$

Proof. (Posted by ***Albanian Eagle****).* It is equivalent to:

$$\sum_{cyc} \frac{a}{\sqrt{a(b+c)}} \geq 2\sqrt{\frac{(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)}}$$

Using the **Jensen inequality**, on $f(x) = \dfrac{1}{\sqrt{x}}$, we get

$$\sum_{cyc} \frac{a}{\sqrt{a(b+c)}} \geq \frac{a+b+c}{\sqrt{\dfrac{\sum_{sym} a^2b}{a+b+c}}}$$

So we need to prove that

$$(a+b+c)^2\left(\sum_{sym} a^2b + 2abc\right) \geq 4(ab+bc+ca)\left(\sum_{sym} a^2b\right)$$

Now let c be the smallest number among a, b, c and we see we can rewrite the above as

$$(a-b)^2(a^2b + b^2a + a^2c + b^2c - ac^2 - bc^2) + c^2(a+b)(c-a)(c-b) \geq 0$$

□

*Proof. (Posted by **Campos**).* The inequality is equivalent to

$$\sum \sqrt{a(a+b)(a+c)} \geq 2\sqrt{(a+b+c)(ab+bc+ca)}$$

After squaring both sides and canceling some terms we have that it is equivalent to

$$\sum a^3 + abc + 2(b+c)\sqrt{bc(a+b)(a+c)} \geq \sum 3a^2b + 3a^2c + 4abc$$

From the **Schur's inequality** we have that it is enough to prove that

$$\sum (b+c)\sqrt{(ab+b^2)(ac+c^2)} \geq \sum a^2b + a^2c + 2abc$$

From the **Cauchy-Schwarz inequality** we have

$$\sqrt{(ab+b^2)(ac+c^2)} \geq a\sqrt{bc} + bc$$

so

$$\sum (b+c)\sqrt{(ab+b^2)(ac+c^2)} \geq \sum a(b+c)\sqrt{bc} + bc(b+c) \geq \sum a^2b + a^2c + 2abc$$

as we wanted to prove. □

*Proof. (Posted by **anas**).* Squaring the both sides , our inequality is equivalent to:

$$\sum a^3 - 3\sum ab(a+b) - 9abc + 2\sum \sqrt{a(a+b)(a+c)}\sqrt{b(b+c)(b+a)} \geq 0$$

But, by the **AM-GM inequality**, we have:

$$\begin{aligned} a(a+b)(a+c)\cdot b(b+c)(b+a) & \\ &= (a^3+a^2c+a^2b+abc)(ab^2+b^2c+b^3+abc) \\ &\geq (a^2b+abc+ab^2+abc)^2 \end{aligned}$$

So we need to prove that:

$$a^3+b^3+c^3-ab(a+b)-ac(a+c)-bc(b+c)+3abc \geq 0$$

which is clearly true by the **Schur inequality** □

∇

Problem 5. ***Macedonian Mathematical Olympiad 2008.*** *Positive numbers a, b, c are such that $(a+b)(b+c)(c+a) = 8$. Prove the inequality*

$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

*Proof. (Posted by **argady**).* By the **AM-GM inequality**, we have

$$(a+b+c)^3 = a^3+b^3+c^3+24 = a^3+b^3+c^3+3+\cdots+3 \geq 9\sqrt[9]{(a^3+b^3+c^3)\cdot 3^8}$$

Q.E.D. □

*Proof. (Posted by **kunny**).* The inequality is equivalent to

$$(a+b+c)^{27} \geq 3^{26}(a^3+b^3+c^3) \ \cdots [*]$$

Let $a+b=2x,\ b+c=2y,\ c+a=2z$, we have that

$$(a+b)(b+c)(c+a)=8 \Longleftrightarrow xyz=1$$

and

$$2(a+b+c)=2(x+y+z) \Longleftrightarrow a+b+c=x+y+z$$

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a) \Longleftrightarrow a^3+b^3+c^3=(x+y+z)^3-24$$

Therefore

$$[*] \Longleftrightarrow (x+y+z)^{27} \geq 3^{26}\{(x+y+z)^3-24\}.$$

Let $t=(x+y+z)^3$, by **AM-GM inequality**, we have that

$$x+y+z \geq 3\sqrt[3]{xyz} \Longleftrightarrow x+y+z \geq 3$$

yielding $t \geq 27$.
Since $y=t^9$ is an increasing and concave up function for $t>0$, the tangent line of $y=t^9$ at $t=3$ is $y=3^{26}(t-27)+3^{27}$.We can obtain

$$t^9 \geq 3^{26}(t-27)+3^{27}$$

yielding $t^9 \geq 3^{26}(t-24)$, which completes the proof. □

*Proof. (Posted by **kunny**).* The inequality is equivalent to

$$\frac{(a+b+c)^{27}}{a^3+b^3+c^3} \geq 3^{26}.$$

Let $x=(a+b+c)^3$, by the **AM-GM inequality**, we have:

$$8=(a+b)(b+c)(c+a) \leq \left(\frac{2(a+b+c)}{3}\right)^3$$

so $a+b+c \geq 3$ The left side of the above inequality

$$f(x) := \frac{x^9}{x-24} \Longrightarrow f'(x)=\frac{8x^8(x-27)}{(x-24)^2} \geq 0$$

We have $f(x) \geq f(27)=3^{26}$. □

∇

Problem 6. ***(Mongolian TST 2008)*** *Find the maximum number C such that for any nonnegative x,y,z the inequality*

$$x^3+y^3+z^3+C(xy^2+yz^2+zx^2) \geq (C+1)(x^2y+y^2z+z^2x).$$

holds.

*Proof. (Posted by **hungkhtn**).* Applying **CID** (**C**yclic **I**nequality of **D**egree 3) [1] theorem, we can let $c = 0$ in the inequality. It becomes

$$x^3 + y^3 + cx^2y \geq (c+1)xy^2.$$

Thus, we have to find the minimal value of

$$f(y) = \frac{y^3 - y^2 + 1}{y^2 - y} = y + \frac{1}{y(y-1)}$$

when $y > 1$. It is easy to find that

$$f'(y) = 0 \Leftrightarrow 2y - 1 = (y(y-1))^2 \Leftrightarrow y^4 - 2y^3 + y^2 - 2y + 1 = 0.$$

Solving this symmetric equation gives us:

$$y + \frac{1}{y} = 1 + \sqrt{2} \Rightarrow y = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}$$

Thus we found the best value of C is

$$y + \frac{1}{y(y-1)} = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2} + \frac{1}{\sqrt{\sqrt{2} + \sqrt{2\sqrt{2} - 1}}} \approx 2.4844$$

□

∇

Problem 7. ***(Federation of Bosnia, 1. Grades 2008.)*** *For arbitrary reals x, y and z prove the following inequality:*

$$x^2 + y^2 + z^2 - xy - yz - zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}.$$

*Proof. (Posted by **delegat**).* Assume that $\frac{3(x-y)^2}{4}$ is max. The inequality is equivalent to

$$4x^2 + 4y^2 + 4z^2 \geq 4xy + 4yz + 4xz + 3x^2 - 6xy + 3y^2$$

$$\Leftrightarrow x^2 + 2xy + y^2 + z^2 \geq 4yz + 4xz$$

$$\Leftrightarrow (x + y - 2z)^2 \geq 0$$

so we are done. □

∇

Problem 8. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove the inequality:*

$$\frac{a^5 + b^5}{ab(a+b)} + \frac{b^5 + c^5}{bc(b+c)} + \frac{c^5 + a^5}{ca(a+b)} \geq 3(ab + bc + ca) - 2$$

[1] You can see here: `http://www.mathlinks.ro/viewtopic.php?p=1130901`

*Proof. (Posted by **Athinaios**).* Firstly, we have

$$(a+b)(a-b)^2(a^2+ab+b^2) \geq 0$$

so

$$a^5+b^5 \geq a^2b^2(a+b).$$

Applying the above inequality, we have

$$LHS \geq ab+bc+ca$$

So we need to prove that

$$ab+bc+ca+2 \geq 3(ab+bc+ca)$$

or

$$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$$

Which is clearly true. □

*Proof. (Posted by **kunny**).* Since $y=x^5$ is an increasing and downwards convex function for $x>0$, by the **Jensen inequality** we have

$$\frac{a^5+b^5}{2} \geq \left(\frac{a+b}{2}\right)^5 \Longleftrightarrow \frac{a^5+b^5}{ab(a+b)} \geq \frac{1}{16}\cdot\frac{(a+b)^4}{ab} = \frac{1}{16}(a+b)^2\cdot\frac{(a+b)^2}{ab}$$

$$\geq \frac{1}{16}(a+b)^2\cdot 4$$

(because $(a+b)^2 \geq 4ab$ for $a>0,\ b>0$)
Thus for $a>0,\ b>0,\ c>0$,

$$\begin{aligned}\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(c+a)} &\geq \frac{1}{4}\{(a+b)^2+(b+c)^2+(c+a)^2\}\\ &= \frac{1}{2}(a^2+b^2+c^2+ab+bc+ca)\\ &\geq ab+bc+ca\end{aligned}$$

Then we are to prove

$$ab+bc+ca \geq 3(ab+bc+ca)-2$$

which can be proved by

$$ab+bc+ca \geq 3(ab+bc+ca)-2 \Leftrightarrow 1 \geq ab+bc+ca \Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca$$

Q.E.D. □

Comment
We can prove the stronger inequality:

$$\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(a+c)} \geq 6-5(ab+bc+ca).$$

*Proof. (Posted by **HTA**).* It is equivalent to

$$\sum \frac{a^5+b^5}{ab(a+b)} - \sum \frac{1}{2}(a^2+b^2) \geq \frac{5}{2}\left(\sum (a-b)^2\right)$$

$$\sum (a-b)^2\left(\frac{2a^2+ab+2b^2}{2ab} - \frac{5}{2}\right) \geq 0$$

$$\sum \frac{(a-b)^4}{ab} \geq 0$$

which is true. □

∇

Problem 9. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals prove inequality:*

$$(1+\frac{4a}{b+c})(1+\frac{4b}{a+c})(1+\frac{4c}{a+b}) > 25$$

*Proof. (Posted by **polskimisiek**).* After multiplying everything out, it is equivalent to:

$$4(\sum_{cyc} a^3) + 23abc > 4(\sum_{cyc} a^2(b+c))$$

which is obvious, because by the **Schur inequality**, we have:

$$(\sum_{cyc} a^3) + 3abc \geq \sum_{cyc} a^2(b+c)$$

So finally we have:

$$4(\sum_{cyc} a^3) + 23abc > 4(\sum_{cyc} a^3) + 12abc \geq 4\sum_{cyc} a^2(b+c)$$

Q.E.D □

∇

Problem 10. ***(Croatian Team Selection Test 2008)*** *Let x, y, z be positive numbers. Find the minimum value of:*

$$(a) \quad \frac{x^2+y^2+z^2}{xy+yz}$$

$$(b) \quad \frac{x^2+y^2+2z^2}{xy+yz}$$

*Proof. (Posted by **nsato**).*
(a) The minimum value is $\sqrt{2}$. Expanding

$$\left(x-\frac{\sqrt{2}}{2}y\right)^2 + \left(\frac{\sqrt{2}}{2}y-z\right)^2 \geq 0,$$

we get $x^2 + y^2 + z^2 - \sqrt{2}xy - \sqrt{2}yz \geq 0$, so

$$\frac{x^2 + y^2 + z^2}{xy + yz} \geq \sqrt{2}.$$

Equality occurs, for example, if $x = 1$, $y = \sqrt{2}$, and $z = 1$.

(b) The minimum value is $\sqrt{8/3}$. Expanding

$$\left(x - \sqrt{\frac{2}{3}}y\right)^2 + \frac{1}{3}\left(y - \sqrt{6}z\right)^2 \geq 0,$$

we get $x^2 + y^2 + 2z^2 - \sqrt{8/3}xy - \sqrt{8/3}yz \geq 0$, so

$$\frac{x^2 + y^2 + z^2}{xy + yz} \geq \sqrt{\frac{8}{3}}.$$

Equality occurs, for example, if $x = 2$, $y = \sqrt{6}$, and $z = 1$. □

∇

Problem 11. ***(Moldova 2008 IMO-BMO Second TST Problem 2)*** *Let $a_1, \ldots, a_n$ be positive reals so that $a_1 + a_2 + \ldots + a_n \leq \frac{n}{2}$. Find the minimal value of*

$$A = \sqrt{a_1^2 + \frac{1}{a_2^2}} + \sqrt{a_2^2 + \frac{1}{a_3^2}} + \ldots + \sqrt{a_n^2 + \frac{1}{a_1^2}}$$

*Proof. (Posted by **NguyenDungTN**).* Using **Minkowski** and **Cauchy-Schwarz** inequalities we get

$$A \geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \left(\frac{1}{a_1} + \frac{1}{a_2} \ldots + \frac{1}{n}\right)^2}$$
$$\geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \frac{n^4}{(a_1 + a_2 + \ldots + a_n)^2}}$$

By the **AM-GM inequality**:

$$(a_1 + a_2 + \ldots + a_n)^2 + \frac{\left(\frac{n}{2}\right)^4}{(a_1 + a_2 + \ldots + a_n)^2} \geq \frac{n^2}{2}$$

Because $a_1 + a_2 + \ldots + a_n \leq \frac{n}{2}$ so

$$\frac{\frac{15n^4}{16}}{(a_1 + a_2 + \ldots + a_n)^2} \geq \frac{15n^2}{4}$$

We obtain

$$A \geq \sqrt{\frac{n^2}{2} + \frac{15n^2}{4}} = \frac{\sqrt{17}n}{2}$$

□

Proof. (Posted by ***silouan****).* Using **Minkowski** and **Cauchy-Schwarz** inequalities we get

$$A \geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \left(\frac{1}{a_1} + \frac{1}{a_2} \ldots + \frac{1}{n}\right)^2}$$

$$\geq \sqrt{(a_1 + a_2 + \ldots + a_n)^2 + \frac{n^4}{(a_1 + a_2 + \ldots + a_n)^2}}$$

Let $a_1 + ... + a_n = s$. Consider the function $f(s) = s^2 + \frac{n^4}{s^2}$
This function is decreasing for $s \in \left(0, \frac{n}{2}\right]$. So it attains its minimum at $s = \frac{n}{2}$ and we are done . □

Proof. (Posted by ***ddlam****).* By the **AM-GM inequality**, we have

$$a_1^2 + \frac{1}{a_2^2} = a_1^2 + \frac{1}{16a_2^2} + ... + \frac{1}{16a_2^2} \geq 17\sqrt[17]{\frac{a_1^2}{(16a_2^2)^{16}}}$$

so

$$A \geq \sqrt{17}\sum_{i=1}^{n}\sqrt[34]{\frac{a_i^2}{16^{16}a_{i+1}^{32}}} \quad (a_{i+1} = a_1)$$

By the **AM-GM inequality** again:

$$\sum_{i=1}^{n}\sqrt[34]{\frac{a_i^2}{16^{16}a_{i+1}^{32}}} \geq \frac{n}{\left(\prod_{i=1}^{n} 16^{16n}x_i^{30}\right)^{34n}}$$

But

$$\prod i = 1^n x_i^n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n \leq \frac{1}{2^n}$$

So

$$A \geq \frac{\sqrt{17}n}{2}$$

□

∇

Problem 12. ***(RMO 2008, Grade 8, Problem 3)*** *Let* $a, b \in [0, 1]$*. Prove that*

$$\frac{1}{1 + a + b} \leq 1 - \frac{a + b}{2} + \frac{ab}{3}.$$

Proof. (Posted by ***Dr Sonnhard Graubner****).* The given inequality is equivalent to

$$3(1 - a)(1 - b)(a + b) + ab(1 - a + 1 - b) \geq 0$$

which is true because of $0 \leq a \leq 1$ and $0 \leq b \leq 1$. □

Proof. (Posted by ***HTA****).* Let

$$f(a,b)=1-\frac{a+b}{2}+\frac{ab}{3}-\frac{1}{1+a+b}$$

Consider the difference between $f(a,b)$ and $f(1,b)$ we see that

$$f(a,b)-f(1,b)=\frac{1}{6}\frac{(b-1)(a+2a(b+1)+3b+2b(b+1))-3a}{(1+a+b)(2+b)}\geq 0$$

it is left to prove that $f(1,b)\geq 0$ which is equivalent to

$$\frac{-1}{6}\frac{b(b-1)}{2+b}\geq 0$$

Which is true . □

∇

Problem 13. ***(Romanian TST 2 2008, Problem 1)*** *Let $n\geq 3$ be an odd integer. Determine the maximum value of*

$$\sqrt{|x_1-x_2|}+\sqrt{|x_2-x_3|}+\ldots+\sqrt{|x_{n-1}-x_n|}+\sqrt{|x_n-x_1|},$$

where x_i are positive real numbers from the interval $[0,1]$

Proof. (Posted by ***Myth****).* We have a continuous function on a compact set $[0,1]^n$, hence there is an optimal point $(x_1,...,x_n)$. Note now that

1. impossible to have $x_{i-1}=x_i=x_{i+1}$;
2. if $x_i\leq x_{i-1}$ and $x_i\leq x_{i+1}$, then $x_i=0$;
3. if $x_i\geq x_{i-1}$ and $x_i\geq x_{i+1}$, then $x_i=1$;
4. if $x_{i+1}\leq x_i\leq x_{i-1}$ or $x_{i-1}\leq x_i\leq x_{i+1}$, then $x_i=\dfrac{x_{i-1}+x_{i+1}}{2}$.

It follows that $(x_1,...,x_n)$ looks like

$$(0,\frac{1}{k_1},\frac{2}{k_1},...,1,\frac{k_2-1}{k_2},...,\frac{2}{k_2},\frac{1}{k_2},0,\frac{1}{k_3},...,\frac{1}{k_l}),$$

where k_1, k_2, ..., k_l are natural numbers, $k_1+k_2+...+k_l=n$, l is even clearly. Then the function is this point equals

$$S=\sqrt{k_1}+\sqrt{k_2}+...\sqrt{k_l}.$$

Using the fact that l is even and $\sqrt{k}<\sqrt{k-1}+1$ we conclude that maximal possible value of S is $n-2+\sqrt{2}$ ($l=n-1$, $k_1=k_2=...=k_{l-1}=1$, $k_l=2$ in this case). □

*Proof. (Posted by **Umut Varolgunes**).* Since n is odd, there must be an i such that both x_i and x_{i+1} are both belong to $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. without loss of generality let $x_1 \leq x_2$ and x_1, x_2 belong to $[0, \frac{1}{2}]$. We can prove that

$$\sqrt{x_2 - x_1} + \sqrt{Ix_3 - x_2I} \leq \sqrt{2}$$

If $x_3 > x_2$, $\sqrt{x_2 - x_1} + \sqrt{x_3 - x_2} \leq 2 \cdot \sqrt{\frac{x_3 - x_1}{2}} \leq \sqrt{2}$;
else x_1, x_2, x_3 are all belong to $[0, \frac{1}{2}]$.
Hence, $\sqrt{x_2 - x_1} + \sqrt{Ix_3 - x_2I} \leq \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}$. Also all of the other terms of the sum are less then or equal to 1. summing them gives the desired result.
Example is $(0, \dfrac{1}{2}, 1, 0, 1, \ldots, 1)$
Note: all the indices are considered in modulo n □

▽

Problem 14. ***(Romania Junior TST Day 3 Problem 2 2008)*** *Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:*

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(b + a)} \leq \frac{1}{abc}.$$

*Proof. (Posted by **silouan**).* Using the **AM-GM inequality**, we derive $\dfrac{ab + bc + ca}{3} \geq \sqrt[3]{(abc)^2}$. Then $abc \leq 1$. Now

$$\sum \frac{1}{1 + a^2(b + c)} \leq \sum \frac{1}{abc + a^2(b + c)} = \sum \frac{1}{3a} = \frac{1}{abc}$$

□

▽

Problem 15. ***(Romanian Junior TST Day 4 Problem 4 2008)*** *Determine the maximum possible real value of the number k, such that*

$$(a + b + c)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - k\right) \geq k$$

for all real numbers $a, b, c \geq 0$ with $a + b + c = ab + bc + ca$.

Proof. (Original solution). Observe that the numbers $a = b = 2, c = 0$ fulfill the condition $a+b+c = ab+bc+ca$. Plugging into the givent inequality, we derive that $4\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - k\right) \geq k$ hence $k \leq 1$.
We claim that the inequality hold for $k = 1$, proving that the maximum value of k is 1. To this end, rewrite the inequality as follows

$$(ab + bc + ca)\left(\frac{1}{a + b} + \frac{1}{c + b} + \frac{1}{a + c} - 1\right) \geq 1$$

$$\Leftrightarrow \sum \frac{ab + bc + ca}{a + b} \geq ab + bc + ca + 1$$

$$\Leftrightarrow \sum \frac{ab}{a+b} + c \geq ab + bc + ca + 1 \Leftrightarrow \sum \frac{ab}{a+b} \geq 1$$

Notice that $\frac{ab}{a+b} \geq \frac{ab}{a+b+c}$, since $a, b, c \geq 0$. Summing over a cyclic permutation of a, b, c we get

$$\sum \frac{ab}{a+b} \geq \sum \frac{ab}{a+b+c} = \frac{ab+bc+ca}{a+b+c} = 1$$

as needed. □

Proof. (Alternative solution). The inequality is equivalent to the following

$$S = \frac{a+b+c}{a+b+c+1}\left(\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c}\right)$$

Using the given condition, we get

$$\begin{aligned}
\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c} &= \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\
&= \frac{a^2+b^2+c^2+2(ab+bc+ca)+(a+b+c)}{(a+b)(b+c)(c+a)} \\
&= \frac{(a+b+c)(a+b+c+1)}{(a+b+c)^2-abc}
\end{aligned}$$

hence

$$S = \frac{(a+b+c)^2}{(a+b+c)^2-abc}$$

It is now clear that $S \geq 1$, and equality hold iff $abc = 0$. Consequently, $k = 1$ is the maximum value. □

∇

Problem 16. ***(2008 Romanian Clock-Tower School Junior Competition)*** *For any real numbers $a, b, c > 0$, with $abc = 8$, prove*

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0$$

Proof. (Original solution). We have:

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0 \Leftrightarrow 3 - 3\sum \frac{1}{a+1} \leq 0 \Leftrightarrow 1 \leq \sum \frac{1}{a+1}$$

We can take $a = 2\dfrac{x}{y}, b = 2\dfrac{y}{z}, c = 2\dfrac{z}{x}$ to have

$$\sum \frac{1}{a+1} = \sum \frac{y^2}{2xy+y^2} \geq \frac{(x+y+z)^2}{x^2+y^2+z^2+2(xy+yz+zx)} = 1$$

(by the **Cauchy-Schwarz** inequality) as needed. □

∇

Problem 17. ***(Serbian National Olympiad 2008)*** *Let a, b, c be positive real numbers such that $x+y+z=1$. Prove inequality:*

$$\frac{1}{yz+x+\frac{1}{x}}+\frac{1}{xz+y+\frac{1}{y}}+\frac{1}{xy+z+\frac{1}{z}}\leq\frac{27}{31}.$$

*Proof. (Posted by **canhang2007**).* Setting $x=\frac{a}{3}$,$y=\frac{b}{3}$,$z=\frac{c}{3}$. The inequality is equivalent to

$$\sum_{cyc}\frac{a}{3a^2+abc+27}\leq\frac{3}{31}$$

By the **Schur Inequality**, we get $3abc\geq 4(ab+bc+ca)-9$. It suffices to prove that

$$\sum\frac{3a}{9a^2+4(ab+bc+ca)+72}\leq\frac{3}{31}$$

$$\Leftrightarrow\sum\left(1-\frac{31a(a+b+c)}{9a^2+4(ab+bc+ca)+72}\right)\geq 0$$

$$\Leftrightarrow\sum\frac{(7a+8c+10b)(c-a)-(7a+8b+10c)(a-b)}{a^2+s}\geq 0$$

(where $s=\dfrac{4(ab+bc+ca)+72}{9}$.)

$$\Leftrightarrow\sum(a-b)^2\frac{8a^2+8b^2+15ab+10c(a+b)+s}{(a^2+s)(b^2+s)}\geq 0$$

which is true. □

∇

Problem 18. ***(Canadian Mathematical Olympiad 2008)*** *Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that*

$$\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab}\leq\frac{3}{2}.$$

*Proof. (Posted by **Altheman**).* We have $a+bc=(a+b)(a+c)$, so apply that, etc. The inequality is

$$\sum(b+c)(a^2+ab+ac-bc)\leq\frac{3}{2}(a+b)(b+c)(c+a)$$

$$\Longleftrightarrow\sum_{cyc}a^2b+b^2a\geq 6abc$$

which is obvious by the **AM-GM inequality**. □

∇

Problem 19. ***(German DEMO 2008)*** *Find the smallest constant C such that for all real x, y*

$$1 + (x + y)^2 \leq C \cdot (1 + x^2) \cdot (1 + y^2)$$

holds.

*Proof. (Posted by **JBL**).* The inequality is equivalent to

$$\frac{x^2 + y^2 + 2xy + 1}{x^2 + y^2 + x^2y^2 + 1} \leq C$$

The greatest value of **LHS** helps us find C in which all real numbers x, y satisfies the inequality.
Let $A = x^2 + y^2$, so

$$\frac{A + 2xy + 1}{A + x^2y^2 + 1} \leq C$$

To maximize the **LHS**, A needs to be minimized, but note that

$$x^2 + y^2 \geq 2xy.$$

So let us set $x^2 + y^2 = 2xy = a \Rightarrow x^2y^2 = \frac{a^2}{4}$
So the inequality becomes

$$L = \frac{8a + 4}{(a + 2)^2} \leq C$$

$$\frac{dL}{dx} = \frac{-8a + 8}{(a + 2)^3} = 0 \Rightarrow a = 1$$

It follows that $max(L) = C = \frac{4}{3}$ □

∇

Problem 20. ***(Irish Mathematical Olympiad 2008)*** *For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that*

$$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq 3/32,$$

and determine the cases of equality.

*Proof. (Posted by **argady**).* We have

$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 = abcd(ab + ac + ad + bc + bd + cd)$$

By the **AM-GM inequality**,

$$a^2 + b^2 + c^2 + d^2 \geq 4\sqrt{abcd}$$

and

$$\frac{a^2 + b^2 + a^2 + c^2 + a^2 + d^2 + b^2 + c^2 + b^2 + d^2 + c^2 + d^2}{2} \geq (ab + ac + ad + bc + bd + cd)$$

so $abcd \leq \dfrac{1}{16}$ and $ab+ac+ad+bc+bd+cd \leq \dfrac{3}{2}$
Multiplying we get

$$a^2b^2cd+ab^2c^2d+abc^2d^2+a^2bcd^2+a^2bc^2d+ab^2cd^2 \leq \frac{1}{16}\cdot\frac{3}{2}=\frac{3}{32}.$$

The equality occurs when $a=b=c=d=\dfrac{1}{2}$. □

∇

Problem 21. ***(Greek national mathematical olympiad 2008, P1)*** *For the positive integers $a_1, a_2, ..., a_n$ prove that*

$$\left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i}\right)^{\frac{kn}{t}} \geq \prod_{i=1}^n a_i$$

where $k = max\{a_1, a_2, ..., a_n\}$ and $t = min\{a_1, a_2, ..., a_n\}$. When does the equality hold?

*Proof. (Posted by **rofler**).* By the **AM-GM** and **Cauchy-Schwarz** inequalities, we easily get that

$$\sqrt[2]{\frac{\sum a_i^2}{n}} \geq \frac{\sum a_i}{n}$$

$$\sum a_i^2 \geq \frac{(\sum a_i)^2}{n}$$

$$\frac{\sum a_i^2}{\sum a_i} \geq \frac{\sum a_i}{n} \geq \sqrt[n]{\prod_{i=1}^n a_i}$$

$$\left(\frac{\sum a_i^2}{\sum a_i}\right)^n \geq \prod_{i=1}^n a_i$$

Now, $\dfrac{\sum a_i^2}{\sum a_i} \geq 1$
So therefore since $\frac{k}{t} \geq 1$

$$\left(\frac{\sum a_i^2}{\sum a_i}\right)^{\frac{kn}{t}} \geq \left(\frac{\sum a_i^2}{\sum a_i}\right)^n$$

Now, the direct application of **AM-GM** required that all terms are equal for equality to occur, and indeed, equality holds when all a_i are equal. □

∇

Problem 22. ***(Greek national mathematical olympiad 2008, P2)***
If x, y, z are positive real numbers with $x, y, z < 2$ and $x^2+y^2+z^2=3$ prove that

$$\frac{3}{2} < \frac{1+y^2}{x+2}+\frac{1+z^2}{y+2}+\frac{1+x^2}{z+2} < 3$$

*Proof. (Posted by **tchebytchev**).* From $x < 2, y < 2$ and $z < 2$ we find

$$\frac{1+y^2}{x+2} + \frac{1+z^2}{y+2} + \frac{1+x^2}{z+2} > \frac{1+y^2}{4} + \frac{1+z^2}{4} + \frac{1+x^2}{4} = \frac{3}{2}$$

and from $x > 0, y > 0$ and $z > 0$ we have

$$\frac{1+y^2}{x+2} + \frac{1+z^2}{y+2} + \frac{1+x^2}{z+2} < \frac{1+y^2}{2} + \frac{1+z^2}{2} + \frac{1+x^2}{2} = 3.$$

□

*Proof. (Posted by **canhang2007**).* Since with $x^2 + y^2 + z^2 = 3$, then we can easily get that $x, y, z \le \sqrt{3} < 2$. Also, we can even prove that

$$\sum \frac{x^2+1}{z+2} \ge 2$$

Indeed, by the **AM-GM** and **Cauchy Schwarz** inequalities, we have

$$\sum \frac{x^2+1}{z+2} \ge \frac{x^2+1}{\frac{z^2+1}{2}+2} = 2\sum \frac{x^2+1}{z^2+5} \ge \frac{2(x^2+y^2+z^2+3)^2}{\sum (x^2+1)(z^2+5)} = \frac{72}{\sum x^2y^2+33}$$

$$\ge \frac{72}{\frac{1}{3}(x^2+y^2+z^2)^2+33} = \frac{72}{3+33} = 2$$

□

∇

Problem 23. ***(Moldova National Olympiad 2008)*** *Positive real numbers a, b, c satisfy inequality $a + b + c \le \frac{3}{2}$. Find the smallest possible value for:*

$$S = abc + \frac{1}{abc}$$

*Proof. (Posted by **NguyenDungTN**).* By the AM-GM inequality, we have

$$\frac{3}{2} \ge a + b + c \ge 3\sqrt[3]{abc}$$

so $abc \le \frac{1}{8}$. By the AM-GM inequality again,

$$S = abc + \frac{1}{abc} = abc + \frac{1}{64abc} + \frac{63}{64abc} \ge 2\sqrt{abc.\frac{1}{64abc}} + \frac{63}{64abc} \ge \frac{1}{4} + \frac{63}{8} = \frac{65}{8}$$

□

∇

Problem 24. ***(British MO 2008)*** *Find the minimum of $x^2 + y^2 + z^2$ where $x, y, z \in \mathbb{R}$ and satisfy $x^3 + y^3 + z^3 - 3xyz = 1$*

*Proof. (Posted by **delegat**).* Condition of problem may be rewritten as:

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=1$$

and since second bracket on LHS is nonnegative we have $x+y+z>0$.
Notice that from last equation we have:

$$x^2+y^2+z^2=\frac{1+(xy+yz+zx)(x+y+z)}{x+y+z}=\frac{1}{x+y+z}+xy+yz+zx$$

and since

$$xy+yz+zx=\frac{(x+y+z)^2-x^2-y^2-z^2}{2}$$

The last equation implies:

$$\begin{aligned}\frac{3(x^2+y^2+z^2)}{2}&=\frac{1}{x+y+z}+\frac{(x+y+z)^2}{2}\\&=\frac{1}{2(x+y+z)}+\frac{1}{2(x+y+z)}+\frac{(x+y+z)^2}{2}\\&\geq\frac{3}{2}\end{aligned}$$

This inequality follows from $AM\geq GM$ so $x^2+y^2+z^2\geq 1$ so minimum of $x^2+y^2+z^2$ is 1 and triple $(1,0,0)$ shows that this value can be achieved. □

Proof. (**Original solution**). Let $x^2+y^2+z^2=r^2$. The volume of the parallelpiped in R^3 with one vertex at $(0,0,0)$ and adjacent vertices at $(x,y,z),(y,z,x),(z,x,y)$ is $|x^3+y^3+z^3-3xyz|=1$ by expanding a determinant. But the volume of a parallelpiped all of whose edges have length r is clearly at most r^3 (actually the volume is $r^3\cos\theta\sin\varphi$ where θ and φ are geometrically significant angles). So $1\leq r^3$ with equality if, and only if, the edges of the parallepiped are perpendicular, where $r=1$. □

Proof. (**Original solution**). Here is an algebraic version of the above solution.

$$\begin{aligned}1&=(x^3+y^3+z^3-3xyz)^2=\left(x(x^2-yz)+y(y^2-zx)+z(z^2-xy)\right)^2\\&\leq(x^2+y^2+z^2)\left((x^2-yz)^2+(y^2-zx)^2+(z^2-xy)^2\right)\\&=(x^2+y^2+z^2)\left(x^4+y^4+z^4+x^2y^2+y^2z^2+z^2x^2-2xyz(x+y+z)\right)\\&=(x^2+y^2+z^2)\left(\left(x^2+y^2+z^2\right)^2-(xy+yz+zx)^2\right)\\&\leq\left(x^2+y^2+z^2\right)^3\end{aligned}$$

□

∇

Problem 25. ***(Zhautykov Olympiad, Kazakhstan 2008, Question 6)*** *Let a,b,c be positive integers for which $abc=1$. Prove that*

$$\sum\frac{1}{b(a+b)}\geq\frac{3}{2}.$$

Proof. (Posted by ***nayel****).* Letting $a = \dfrac{x}{y}, b = \dfrac{y}{z}, c = \dfrac{z}{x}$ implies

$$LHS = \sum_{cyc} \frac{x^2}{z^2 + xy} \geq \frac{(x^2 + y^2 + z^2)^2}{x^2y^2 + y^2z^2 + z^2x^2 + x^3y + y^3z + z^3x}$$

Now it remains to prove that

$$2(x^2 + y^2 + z^2)^2 \geq 3\sum_{cyc} x^2y^2 + 3\sum_{cyc} x^3y$$

Which follows by adding the two inequalities

$$x^4 + y^4 + z^4 \geq x^3y + y^3z + z^3x$$

$$\sum_{cyc}(x^4 + x^2y^2) \geq \sum_{cyc} 2x^3y$$

□

∇

Problem 26. ***(Ukraine National Olympiad 2008, P1)*** *Let x, y and z are non-negative numbers such that $x^2 + y^2 + z^2 = 3$. Prove that:*

$$\frac{x}{\sqrt{x^2 + y + z}} + \frac{y}{\sqrt{x + y^2 + z}} + \frac{z}{\sqrt{x + y + z^2}} \leq \sqrt{3}$$

Proof. (Posted by ***nayel****).* By Cauchy Schwarz we have

$$(x^2 + y + z)(1 + y + z) \geq (x + y + z)^2$$

so we have to prove that

$$\frac{x\sqrt{1 + y + z} + y\sqrt{1 + x + z} + z\sqrt{1 + x + y}}{x + y + z} \leq \sqrt{3}$$

But again by the **Cauchy Schwarz inequality** we have

$$x\sqrt{1 + y + z} + y\sqrt{1 + x + z} + z\sqrt{1 + x + y} = \sum \sqrt{x}\sqrt{x + xy + xz}$$

$$\leq \sqrt{(x + y + z)(x + y + z + 2(xy + yz + zx)}$$

and also

$$\sqrt{(x + y + z)(x + y + z + 2(xy + yz + zx)}$$

$$\leq \sqrt{(x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)} = s\sqrt{s}$$

where $s = x + y + z$ so we have to prove that $\sqrt{s} \leq \sqrt{3}$ which is trivially true so QED □

*Proof. (Posted by **argady**).* We have

$$\sum_{cyc}\frac{x}{\sqrt{x^2+y+z}}=\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\sqrt{\frac{x^2+y^2+z^2}{3}}}}\leq\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}$$

Thus, it remains to prove that

$$\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}\leq\sqrt{3}.$$

Let $x+y+z=3$. Hence,

$$\sum_{cyc}\frac{x}{\sqrt{x^2+(y+z)\frac{x+y+z}{3}}}\leq\sqrt{3}\Leftrightarrow\sum_{cyc}\left(\frac{1}{\sqrt{3}}-\frac{x}{\sqrt{x^2-x+3}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{1}{\sqrt{3}}-\frac{x}{\sqrt{x^2-x+3}}+\frac{5(x-1)}{6\sqrt{3}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{(x-1)^2(25x^2+35x+3)}{((5x+1)\sqrt{x^2-x+3}+6\sqrt{3}x)\sqrt{x^2-x+3}}\geq 0.$$

□

∇

Problem 27. ***(Ukraine National Olympiad 2008, P2)*** *For positive a,b,c,d prove that*

$$(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4\geq 16abcd(1+a)(1+b)(1+c)(1+d)$$

*Proof. (Posted by **Yulia**).* Let's rewrite our inequality in the form

$$\frac{(a+b)(b+c)(c+d)(d+a)}{(1+a)(1+b)(1+c)(1+d)}\geq\frac{16abcd}{(1+\sqrt[4]{abcd})^4}$$

We will use the following obvious lemma

$$\frac{x+y}{(1+x)(1+y)}\geq\frac{2\sqrt{xy}}{(1+\sqrt{xy})^2}$$

By lemma and Cauchy-Schwarz

$$\frac{a+b}{(1+a)(1+b)}\frac{c+d}{(1+c)(1+d)}(b+c)(a+d)\geq\frac{4\sqrt{abcd}(\sqrt{ab}+\sqrt{cd})^2}{(1+\sqrt{ab})^2(1+\sqrt{cd})^2}\geq\frac{16abcd}{(1+\sqrt[4]{abcd})^4}$$

Last one also by lemma for $x=\sqrt{ab},y=\sqrt{cd}$ □

*Proof. (Posted by **argady**).* The inequality equivalent to
$(a+b)(b+c)(c+d)(d+a)-16abcd+$
$+4\sqrt[4]{abcd}\left((a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{a^3b^3c^3d^3}(a+b+c+d)\right)+$
$+2\sqrt{abcd}\left(3(a+b)(b+c)(c+d)(d+a)-8\sqrt{abcd}(ab+ac+ad+bc+bd+cd)\right)$

$+4\sqrt[4]{a^3b^3c^3d^3}((a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{abcd}(abc+abd+acd+bcd))\geq 0,$
which obvious because

$$(a+b)(b+c)(c+d)(d+a)-16abcd\geq 0$$

is true by AM-GM;

$$(a+b)(b+c)(c+d)(d+a)-4\sqrt[4]{a^3b^3c^3d^3}(a+b+c+d)\geq 0$$

is true since,

$$(a+b)(b+c)(c+d)(d+a)\geq(abc+abd+acd+bcd)(a+b+c+d)\Leftrightarrow(ac-bd)^2\geq 0$$

and

$$abc+abd+acd+bcd\geq 4\sqrt[4]{a^3b^3c^3d^3}$$

is true by AM-GM;

$$(a+b)(b+c)(c+d)(d+a)\geq 4\sqrt[4]{abcd}(abc+abd+acd+bcd)$$

is true because

$$a+b+c+d\geq 4\sqrt[4]{abcd}$$

is true by AM-GM;

$$3(a+b)(b+c)(c+d)(d+a)\geq 8\sqrt{abcd}(ab+ac+ad+bc+bd+cd)$$

follows from three inequalities:

$$(a+b)(b+c)(c+d)(d+a)\geq(abc+abd+acd+bcd)(a+b+c+d);$$

by Maclaren we obtain:

$$\frac{a+b+c+d}{4}\geq\sqrt{\frac{ab+ac+bc+ad+bd+cd}{6}}$$

and

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}{4}\geq\sqrt{\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}}{6}},$$

which equivalent to

$$abc+abd+acd+bcd\geq\sqrt{\frac{8}{3}(ab+ac+bc+ad+bd+cd)abcd}.$$

□

∇

Problem 28. ***(Polish MO 2008, Pro 5)*** *Show that for all nonnegative real values an inequality occurs:*

$$4(\sqrt{a^3b^3}+\sqrt{b^3c^3}+\sqrt{c^3a^3})\leq 4c^3+(a+b)^3.$$

*Proof. (Posted by **NguyenDungTN**).* We have:

$$RHS - LHS = \left(\sqrt{a^3} + \sqrt{b^3} - 2\sqrt{c^3})^2 + 3ab(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$$

Thus we are done. Equality occurs for $a = b = c$ or $a = 0, b = \sqrt[3]{4}c$ or $a = \sqrt[3]{4}c, b = 0$ □

∇

Problem 29. ***(Brazilian Math Olympiad 2008, Problem 3).*** *Let x, y, z real numbers such that $x + y + z = xy + yz + zx$. Find the minimum value of*

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1}$$

*Proof. (Posted by **crazyfehmy**).* We will prove that this minimum value is $-\frac{1}{2}$. If we take $x = y = -1$, $z = 1$, the value is $-\frac{1}{2}$.
Let's prove that

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} + \frac{1}{2} \geq 0$$

We have

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} + \frac{1}{2} = \frac{(1+x)^2}{2+2x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{y}{1+y^2} + \frac{z}{1+z^2}$$

$$\frac{y}{1+y^2} + \frac{z}{1+z^2} < 0$$

then $(y+z)(yz+1) < 0$ and by similar way $(x+z)(xz+1) < 0$ and $(y+z)(yz+1) < 0$.
Let all of x , y , z are different from 0.

- All of $x + y$, $y + z$, $x + z$ is ≥ 0 Then $x + y + z \geq 0$ and $xy + yz + xz \leq -3$.It's a contradiction.
- Exactly one of the $(x + y)$, $y + z$, $x + z$ is < 0.
  W.L.O.G, Assuming $y + z < 0$.
  Because $x + z > 0$ and $x + y > 0$ so $x > 0$.
  $xz + 1 < 0$ and $xy + 1 < 0$ hence y and z are < 0.
  Let $y = -a$ and $z = -b$. $x = \dfrac{ab+a+b}{a+b+1}$ and $x > a > \dfrac{1}{x}$ and $x > b > \dfrac{1}{x}$.
  So $x > 1$ and $ab > 1$.
  Otherwise because of $x > a$ and $x > b$ hence $\frac{ab+a+b}{a+b+1} > a$ and $\frac{ab+a+b}{a+b+1} > b$.
  So $b > a^2$ and $a > b^2$. So $ab < 1$. It's a contradiction.
- Exactly two of them $(x + y)$,$(y + z)$,$(x + z)$ are < 0.
  W.L.O.G, Assuming $y + z$ and $x + z$ are < 0.
  Because $x + y > 0$ so $z < 0$. Because $xy < 0$ we can assume $x < 0$ and $y > 0$.
  Let $x = -a$ and $z = -c$ and because $xz + 1 > 0$ and $xy + 1 < 0$ so $c < \frac{1}{y}$ and $a > \frac{1}{y}$.
  Because $y + z < 0$ and $x + y > 0$ hence $a < y < c$ and so $\frac{1}{y} < a < y < c < \frac{1}{y}$. It's a contradiction.

- All of them are < 0 .So $x+y+z<0$ and $xy+yz+xz>0$. It's a contradiction.
- Some of x , y , z are $=0$
  W.L.O.G, Assuming $x=0$. So $y+z=yz=K$ and

$$\frac{y}{1+y^2}+\frac{z}{1+z^2}=\frac{K^2+K}{2K^2-2K+1}\geq -\frac{1}{2}\Longleftrightarrow 4K^2+1\geq 0$$

  which is obviously true.

The proof is ended. □

∇

Problem 30. ***(Kiev 2008, Problem 1)***. *Let $a,b,c\geq 0$. Prove that*

$$\frac{a^2+b^2+c^2}{5}\geq min((a-b)^2,(b-c)^2,(c-a)^2)$$

Proof. (Posted by ***canhang2007****).* Assume that $a\geq b\geq c$, then

$$\min\left\{(a-b)^2,(b-c)^2,(c-a)^2\right\}=\min\{(a-b)^2,(b-c)^2\}$$

If $a+c\geq 2b$, then $(b-c)^2=\min\{(a-b)^2,(b-c)^2\}$, we have to prove

$$a^2+b^2+c^2\geq 5(b-c)^2$$

which is true because

$$a^2+b^2+c^2-5(b-c)^2\geq (2b-c)^2+b^2+c^2-5(b-c)^2=3c(2b-c)\geq 0$$

If $a+c\leq 2b$, then $(a-b)^2=\min\{(a-b)^2,(b-c)^2\}$, we have to prove

$$a^2+b^2+c^2\geq 5(a-b)^2$$

which is true because

$$a^2+b^2-5(a-b)^2=2(2a-b)(2b-a)\geq 0$$

This ends the proof. □

∇

Problem 31. ***(Kiev 2008, Problem 2)***. *Let $x_1,x_2,\cdots,x_n\geq 0,n>3$ and $x_1+x_2+\cdots+x_n=2$ Find the minimum value of*

$$\frac{x_2}{1+x_1^2}+\frac{x_3}{1+x_2^2}+...+\frac{x_1}{1+x_n^2}$$

Proof. (Posted by ***canhang2007****).* By AM-GM Inequality, we have that

$$\frac{x_2}{x_1^2+1}=x_2-\frac{x_1^2x_2}{x_1^2+1}\geq x_2-\frac{1}{2}x_1x_2$$

Apply this for the similar terms and adding them up to obtain

$$LHS \geq 2 - \frac{1}{2}(x_1x_2 + x_2x_3 + \cdots + x_nx_1)$$

Moreover, we can easily show that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leq x_k(x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_n) \leq 1$$

for k is a number such that $x_k = \max\{x_1, x_2, \ldots, x_n\}$. Hence

$$LHS \geq 2 - \frac{1}{2} = \frac{3}{2}$$

□

∇

Problem 32. ***(Hong Kong TST1 2009, Problem 1)*** *Let $\theta_1, \theta_2, \ldots, \theta_{2008}$ be real numbers. Find the maximum value of*

$$\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_3 + \ldots + \sin\theta_{2007}\cos\theta_{2008} + \sin\theta_{2008}\cos\theta_1$$

*Proof. (Posted by **brianchung11**).* By the AM-GM Inequality, we have

$$\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_3 + \ldots + \sin\theta_{2007}\cos\theta_{2008} + \sin\theta_{2008}\cos\theta_1 \leq \frac{1}{2}\sum(\sin^2\theta_i + \cos^2\theta_{i+1}) = 1004$$

Equality holds when θ_i is constant. □

∇

Problem 33. ***(Hong Kong TST1 2009, Problem 5)***. *Let a, b, c be the three sides of a triangle. Determine all possible values of*

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

*Proof. (Posted by **Hong Quy**).* We have

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

and $|a - b| < c$ then $a^2 + b^2 - c^2 < 2ab$.
Thus,

$$a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

$$1 \leq F = \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$$

□

∇

Problem 34. ***(Indonesia National Science Olympiad 2008)*** *Prove that for x and y positive reals,*

$$\frac{1}{(1+\sqrt{x})^2} + \frac{1}{(1+\sqrt{y})^2} \geq \frac{2}{x+y+2}.$$

*Proof. (Posted by **Dr Sonnhard Graubner**).* This inequality is equivalent to

$$2+2x+2y+x^2+y^2-2\sqrt{x}y-2x\sqrt{y}+2x^{\frac{3}{2}}+2y^{\frac{3}{2}}-8\sqrt{x}\sqrt{y}\geq 0$$

We observe that the following inequalities hold

1. $x+y\geq 2\sqrt{xy}$
2. $x+y^2\geq 2y\sqrt{x}$
3. $y+x^2\geq 2x\sqrt{y}$
4. $2+2y^{\frac{3}{2}}+2x^{\frac{3}{2}}\geq 6\sqrt{xy}$.

Adding $(1),(2),(3)$ and (4) we get the desired result. □

*Proof. (Posted by **limes123**).* We have

$$(1+xy)(1+\frac{x}{y})\geq(1+x)^2 \iff \frac{1}{(1+x)^2}\geq\frac{1}{1+xy}\cdot\frac{y}{x+y}$$

and analogously

$$\frac{1}{(1+y)^2}\geq\frac{1}{1+xy}\cdot\frac{x}{x+y}$$

as desired. □

∇

Problem 35. ***(Baltic Way 2008)****. Prove that if the real numbers a, b and c satisfy $a^2+b^2+c^2=3$ then*

$$\sum\frac{a^2}{2+b+c^2}\geq\frac{(a+b+c)^2}{12}.$$

When does the inequality hold?

*Proof. (Posted by **Raja Oktovin**).* By the Cauchy-Schwarz Inequality, we have

$$\frac{a^2}{2+b+c^2}+\frac{b^2}{2+c+a^2}+\frac{c^2}{2+a+b^2}\geq\frac{(a+b+c)^2}{6+a+b+c+a^2+b^2+c^2}.$$

So it suffices to prove that

$$6+a+b+c+a^2+b^2+c^2\leq 12.$$

Note that $a^2+b^2+c^2=3$, then we only need to prove that

$$a+b+c\leq 3$$

But

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\leq a^2+b^2+c^2+2(a^2+b^2+c^2)=3(a^2+b^2+c^2)=9.$$

Hence $a+b+c\leq 3$ which completes the proof. □

∇

Problem 36. ***(Turkey NMO 2008 Problem 3)***. *Let a.b.c be positive reals such that their sum is 1. Prove that*

$$\frac{a^2b^2}{c^3(a^2-ab+b^2)}+\frac{b^2c^2}{a^3(b^2-bc+c^2)}+\frac{a^2c^2}{b^3(a^2-ac+c^2)}\geq\frac{3}{ab+bc+ac}$$

*Proof. (Posted by **canhang2007**).* The inequality is equivalent to

$$\sum\frac{a^2b^2}{c^3(a^2-ab+b^2)}\geq\frac{3(a+b+c)}{ab+bc+ca}$$

Put $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$, then the above inequality becomes

$$\sum\frac{z^3}{x^2-xy+y^2}\geq\frac{3(xy+yz+zx)}{x+y+z}$$

This is a very known inequality. □

*Proof. (Posted by **mehdi cherif**).* The inequality is equivalent to :

$$\sum\frac{(ab)^2}{c^3(a^2-ab+b^2)}\geq\frac{3(a+b+c)}{ab+ac+bc}$$
$$\iff\sum\frac{(ab)^5}{a^2-ab+b^2}\geq\frac{3(abc)^3(a+b+c)}{ab+ac+bc}$$

But

$$3(abc)^3(a+b+c)=3abc(\sum a)(abc)^2\leq(\sum ab)^2(abc)^2(AM-GM)$$

Hence it suffices to prove that :

$$\sum\frac{(ab)^5}{a^2-ab+b^2}\geq(abc)^2(\sum ab)$$
$$\iff\sum\frac{(ab)^3}{c^2(a^2-ab+b^2)}\geq\sum ab$$
$$\iff\sum\frac{(ab)^3}{c^2(a^2-ab+b^2)}+\sum c(a+b)\geq 3\sum ab$$
$$\iff\sum\frac{(ab)^3+(bc)^3+(ca)^3}{c^2(a^2-ab+b^2)}\geq 3\sum ab$$

On the other hands,

$$\sum\frac{(ab)^3+(bc)^3+(ca)^3}{c^2(a^2-ab+b^2)}\geq 9\frac{(ab)^3+(ac)^3+(bc)^3}{2\sum(ab)^2-abc(\sum a)}$$

It suffices to prove that :

$$9\frac{(ab)^3+(ac)^3+(bc)^3}{2\sum(ab)^2-abc(\sum a)}\geq 3\sum ab$$

Denote that $x=ab,y=ac$ and $z=bc$

$$3\frac{x^3+y^3+z^3}{2(x^2+y^2+z^2)-xy+yz+zx}\geq x+y+z$$
$$\iff\sum x^3+3xyz\geq\sum xy(x+y)$$

which is Schur inequality ,and we have done. □

∇

Problem 37. ***(China Western Mathematical Olympiad 2008)***. *Given $x, y, z \in (0,1)$ satisfying that*

$$\sqrt{\frac{1-x}{yz}} + \sqrt{\frac{1-y}{xz}} + \sqrt{\frac{1-z}{xy}} = 2.$$

Find the maximum value of xyz.

*Proof. (Posted by **Erken**).* Let's make the following substitution: $x = \sin^2 \alpha$ and so on... It follows that

$$2\sin\alpha\sin\beta\sin\gamma = \sum \cos\alpha\sin\alpha$$

But it means that $\alpha + \beta + \gamma = \pi$, then obviously

$$(\sin\alpha\sin\beta\sin\gamma)^2 \le \frac{27}{64}$$

□

Proof. (Posted by turcas$_c$). We have that

$$2\sqrt{xyz} = \frac{1}{\sqrt{3}}\sum\sqrt{x(3-3x)} \le \frac{1}{\sqrt{3}}\sum\frac{x+3(1-x)}{2} =$$

$$= \frac{3\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\sum x.$$

So $2\sqrt{xyz} \le \frac{3\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\sum x \le \frac{3\sqrt{3}}{2} - \sqrt{3}\cdot\sqrt[3]{xyz}$.

If we denote $p = \sqrt[6]{xyz}$ we get that $2p^3 \le \frac{3\sqrt{3}}{2} - \sqrt{3}p^2$. This is equivalent to

$$4p^3 + 2\sqrt{3}p^2 - 3\sqrt{3} \le 0 \Rightarrow (2p - \sqrt{3})(2p^2 + 2\sqrt{3}p + 3) \le 0,$$

then $p \le \frac{\sqrt{3}}{2}$. So $xyz \le \frac{27}{64}$. The equality holds for $x = y = z = \frac{3}{4}$. □

∇

Problem 38. ***(Chinese TST 2008 P5)*** *For two given positive integers $m, n > 1$, let $a_{ij}(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where*

$$f = \frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij})^2 + m\sum_{j=1}^{m}(\sum_{i=1}^{n} a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij})^2 + mn\sum_{i=1}^{n}\sum_{i=j}^{m} a_{ij}^2}$$

*Proof. (Posted by **tanpham**).* We will prove that the maximum value of f is 1.

- For $n = m = 2$. Setting $a_{11} = a, a_{21} = b, a_{12} = x, a_{21} = y$. We have

$$f = \frac{2\left((a+b)^2 + (x+y)^2 + (a+x)^2 + (b+y)^2\right)}{(a+b+x+y)^2 + 4\left(a^2+b^2+x^2+y^2\right)} \le 1$$

$$\Leftrightarrow = (x+b-a-y)^2 \ge 0$$

as needed.

- For $n = 2, m = 3$. Using the similar substitution:

$$(x, y, z), (a, b, c)$$

We have

$$f = \frac{2(a+b+c)^2 + 2(x+y+z)^2 + 3(a+x)^2 + 3(b+y)^2 + 3(c+z)^2}{6(a^2+b^2+c^2+x^2+y^2+z^2) + (a+b+c+x+y+z)^2} \leq 1$$

$$\Leftrightarrow (x+b-y-a)^2 + (x+c-z-a)^2 + (y+c-b-z)^2 \geq 0$$

as needed.

- For $n = 3, m = 4$. With

$$(x, y, z, t), (a, b, c, d), (k, l, m, n)$$

The inequality becomes

$$(x+b-a-y)^2 + (x+c-a-z)^2 + (x+d-a-t)^2 + (x+l-k-y)^2 +$$

$$+ (x+m-k-z)^2 + (x+n-k-t)^2 + (y+c-b-z)^2 + (y+d-b-t)^2 +$$

$$+ (y+m-l-z)^2 + (y+n-l-t)^2 + \cdots \geq 0$$

as needed.
By induction, the inequality is true for every integer numbers $m, n > 1$

□

∇

Chapter 3

The inequality from IMO 2008

In this chapter, we will introduce 11 solutions for the inequality from **IMO 2008**.
Problem.
(i). If x, y and z are three real numbers, all different from 1 , such that $xyz = 1$, then prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

With the sign $\sum$ for cyclic summation, this inequality could be rewritten as

$$\sum \frac{x^2}{(x-1)^2} \geq 1$$

(ii). Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.
Solution.

*Proof. (Posted by **vothanhvan**).* We have

$$\sum_{cyc} \left(1-\frac{1}{y}\right)^2 \left(1-\frac{1}{z}\right)^2 \geq \left(1-\frac{1}{x}\right)^2 \left(1-\frac{1}{y}\right)^2 \left(1-\frac{1}{z}\right)^2 \Leftrightarrow \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-3\right)^2 \geq 0$$

We conclude that

$$\sum_{cyc} x^2 (y-1)^2 (z-1)^2 \geq (x-1)^2 (y-1)^2 (z-1)^2 \Leftrightarrow \frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2} \geq 1$$

Q.E.D $\square$

∇

*Proof. (Posted by **TTsphn**).* Let

$$a = \frac{x}{1-x}, b = \frac{y}{1-y}, c = \frac{z}{1-z}$$

Then we have :

$$bc = (a+1)(b+1)(c+1) = \frac{1}{(x-1)(y-1)(z-1)} \Leftrightarrow ab+ac+bc+a+b+c+1 = 0$$

Therefore :

$$a^2+b^2+c^2 = a^2+b^2+c^2+2(ab+ac+bc)+2(a+b+c)+2 \Leftrightarrow a^2+b^2+c^2 = (a+b+c+1)^2+1 \geq 1$$

So problem a claim .
The equality hold if and only if $a+b+c+1=0$.
This is equivalent to

$$xy+zx+zx = 3$$

From $x = \frac{1}{yz}$ we have

$$\frac{1}{z}+\frac{1}{y}+yz = 3 \Leftrightarrow z^2y^2 - y(3z-1)+z = 0$$

$$\Delta = (3z-1)^2 - 4z^3 = (z-1)^2(1-4z)$$

We only chose $z = \dfrac{1-m^2}{4}, |m| > 0$ then the equation has rational solution y. Because $x = \dfrac{1}{yz}$ so it also a rational .
Problem claim . □

∇

*Proof. (Posted by **Darij Grinberg**).* We have

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}-1 = \frac{(yz+zx+xy-3)^2}{(x-1)^2(y-1)^2(z-1)^2}$$

For part (ii) you are looking for rational x, y, z with $xyz=1$ and $x+y+z=3$.
In other words, you are looking for rational x and y with $x+y+\frac{1}{xy} = 3$. This rewrites as $y^2+(x-3)\,y+\frac{1}{x} = 0$, what is a quadratic equation in y. So for a given x, it has a rational solution y if and only if its determinant $(x-3)^2 - 4\cdot\frac{1}{x}$ is a square. But $(x-3)^2-4\cdot\frac{1}{x} = \frac{x-4}{x}(x-1)^2$, so this is equivalent to $\frac{x-4}{x}$ being a square.
Parametrize... □

∇

*Proof. (Posted by **Erken**).* Let $a = 1-\frac{1}{x}$ and so on... Then our inequality becomes:

$$a^2b^2+b^2c^2+c^2a^2 \geq a^2b^2c^2$$

while $(1-a)(1-b)(1-c) = 1$.
Second condition gives us that:

$$a^2b^2+b^2c^2+c^2a^2 = a^2b^2c^2+(a+b+c)^2 \geq a^2b^2c^2$$

□

∇

Proof. (Posted by ***Sung-yoon Kim****).* First letting $x = \frac{q}{p}, y = \frac{r}{q}, z = \frac{p}{r},$. We have to show that

$$\sum \frac{q^2}{(p-q)^2} \geq 1$$

Define $f(t)$ to be

$$\sum \frac{(t+q)^2}{(p-q)^2} = (\sum \frac{1}{(p-q)^2})t^2 + 2(\sum \frac{q}{(p-q)^2})t + \sum \frac{q^2}{(p-q)^2} = At^2 + 2Bt + C$$

This is a quadratic function of t and we know that this has minimum at t_0 such that $At_0 + B = 0$.
Hence,

$$f(t) \geq f(t_0) = At_0^2 + 2Bt_0 + C = Bt_0 + C = \frac{AC - B^2}{A}$$

Since

$$AC - B^2 = (\sum \frac{1}{(p-q)^2})(\sum \frac{q^2}{(p-q)^2}) - (\sum \frac{q}{(p-q)^2})^2$$

and we have

$$(a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 = \sum (ae - bd)^2,$$

We obtain

$$AC - B^2 = \sum (\frac{r-q}{(p-q)(q-r)})^2 = \sum \frac{1}{(p-q)^2} = A$$

This makes $f(t) \geq 1$, as desired.
The second part is trivial, since we can find (p, q, r) with fixed $p - q$ and any various $q - r$, which would give different (x, y, z) satisfying the equality. $\square$

∇

Proof. (Posted by ***Ji Chen****).* We have

$$\begin{aligned} &\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} - 1 \\ &\equiv \frac{a^6}{(a^3 - abc)^2} + \frac{b^6}{(b^3 - abc)^2} + \frac{c^6}{(c^3 - abc)^2} - 1 \\ &= \frac{(bc + ca + ab)^2 (b^2c^2 + c^2a^2 + a^2b^2 - a^2bc - b^2ca - c^2ab)^2}{(a^2 - bc)^2 (b^2 - ca)^2 (c^2 - ab)^2} \geq 0 \end{aligned}$$

$\square$

∇

*Proof. (Posted by **kunny**).* By $xyz = 1$, we have

$$\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-\left\{\frac{xy}{(x-1)(y1)}+\frac{yz}{(y-1)(z-1)}+\frac{zx}{(z-1)(x-1)}\right\}$$
$$=\frac{x(y-1)(z-1)+y(z-1)(x-1)+z(x-1)(y-1)-xy(z-1)-yz(x-1)-zx(y-1)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x(y-1)(z-1-z)+y(z-1)(x-1-x)+zx(y-1-y)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x+y+z-(xy+yz+zx)}{(x-1)(y-1)(z-1)}$$
$$=\frac{x+y+z-(xy+yz+zx)+xyz-1}{(x-1)(y-1)(z-1)}$$
$$=\frac{(x-1)(y-1)(z-1)}{(x-1)(y-1)(z-1)}=1$$

(Because $x \neq 1,\ y \neq 1,\ z \neq 1$).)
Therefore

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}\right)^2-2\left\{\frac{xy}{(x-1)(y-1)}+\frac{yz}{(y-1)(z-1)}+\frac{zx}{(z-1)(x-1)}\right\}$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}\right)^2-2\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-1\right)$$
$$=\left(\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}-1\right)^2+1\geq 1$$

The equality holds when

$$\boxed{\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}=1 \Longleftrightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3}$$

□

∇

*Proof. (Posted by **kunny**).* Let $x+y+z=a,\ xy+yz+zx=b,\ xyz=1,\ x,\ y,\ z$ are the roots of the cubic equation :

$$t^3-at^2+bt-1=0$$

If $t=1$ is the roots of the equation, then we have

$$1^3-a\cdot 1^2+b\cdot 1-1=0 \Longleftrightarrow a-b=0$$

Therefore $t \neq 1 \Longleftrightarrow a-b \neq 0$.
Thus the cubic equation with the roots $\alpha = \frac{x}{x-1},\ \beta = \frac{y}{y-1},\ \gamma = \frac{z}{z-1}$ is

$$\left(\frac{t}{t-1}\right)^3 - a\left(\frac{t}{t-1}\right) + b\cdot\frac{t}{t-1} - 1 = 0$$
$$\Longleftrightarrow (a-b)t^3 - (a-2b+3)t^2 - (b-3)t - 1 = 0 \ \cdots [*]$$

Let $a-b = p \neq 0,\ b-3 = q$, we can rerwite the equation as

$$pt^3 - (p-q)t^2 - qt - 1 = 0$$

By Vieta's formula, we have

$$\alpha+\beta+\gamma = \frac{p-q}{p} = 1 - \frac{q}{p},\ \alpha\beta+\beta\gamma+\gamma\alpha = -\frac{q}{p}$$

Therefore

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} = \alpha^2+\beta^2+\gamma^2$$
$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= \left(1-\frac{q}{p}\right)^2 - 2\left(-\frac{q}{p}\right)$$
$$= \left(\frac{q}{p}\right)^2 + 1 \geq 1,$$

The equality holds when $\frac{q}{p} = 0 \Longleftrightarrow q = 0 \Longleftrightarrow b = 3$, which completes the proof. $\square$

∇

Proof. (Posted by ***kunny****).* Since $x,\ y,\ z$ aren't equal to 1, we can set $x = a+1,\ y = b+1,\ z = c+1\ (abc \neq 0)$.

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} = \frac{(a+1)^2}{a^2} + \frac{(b+1)^2}{b^2} + \frac{(c+1)^2}{c^2}$$
$$= 3 + 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$
$$= 3 + \frac{2(ab+bc+ca)}{abc} + \frac{(ab+bc+ca)^2 - 2abc(a+b+c)}{(abc)^2} \cdots [*]$$

Let

$$a+b+c = p,\ ab+bc+ca = q,\ abc = r \neq 0$$

we have $xyz = 1 \Longleftrightarrow p+q+r = 0$, since $r \neq 0$, we have

$$[*] = 3 + \frac{2q}{r} + \frac{q^2-2rp}{r^2} = 3 + \frac{2q}{r} + \left(\frac{q}{r}\right)^2 - 2\frac{p}{r}$$
$$= 3 + \frac{2q}{r} + \left(\frac{q}{r}\right)^2 + 2\frac{q+r}{r} = \left(\frac{q}{r}\right)^2 + 4\frac{q}{r} + 5$$
$$= \left(\frac{q}{r}+2\right)^2 + 1 \geq 1.$$

The equality holds when $q = -2r$ and $p+q+r = 0$ $(r > 0) \Longleftrightarrow p : q : r = 1 : (-2) : 1$.
Q.E.D. □

∇

*Proof. (Posted by **Allnames**).* The inequality can be rewritten in this form

$$\sum \frac{1}{(1-a)^2} \geq 1$$

or

$$\sum (1-a)^2(1-b)^2 \geq ((1-a)(1-b)(1-c))^2$$

where $x = \dfrac{1}{a}$ and $abc = 1$.
We set $a + b + c = p, ab + bc + ca = q, abc = r = 1$. So the above inequality is equivalent to

$$(p-3)^2 \geq 0$$

which is clearly true. □

∇

*Proof. (Posted by **tchebytchev**).* Let $x = \frac{1}{a}, y = \frac{1}{b}$ and $z = \frac{1}{c}$. We have

$$\begin{aligned}
&\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \\
&= \frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2} \\
&= \left[\frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)}\right]^2 - 2\left[\frac{1}{(1-a)(1-b)} + \frac{1}{(1-b)(1-c)} + \frac{1}{(1-c)(1-a)}\right] \\
&= \left[\frac{3-2(a+b+c)+ab+bc+ca}{ab+bc+ca-(a+b+c)}\right]^2 - 2\left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right] \\
&= \left[1 + \frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right]^2 - 2\left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right] \\
&= 1 + \left[\frac{3-(a+b+c)}{ab+bc+ca-(a+b+c)}\right]^2 \geq 1
\end{aligned}$$

□

∇

★★★★★

Glossary

1. **AM-GM inequality**
   For all non-negative real number $a_1, a_2, \cdots, a_n$ then
   $$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n}$$

2. **Cauchy-Schwarz inequality**
   For all real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ then
   $$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + n_n^2\right) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)$$

3. **Jensen Inequality**
   If f is convex on $\mathbb{I}$ then for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ we have
   $$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

4. **Schur Inequality**
   For all non-negative real numbers a, b, c and positive real number numbers r
   $$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$
   Moreover, if a, b, c are positive real numbers then the above results still holds for all **real** number r

5. **The extension of Schur Inequality (We often call 'Vornicu-Schur inequality')**
   For $x \geq y \geq z$ and $a \geq b \geq c$ then
   $$a(x-y)(x-z) + b(y-z)(y-x) + c(z-x)(z-y) \geq 0$$

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INFINITY

Hojoo Lee, Tom Lovering, and Cosmin Pohoață

Foreword by Dr. Geoff Smith

The first edition (Oct. 2008)

Foreword

The International Mathematical Olympiad is the largest and most prestigious mathematics competition in the world. It is held each July, and the host city changes from year to year. It has existed since 1959.

Originally it was a competition between students from a small group of communist countries, but by the late 1960s, social-democratic nations were starting to send teams. Over the years the enthusiasm for this competition has built up so much that very soon (I write in 2008) there will be an IMO with students participating from over 100 countries. In recent years, the format has become stable. Each nation can send a team of up to six students. The students compete as individuals, and must try to solve 6 problems in 9 hours of examination time, spread over two days.

The nations which do consistently well at this competition must have at least one (and probably at least two) of the following attributes:

(a) A large population.
(b) A significant proportion of its population in receipt of a good education.
(c) A well-organized training infrastructure to support mathematics competitions.
(d) A culture which values intellectual achievement.

Alternatively, you need a cloning facility and a relaxed regulatory framework.

Mathematics competitions began in the Austro-Hungarian Empire in the 19^{th} century, and the IMO has stimulated people into organizing many other related regional and world competitions. Thus there are quite a few opportunities to take part in international mathematics competitions other than the IMO.

The issue arises as to where talented students can get help while they prepare themselves for these competitions. In some countries the students are lucky, and there is a well-developed training regime. Leaving aside the coaching, one of the most important features of these regimes is that they put talented young mathematicians together. This is very important, not just because of the resulting exchanges of ideas, but also for mutual encouragment in a world where interest in mathematics is not always widely understood. There are some very good books available, and a wealth of resources on the internet, including this excellent book Infinity.

The principal author of Infinity is Hojoo Lee of Korea. He is the creator of many beautiful problems, and IMO juries have found his style most alluring. Since 2001 they have chosen 8 of his problems for IMO papers. He has some way to go to catch up with the sage of Scotland, David Monk, who has had 14 problems on IMO papers. These two gentlemen are reciprocal Nemeses, dragging themselves out of bed every morning to face the possibility that the other has just had a good idea. What they each need is a framed picture of the other, hung in their respective studies. I will organize this.

The other authors of Infinity are the young mathematicians Tom Lovering of the United Kingdom and Cosmin Pohoaţă of Romania. Tom is an alumnus of the UK IMO team, and is now starting to read mathematics at Newton's outfit, Trinity College Cambridge. Cosmin has a formidable internet presence, and is a PEN activist (Problems in Elementary Number theory).

One might wonder why anyone would spend their time doing mathematics, when there are so many other options, many of which are superficially more attractive. There are a whole range of opportunities for an enthusiastic Sybarite, ranging from full scale debauchery down to gentle dissipation. While not wishing to belittle these interesting hobbies, mathematics can be more intoxicating.

There is danger here. Many brilliant young minds are accelerated through education, sometimes graduating from university while still under 20. I can think of people for whom this has worked out well, but usually it does not. It is not sensible to deprive teenagers of the company of their own kind. Being a teenager is very stressful; you have to cope with hormonal poisoning, meagre income, social incompetance and the tyranny of adults. If you find yourself with an excellent mathematical mind, it just gets worse, because you have to endure the approval of teachers.

Olympiad mathematics is the sensible alternative to accelerated education. Why do lots of easy courses designed for older people, when instead you can do mathematics which is off the contemporary mathematics syllabus because it is too interesting and too hard? Euclidean and projective geometry and the theory of inequalities (laced with some number theory and combinatorics) will keep a bright young mathematician intellectually engaged, off the streets, and able to go school discos with other people in the same unfortunate teenaged state.

The authors of Infinity are very enthusiastic about MathLinks, a remarkable internet site. While this is a fantastic resource, in my opinion the atmosphere of the Olympiad areas is such that newcomers might feel a little overwhelmed by the extraordinary knowledge and abilities of many of the people posting. There is a kinder, gentler alternative in the form of the nRich site based at the University of Cambridge. In particular the Onwards and Upwards section of their Ask a Mathematician service is MathLinks for herbivores. While still on the theme of material for students at the beginning of their maths competition careers, my accountant would not forgive me if I did not mention A Mathematical Olympiad Primer available on the internet from the United Kingdom Mathematics Trust, and also through the Australian Mathematics Trust.

Returning to this excellent weblished document, Infinity is an wonderful training resource, and is brim full of charming problems and exercises. The mathematics competition community owes the authors a great debt of gratitude.

Dr. Geoff Smith (Dept. of Mathematical Sciences, Univ. of Bath, UK)
UK IMO team leader & Chair of the British Mathematical Olympiad
October 2008

Overture

It was a dark decade until MathLinks was born. However, after Valentin Vornicu founded MathLinks, everything has changed. As the best on-line community, MathLinks helps young students around the worlds to develop problem-solving strategies and broaden their mathematical backgrounds. Nowadays, students, as young mathematicians, use the LaTeX typesetting system to upload recent olympiad problems or their own problems and enjoy mathematical friendship by sharing their creative solutions with each other. In other words, MathLinks encourages and challenges young people in all countries, foster friendships between young mathematicians around the world. Yes, it exactly coincides with the aim of the IMO. Actually, MathLinks is even better than IMO. Simply, it is because everyone can join MathLinks!

In this never-ending project, which bears the name Infinity, we offer a delightful playground for young mathematicians and try to continue the beautiful spirit of IMO and MathLinks. Infinity begins with a chapter on elementary number theory and mainly covers Euclidean geometry and inequalities. We re-visit beautiful well-known theorems and present heuristics for elegant problem-solving. Our aim in this weblication is not just to deliver *must-know* techniques in problem-solving. Young readers should keep in mind that our aim in this project is to present the beautiful aspects of Mathematics. Eventually, Infinity will admit bridges between Olympiads Mathematics and undergraduate Mathematics.

Here goes the reason why we focus on the algebraic and trigonometric methods in geometry. It is a *cliché* that, in the IMOs, some students from hard-training countries used to employ the brute-force algebraic techniques, such as employing trigonometric methods, to attack hard problems from classical triangle geometry or to trivialize easy problems. Though MathLinks already has been contributed to the distribution of the power of algebraic methods, it seems that still many people do not feel the importance of such techniques. Here, we try to destroy such situations and to deliver a friendly introduction on algebraic and trigonometric methods in geometry.

We have to confess that many materials in the first chapter are stolen from PEN (Problems in Elementary Number theory). Also, the lecture note on inequalities is a continuation of the weblication TIN (Topic in INequalities). We are indebted to Orlando Döhring and Darij Grinberg for providing us with TeX files including collections of interesting problems. We owe great debts to Stanley Rabinowitz who kindly sent us his paper. We'd also like to thank Marian Muresan for his excellent collection of problems. We are pleased that Cao Minh Quang sent us various Vietnam problems and nice proofs of Nesbitt's Inequality.

Infinity is a joint work of three coauthors: Hojoo Lee (Korea), Tom Lovering (United Kingdom), and Cosmin Pohoaţă (Romania). We would greatly appreciate hearing about comments and corrections from our readers. Have fun!

> Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.
>
> \- P. Erdős

1.1. **Fundamental Theorem of Arithmetic.** In this chapter, we meet various inequalities and estimations which appears in number theory. Throughout this section, we denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ the set of positive integers, integers, rational numbers, respectively. For integers a and b, we write $a \mid b$ if there exists an integer k such that $b = ka$. Our starting point In this section is the cornerstone theorem that every positive integer $n \neq 1$ admits a unique factorization of prime numbers.

Theorem 1.1. (The Fundamental Theorem of Arithmetic in $\mathbb{N}$) Let $n \neq 1$ be a positive integer. Then, n is a product of primes. If we ignore the order of prime factors, the factorization is unique. Collecting primes from the factorization, we obtain a standard factorization of n:

$$n = {p_1}^{e_1} \cdots {p_l}^{e_l}.$$

The distinct prime numbers $p_1, \cdots, p_l$ and the integers $e_1, \cdots, e_l \geq 0$ are uniquely determined by n.

We define $ord_p(n)$, the order of $n \in \mathbb{N}$ at a prime p,[1] by the nonnegative integer k such that $p^k \mid n$ but $p^{k+1} \not| \, n$. Then, the standard factorization of positive integer n can be rewritten as the form

$$n = \prod_{\text{p : prime}} p^{ord_p(n)}.$$

One immediately has the following simple and useful criterion on divisibility.

Proposition 1.1. *Let A and B be positive integers. Then, A is a multiple of B if and only if the inequality*

$$ord_p(A) \geq ord_p(B)$$

holds for all primes p.

Epsilon 1. [NS] Let a and b be positive integers such that

$$a^k \mid b^{k+1}$$

for all positive integers k. Show that b is divisible by a.

We now employ a formula for the prime factorization of $n!$. Let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to the real number x.

Delta 1. (De Polignac's Formula) Let p be a prime and let n be a nonnegative integer. Then, the largest exponent e of $n!$ such that $p^e \mid n!$ is given by

$$ord_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

[1] Here, we do not assume that $n \neq 1$.

Example 1. *Let $a_1, \cdots, a_n$ be nonnegative integers. Then, $(a_1 + \cdots + a_n)!$ is divisible by $a_1! \cdots a_n!$.*

Proof. Let p be a prime. Our job is to establish the inequality

$$ord_p\left(\,(a_1 + \cdots + a_n)!\,\right) \geq ord_p\left(a_1!\right) + \cdots ord_p\left(a_n!\right).$$

or

$$\sum_{k=1}^{\infty} \left\lfloor \frac{a_1 + \cdots + a_n}{p^k} \right\rfloor \geq \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a_1}{p^k} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{p^k} \right\rfloor \right).$$

However, the inequality

$$\lfloor x_1 + \cdots + x_n \rfloor \geq \lfloor x_1 \rfloor + \cdots \lfloor x_n \rfloor,$$

holds for all real numbers $x_1, \cdots, x_n$. □

Epsilon 2. [IMO 1972/3 UNK] *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Epsilon 3. *Let $n \in \mathbb{N}$. Show that $\mathcal{L}_n := lcm(1, 2, \cdots, 2n)$ is divisible by $\mathcal{K}_n := \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.*

Delta 2. (Canada 1987) Show that, for all positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

Delta 3. (Iran 1996) Prove that, for all positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor.$$

1.2. **Fermat's Infinite Descent.** In this section, we learn Fermat's trick, which bears the name method of infinite descent. It is extremely useful for attacking many Diophantine equations. We first present a proof of Fermat's Last Theorem for $n = 4$.

Theorem 1.2. (The Fermat-Wiles Theorem) Let $n \geq 3$ be a positive integer. The equation

$$x^n + y^n = z^n$$

has no solution in positive integers.

Lemma 1.1. *Let σ be a positive integer. If we have a factorization $\sigma^2 = AB$ for some relatively integers A and B, then the both factors A and B are also squares. There exist positive integers a and b such that*

$$\sigma = ab, \ \ A = a^2, \ \ B = b^2, \ \ gcd(a, b) = 1.$$

Proof. Use The Fundamental Theorem of Arithmetic. □

Lemma 1.2. (Primitive Pythagoras Triangles) Let $x, y, z \in \mathbb{N}$ with $x^2 + y^2 = z^2$, $gcd(x, y) = 1$, and $x \equiv 0 \ (mod\ 2)$ Then, there exists positive integers p and q such that $gcd(p, q) = 1$ and

$$(x,\ y,\ z) = \left(2pq,\ p^2 - q^2,\ p^2 + q^2\right).$$

Proof. The key observation is that the equation can be rewritten as

$$\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right).$$

Reading the equation $x^2 + y^2 = z^2$ modulo 2, we see that both y and z are odd. Hence, $\frac{z+y}{2}$, $\frac{z-y}{2}$, and $\frac{x}{2}$ are positive integers. We also find that $\frac{z+y}{2}$ and $\frac{z-y}{2}$ are relatively prime. Indeed, if $\frac{z+y}{2}$ and $\frac{z-y}{2}$ admits a common prime divisor p, then p also divides both $y = \frac{z+y}{2} - \frac{z-y}{2}$ and $\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$, which means that the prime p divides both x and y. This is a contradiction for $gcd(x, y) = 1$. Now, applying the above lemma, we obtain

$$\left(\frac{x}{2},\ \frac{z+y}{2},\ \frac{z-y}{2}\right) = \left(pq,\ p^2,\ q^2\right)$$

for some positive integers p and q such that $\gcd(p, q) = 1$. □

Theorem 1.3. *The equation $x^4 + y^4 = z^2$ has no solution in positive integers.*

Proof. Assume to the contrary that there exists a bad triple (x, y, z) of positive integers such that $x^4 + y^4 = z^2$. Pick a bad triple $(A, B, C) \in \mathcal{D}$ so that $A^4 + B^4 = C^2$. Letting d denote the greatest common divisor of A and B, we see that $C^2 = A^4 + B^4$ is divisible by d^4, so that C is divisible by d^2. In the view of $\left(\frac{A}{d}\right)^4 + \left(\frac{B}{d}\right)^4 = \left(\frac{C}{d^2}\right)^2$, we find that $(a, b, c) = \left(\frac{A}{d}, \frac{B}{d}, \frac{C}{d^2}\right)$ is also in $\mathcal{D}$, that is,

$$a^4 + b^4 = c^2.$$

Furthermore, since d is the greatest common divisor of A and B, we have $\gcd(a, b) = \gcd\left(\frac{A}{d}, \frac{B}{d}\right) = 1$. Now, we do the parity argument. If both a and b are odd, we find that $c^2 \equiv a^4 + b^4 \equiv 1 + 1 \equiv 2 \ (mod\ 4)$, which is impossible. By symmetry, we may assume that a is even and that b is odd. Combining results, we see that a^2 and b^2

are relatively prime and that a^2 is even. Now, in the view of $\left(a^2\right)^2 + \left(b^2\right)^2 = c^2$, we obtain

$$\left(a^2,\ b^2,\ c\right) = \left(2pq,\ p^2 - q^2,\ p^2 + q^2\right).$$

for some positive integers p and q such that $\gcd(p, q) = 1$. It is clear that p and q are of opposite parity. We observe that

$$q^2 + b^2 = p^2.$$

Since b is odd, reading it modulo 4 yields that q is even and that p is odd. If q and b admit a common prime divisor, then $p^2 = q^2 + b^2$ guarantees that p also has the prime, which contradicts for $\gcd(p, q) = 1$. Combining the results, we see that q and b are relatively prime and that q is even. In the view of $q^2 + b^2 = p^2$, we obtain

$$(q,\ b,\ p) = \left(2mn,\ m^2 - n^2,\ m^2 + n^2\right).$$

for some positive integers m and n such that $\gcd(m, n) = 1$. Now, recall that $a^2 = 2pq$. Since p and q are relatively prime and since q is even, it guarantees the existence of the pair (P, Q) of positive integers such that

$$a = 2PQ,\ p = P^2,\ q = 2Q^2,\ \gcd(P, Q) = 1.$$

It follows that $2Q^2 = 2q = 2mn$ so that $Q = mn$. Since $\gcd(m, n) = 1$, this guarantees the existence of the pair (M, N) of positive integers such that

$$Q = MN,\ m = M^2,\ n = N^2,\ \gcd(M, N) = 1.$$

Combining the results, we find that $P^2 = p = m^2 + n^2 = M^4 + N^4$ so that (M, N, P) is a bad triple. Recall the starting equation $A^4 + B^4 = C^2$. Now, let's summarize up the results what we did. The bad triple (A, B, C) produces a new bad triple (M, N, P). However, we need to check that it is indeed new. We observe that $P < C$. Indeed, we deduce

$$P \leq P^2 = p < p^2 + q^2 = c = \frac{C}{d^2} \leq C.$$

In words, from a solution of $x^4 + y^4 = z^2$, we are able to find another solution with smaller positive integer z. The key point is that this reducing process can be repeated. Hence, it produces to an infinite sequence of strictly decreasing positive integers. However, it is clearly impossible. We therefore conclude that there exists no bad triple. $\square$

Corollary 1.1. *The equation $x^4 + y^4 = z^4$ has no solution in positive integers.*

Proof. Letting $w = z^2$, we obtain $x^4 + y^4 = w^2$. $\square$

We now include a recent problem from IMO as another working example.

Example 2. [IMO 2007/5 IRN] *Let a and b be positive integers. Show that if $4ab - 1$ divides $\left(4a^2 - 1\right)^2$, then $a = b$.*

First Solution. (by NZL at IMO 2007) When $4ab - 1$ divides $\left(4a^2 - 1\right)^2$ for two distinct positive integers a and b, we say that (a, b) is a bad pair. We want to show that there is no bad pair. Suppose that $4ab - 1$ divides $\left(4a^2 - 1\right)^2$. Then, $4ab - 1$ also divides

$$b\left(4a^2 - 1\right)^2 - a\left(4ab - 1\right)\left(4a^2 - 1\right) = (a - b)\left(4a^2 - 1\right).$$

The converse also holds as $\gcd(b, 4ab-1) = 1$. Similarly, $4ab-1$ divides $(a-b)\left(4a^2-1\right)^2$ if and only if $4ab-1$ divides $(a-b)^2$. So, the original condition is equivalent to the condition

$$4ab - 1 \mid (a-b)^2 .$$

This condition is symmetric in a and b, so (a, b) is a bad pair if and only if (b, a) is a bad pair. Thus, we may assume without loss of generality that $a > b$ and that our bad pair of this type has been chosen with the smallest possible vales of its first element. Write $(a-b)^2 = m(4ab-1)$, where m is a positive integer, and treat this as a quadratic in a:

$$a^2 + (-2b - 4ma)a + \left(b^2 + m\right) = 0.$$

Since this quadratic has an integer root, its discriminant

$$(2b + 4mb)^2 - 4\left(b^2 + m\right) = 4\left(4mb^2 + 4m^2b^2 - m\right)$$

must be a perfect square, so $4mb^2 + 4m^2b^2 - m$ is a perfect square. Let his be the square of $2mb + t$ and note that $0 < t < b$. Let $s = b - t$. Rearranging again gives:

$$\begin{gathered} 4mb^2 + 4m^2b^2 - m = (2mb + t)^2 \\ m\left(4b^2 - 4bt - 1\right) = t^2 \\ m\left(4b^2 - 4b(b-s) - 1\right) = (b-s)^2 \\ m(4bs - 1) = (b-s)^2. \end{gathered}$$

Therefore, (b, s) is a bad pair with a smaller first element, and we have a contradiction. □

Second Solution. (by UNK at IMO 2007) This solution is inspired by the solution of NZL7 and Atanasov's special prize solution at IMO 1988 in Canberra. We begin by copying the argument of NZL7. A counter-example (a, b) is called a bad pair. Consider a bad pair (a, b) so $4ab-1 | \left(4a^2-1\right)^2$. Notice that $b\left(4a^2-1\right) - (4ab-1)a = a - b$ so working modulo $4ab-1$ we have $b^2\left(4a^2-1\right) \equiv (a-b)^2$. Now, b^2 an $4ab-1$ are coprime so $4ab-1$ divides $\left(4a^2-1\right)^2$ if and only if $4ab-1$ divides $(a-b)^2$. This condition is symmetric in a and b, so we learn that (a, b) is a bad pair if and only if (b, a) is a bad pair. Thus, we may assume that $a > b$ and we may as well choose a to be minimal among all bad pairs where the first component is larger than the second. Next, we deviate from NZL7's solution. Write $(a-b)^2 = m(4ab-1)$ and treat it as a quadratic so a is a root of

$$x^2 + (-2b - 4mb)x + \left(b^2 + m\right) = 0.$$

The other root must be an integer c since $a + c = 2b + 4mb$ is an integer. Also, $ac = b^2 + m > 0$ so c is positive. We will show that $c < b$, and then the pair (b, c) will violate the minimality of (a, b). It suffices to show that $2b + 4mb < a + b$, i.e., $4mb < a - b$. Now,

$$4b(a-b)^2 = 4mb(4ab-1)$$

so it suffices to show that $4b(a-b) < 4ab - 1$ or rather $1 < 4b^2$ which is true. □

Delta 4. [IMO 1988/6 FRG] *Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that*

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Delta 5. (Canada 1998) Let m be a positive integer. Define the sequence $\{a_n\}_{n\geq 0}$ by

$$a_0 = 0,\ a_1 = m,\ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2+b^2}{ab+1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

Delta 6. *Let x and y be positive integers such that xy divides x^2+y^2+1. Show that*

$$\frac{x^2+y^2+1}{xy} = 3.$$

Delta 7. *Find all triple (x, y, z) of integers such that*

$$x^2+y^2+z^2 = 2xyz.$$

Delta 8. (APMO 1989) Prove that the equation

$$6\left(6a^2+3b^2+c^2\right) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

1.3. **Monotone Multiplicative Functions.** In this section, we study when multiplicative functions has the monotonicity.

Example 3. (Canada 1969) Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$: $f(2) = 2$, $f(mn) = f(m)f(n)$, $f(n+1) > f(n)$.

First Solution. We first evaluate $f(n)$ for small n. It follows from $f(1 \cdot 1) = f(1) \cdot f(1)$ that $f(1) = 1$. By the multiplicity, we get $f(4) = f(2)^2 = 4$. It follows from the inequality $2 = f(2) < f(3) < f(4) = 4$ that $f(3) = 3$. Also, we compute $f(6) = f(2)f(3) = 6$. Since $4 = f(4) < f(5) < f(6) = 6$, we get $f(5) = 5$. We prove by induction that $f(n) = n$ for all $n \in \mathbb{N}$. It holds for $n = 1, 2, 3$. Now, let $n > 2$ and suppose that $f(k) = k$ for all $k \in \{1, \cdots, n\}$. We show that $f(n+1) = n+1$.

Case 1. $n+1$ is composite. One may write $n+1 = ab$ for some positive integers a and b with $2 \leq a \leq b \leq n$. By the inductive hypothesis, we have $f(a) = a$ and $f(b) = b$. It follows that $f(n+1) = f(a)f(b) = ab = n+1$.

Case 2. $n+1$ is prime. In this case, $n+2$ is even. Write $n+2 = 2k$ for some positive integer k. Since $n \geq 2$, we get $2k = n+2 \geq 4$ or $k \geq 2$. Since $k = \frac{n+2}{2} \leq n$, by the inductive hypothesis, we have $f(k) = k$. It follows that $f(n+2) = f(2k) = f(2)f(k) = 2k = n+2$. From the inequality

$$n = f(n) < f(n+1) < f(n+2) = n+2$$

we conclude that $f(n+1) = n+1$. By induction, $f(n) = n$ holds for all positive integers n. $\square$

Second Solution. As in the previous solution, we get $f(1) = 1$. We find that

$$f(2n) = f(2)f(n) = 2f(n)$$

for all positive integers n. This implies that, for all positive integers k,

$$f\left(2^k\right) = 2^k$$

Let $k \in \mathbb{N}$. From the assumption, we obtain the inequality

$$2^k = f\left(2^k\right) < f\left(2^k+1\right) < \cdots < f\left(2^{k+1}-1\right) < f\left(2^{k+1}\right) = 2^{k+1}.$$

In other words, the increasing sequence of $2^k + 1$ positive integers

$$f\left(2^k\right), f\left(2^k+1\right), \cdots, f\left(2^{k+1}-1\right), f\left(2^{k+1}\right)$$

lies in the set of $2^k + 1$ consecutive integers $\{2^k, 2^k+1, \cdots, 2^{k+1}-1, 2^{k+1}\}$. This means that $f(n) = n$ for all $2^k \leq n \leq 2^{k+1}$. Since this holds for all positive integers k, we conclude that $f(n) = n$ for all $n \geq 2$. $\square$

The conditions in the problem are too restrictive. Let's throw out the condition $f(2) = 2$.

Epsilon 4. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ for all positive integers m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there is a constant $\alpha \in \mathbb{R}$ such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

We can weaken the assumption that f is completely multiplicative, but we bring back the condition $f(2) = 2$.

Epsilon 5. (Putnam 1963/A2) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all relatively prime m and n. Then, f is the identity function on $\mathbb{N}$.

In fact, we can completely drop the constraint $f(2) = 2$. In 1946, P. Erdős proved the following result in [PE]:

Theorem 1.4. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function satisfying the conditions:*
(a) $f(mn) = f(m) + f(n)$ for all relatively prime m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there exists a constant $\alpha \in \mathbb{R}$ such that $f(n) = \alpha \ln n$ for all $n \in \mathbb{N}$.

This implies the following multiplicative result.

Theorem 1.5. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ for all relatively prime m and n, and
(b) $f(n+1) \geq f(n)$ for all positive integers n.
Then, there is a constant $\alpha \in \mathbb{R}$ such that $f(n) = n^\alpha$ for all $n \in \mathbb{N}$.

Proof. [2] It is enough to show that the function f is completely multiplicative: $f(mn) = f(m)f(n)$ for all m and n. We split the proof in three steps.

Step 1. Let $a \geq 2$ be a positive integer and let $\Omega_a = \{x \in \mathbb{N} \mid \gcd(x, a) = 1\}$. Then, we find that

$$L := \inf_{x \in \Omega_a} \frac{f(x+a)}{f(x)} = 1$$

and

$$f\left(a^{k+1}\right) \leq f\left(a^k\right) f(a)$$

for all positive integers k.

Proof of Step 1. Since f is monotone increasing, it is clear that $L \geq 1$. Now, we notice that $f(k+a) \geq Lf(k)$ whenever $k \in \Omega_a$. Let m be a positive integer. We take a sufficiently large integer $x_0 > ma$ with $\gcd(x_0, a) = \gcd(x_0, 2) = 1$ to obtain

$$f(2)f(x_0) = f(2x_0) \geq f(x_0 + ma) \geq Lf(x_0 + (m-1)a) \geq \cdots \geq L^m f(x_0)$$

or

$$f(2) \geq L^m.$$

Since m is arbitrary, this and $L \geq 1$ force to $L = 1$. Whenever $x \in \Omega_a$, we obtain

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} = \frac{f\left(a^{k+1} x\right)}{f\left(a^k\right)} \leq \frac{f\left(a^{k+1} x + a^k\right)}{f\left(a^k\right)} = f(ax+1) \leq f\left(ax + a^2\right)$$

or

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} \leq f(a)f(x+a)$$

or

$$\frac{f(x+a)}{f(x)} \geq \frac{f\left(a^{k+1}\right)}{f(a) f\left(a^k\right)}.$$

[2] We present a slightly modified proof in [EH]. For another short proof, see [MJ].

It follows that

$$1 = \inf_{x \in \Omega_a} \frac{f(x+a)}{f(x)} \geq \frac{f\left(a^{k+1}\right)}{f(a) f\left(a^k\right)}$$

so that

$$f\left(a^{k+1}\right) \leq f\left(a^k\right) f(a),$$

as desired.

Step 2. Similarly, we have

$$U := \sup_{x \in \Omega_a} \frac{f(x)}{f(x+a)} = 1$$

and

$$f\left(a^{k+1}\right) \geq f\left(a^k\right) f(a)$$

for all positive integers k.

Proof of Step 2. The first result immediately follows from Step 1.

$$\sup_{x \in \Omega_a} \frac{f(x)}{f(x+a)} = \frac{1}{\inf_{x \in \Omega_a} \frac{f(x+a)}{f(x)}} = 1.$$

Whenever $x \in \Omega_a$ and $x > a$, we have

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} = \frac{f\left(a^{k+1} x\right)}{f\left(a^k\right)} \geq \frac{f\left(a^{k+1} x - a^k\right)}{f\left(a^k\right)} = f\left(ax - 1\right) \geq f\left(ax - a^2\right)$$

or

$$\frac{f\left(a^{k+1}\right) f(x)}{f\left(a^k\right)} \geq f(a) f(x-a).$$

It therefore follows that

$$1 = \sup_{x \in \Omega_a} \frac{f(x)}{f(x+a)} = \sup_{x \in \Omega_a,\, x > a} \frac{f(x-a)}{f(x)} \leq \frac{f\left(a^{k+1}\right)}{f(a) f\left(a^k\right)},$$

as desired.

Step 3. From the two previous results, whenever $a \geq 2$, we have

$$f\left(a^{k+1}\right) = f\left(a^k\right) f(a).$$

Then, the straightforward induction gives that

$$f\left(a^k\right) = f(a)^k$$

for all positive integers a and k. Since f is multiplicative, whenever

$$n = {p_1}^{k_1} \cdots {p_l}^{k_l}$$

gives the standard factorization of n, we obtain

$$f(n) = f\left({p_1}^{k_1}\right) \cdots f\left({p_l}^{k_l}\right) = f\left(p_1\right)^{k_1} \cdots f\left(p_l\right)^{k_l}.$$

We therefore conclude that f is completely multiplicative. $\square$

1.4. **There are Infinitely Many Primes.** The purpose of this subsection is to offer various proofs of Euclid's Theorem.

Theorem 1.6. (Euclid's Theorem) The number of primes is infinite.

Proof. Assume to the contrary $\{p_1 = 2, p_2 = 3, \cdots, p_n\}$ is the set of all primes. Consider the positive integer

$$P = p_1 \cdots p_n + 1.$$

Since $P > 1$, P must admit a prime divisor p_i for some $i \in \{1, \cdots, n\}$. Since both P and $p_1 \cdots p_n$ are divisible by p_i, we find that $1 = P - p_1 \cdots p_n$ is also divisible by p_i, which is a contradiction. □

In fact, more is true. We now present four proofs of Euler's Theorem that the sum of the reciprocals of all prime numbers diverges.

Theorem 1.7. (Euler's Theorem, PEN E24) Let p_n denote the nth prime number. The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

First Proof. [NZM, pp.21-23] We first prepare a lemma. Let $\varrho(n)$ denote the set of prime divisors of n. Let $\mathcal{S}_n(N)$ denote the set of positive integers $i \leq N$ satisfying that $\varrho(i) \subset \{p_1, \cdots, p_n\}$.

Lemma 1.3. *We have* $|\mathcal{S}_n(N)| \leq 2^n\sqrt{N}$.

Proof of Lemma. It is because every positive integer $i \in \mathcal{S}_n(N)$ has a unique factorization $i = st^2$, where s is a divisor of $p_1 \cdots p_n$ and $t \leq \sqrt{N}$. In other words, $i \mapsto (s, t)$ is an injective map from $\mathcal{S}_n(N)$ to $\mathcal{T}_n(N) = \{\, (s,t) \mid s \,|\, p_1 \cdots p_n,\ t \leq \sqrt{N}\}$, which means that $|\mathcal{S}_n(N)| \leq |\mathcal{T}_n(N)| \leq 2^n\sqrt{N}$.

Now, assume to the contrary that the infinite series $\frac{1}{p_1} + \frac{1}{p_2} + \cdots$ converges. Then we can take a sufficiently large positive integer n satisfying that

$$\frac{1}{2} \geq \sum_{i>n}^{\infty} \frac{1}{p_i} = \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \cdots.$$

Take a sufficiently large positive integer N so that $N > 4^{n+1}$. By its definition of $\mathcal{S}_n(N)$, we see that each element i in $\{1, \cdots, N\} - \mathcal{S}_n(N)$ is divisible by at least one prime p_j for some $j > n$. Since the number of multiples of p_j not exceeding N is $\left\lfloor \frac{N}{p_j} \right\rfloor$, we have

$$|\{1, \cdots, N\} - \mathcal{S}_n(N)| \leq \sum_{j>n} \left\lfloor \frac{N}{p_j} \right\rfloor$$

or

$$N - |\mathcal{S}_n(N)| \leq \sum_{j>n} \left\lfloor \frac{N}{p_j} \right\rfloor \leq \sum_{j>n} \frac{N}{p_j} \leq \frac{N}{2}$$

or

$$\frac{N}{2} \leq |\mathcal{S}_n(N)|.$$

It follows from this and from the lemma that $\frac{N}{2} \leq 2^n\sqrt{N}$ so that $N \leq 4^{n+1}$. However, it is a contradiction for the choice of N. $\square$

Second Proof. We employ an auxiliary inequality without a proof.

Lemma 1.4. *The inequality $1+t \leq e^t$ holds for all $t \in \mathbb{R}$.*

Let $n > 1$. Since each positive integer $i \leq n$ has a unique factorization $i = st^2$, where s is square free and $t \leq \sqrt{n}$, we obtain

$$\sum_{k=1}^{n} \frac{1}{k} \leq \prod_{\substack{p:prime,\\ p\leq n}} \left(1+\frac{1}{p}\right) \sum_{t\leq\sqrt{n}} \frac{1}{t^2}.$$

Together with the estimation

$$\sum_{t=1}^{\infty} \frac{1}{t^2} \leq 1 + \sum_{t=2}^{\infty} \frac{1}{t(t-1)} = 1 + \sum_{t=2}^{\infty} \left(\frac{1}{t-1} - \frac{1}{t}\right) = 2,$$

we conclude that

$$\sum_{k=1}^{n} \frac{1}{k} \leq 2 \prod_{\substack{p:prime,\\ p\leq n}} \left(1+\frac{1}{p}\right) \leq 2 \prod_{\substack{p:prime\\ p\leq n}} e^{\frac{1}{p}}$$

or

$$\sum_{\substack{p:prime\\ p\leq n}} \frac{1}{p} \geq \ln\left(\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\right).$$

Since the divergence of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is well-known, by Comparison Test, the series diverges. $\square$

Third Proof. [NZM, pp.21-23] We exploit an auxiliary inequality without a proof.

Lemma 1.5. *The inequality $\frac{1}{1-t} \leq e^{t+t^2}$ holds for all $t \leq \left[0, \frac{1}{2}\right]$.*

Let $l \in \mathbb{N}$. By The Fundamental Theorem of Arithmetic, each positive integer $i \leq p_l$ has a unique factorization $i = {p_1}^{e_1} \cdots {p_1}^{e_l}$ for some $e_1, \cdots, e_l \in \mathbb{Z}_{\geq 0}$. It follows that

$$\sum_{i=1}^{p_l} \frac{1}{i} \leq \sum_{e_1,\cdots,e_l\in\mathbb{Z}_{\geq 0}} \frac{1}{{p_1}^{e_1}\cdots {p_l}^{e_l}} = \prod_{j=1}^{l}\left(\sum_{k=0}^{\infty}\frac{1}{{p_j}^k}\right) = \prod_{j=1}^{l}\frac{1}{1-\frac{1}{p_j}} \leq \prod_{j=1}^{l} e^{\frac{1}{p_j}+\frac{1}{{p_j}^2}}$$

so that

$$\sum_{j=1}^{l}\left(\frac{1}{p_j}+\frac{1}{{p_j}^2}\right) \geq \ln\left(\sum_{i=1}^{p_l}\frac{1}{i}\right).$$

Together with the estimation

$$\sum_{j=1}^{\infty}\frac{1}{{p_j}^2} \leq \sum_{j=1}^{l}\frac{1}{(j+1)^2} \leq \sum_{j=1}^{l}\frac{1}{(j+1)j} = \sum_{j=1}^{l}\left(\frac{1}{j}-\frac{1}{j+1}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1,$$

we conclude that

$$\sum_{j=1}^{l}\frac{1}{p_j} \geq \ln\left(\sum_{i=1}^{p_l}\frac{1}{i}\right) - 1.$$

Since the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, by The Comparison Test, we get the result. $\square$

Fourth Proof. [DB, p.334] It is a consequence of The Prime Number Theorem. Let $\pi(x)$ denote the prime counting function. Since The Prime Number Theorem says that $\pi(x) \to \frac{x}{\ln x}$ as $x \to \infty$, we can find a constant $\lambda > 0$ satisfying that $\pi(x) > \lambda \frac{x}{\ln x}$ for all sufficiently large positive real numbers x. This means that $n > \lambda \frac{p_n}{\ln p_n}$ when n is sufficiently large. Since $\lambda \frac{x}{\ln x} > \sqrt{x}$ for all sufficiently large $x > 0$, we also have

$$n > \lambda \frac{p_n}{\ln p_n} > \sqrt{p_n}$$

or

$$n^2 > p_n$$

for all sufficiently large n. We conclude that, when n is sufficiently large,

$$n > \lambda \frac{p_n}{\ln p_n} > \lambda \frac{p_n}{\ln (n^2)},$$

or equivalently,

$$\frac{1}{p_n} > \frac{\lambda}{2n \ln n}.$$

Since we have $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$, The Comparison Test yields the desired result. $\square$

We close this subsection with a striking result establish by Viggo Brun.

Theorem 1.8. (Brun's Theorem) The sum of the reciprocals of the twin primes converges:

$$\mathcal{B} = \sum_{\text{p, p+2: prime}} \left(\frac{1}{p} + \frac{1}{p+2} \right) = \left(\frac{1}{3} + \frac{1}{5} \right) + \left(\frac{1}{5} + \frac{1}{7} \right) + \left(\frac{1}{11} + \frac{1}{13} \right) + \cdots < \infty$$

The constant $\mathcal{B} = 1.90216 \cdots$ is called Brun's Constant.

1.5. **Towards $1 Million Prize Inequalities.** In this section, we follow [JL]. We consider two conjectures.

Open Problem 1.1. (J. C. Lagarias) Given a positive integer n, let $\mathcal{H}_n$ denote the n-th harmonic number

$$\mathcal{H}_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \cdots + \frac{1}{n}$$

and let $\sigma(n)$ denote the sum of positive divisors of n. Prove that that the inequality

$$\sigma(n) \leq \mathcal{H}_n + e^{\mathcal{H}_n} \ln \mathcal{H}_n$$

holds for all positive integers n.

Open Problem 1.2. *Let π denote the prime counting function, that is, $\pi(x)$ counts the number of primes p with $1 < p \leq x$. Let $\varepsilon > 0$. Prove that that there exists a positive constant C_ε such that the inequality*

$$\left| \pi(x) - \int_2^x \frac{1}{\ln t} dt \right| \leq C_\varepsilon x^{\frac{1}{2}+\varepsilon}$$

holds for all real numbers $x \geq 2$.

These two unseemingly problems are, in fact, equivalent. Furthermore, more strikingly, they are equivalent to The Riemann Hypothesis from complex analysis. In 2000, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven prize problems. If you knock them down, you earn at least $1 Million.[3] For more info, visit the CMI website at

http://www.claymath.org/millennium

Wir müssen wissen. Wir werden wissen.

- D. Hilbert

[3]However, in that case, be aware that Mafias can knock you down and take money from you :]

2. Symmetries

> Each problem that I solved became a rule, which served afterwards to solve other problems.
>
> - R. Descartes

2.1. **Exploiting Symmetry.** We begin with the following example.

Example 4. *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a}\geq a^3+b^3+c^3.$$

First Solution. After brute-force computation, i.e, clearing denominators, we reach

$$a^5b+a^5c+b^5c+b^5a+c^5a+c^5b\geq a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2.$$

Now, we deduce

$$\begin{aligned}
& a^5b+a^5c+b^5c+b^5a+c^5a+c^5b\\
= \; & a\left(b^5+c^5\right)+b\left(c^5+a^5\right)+c\left(a^5+b^5\right)\\
\geq \; & a\left(b^3c^2+b^2c^3\right)+b\left(c^3a^2+c^2b^3\right)+c\left(c^3a^2+c^2b^3\right)\\
= \; & a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2.
\end{aligned}$$

Here, we used the the auxiliary inequality

$$x^5+y^5\geq x^3y^2+x^2y^3,$$

where $x, y \geq 0$. Indeed, we obtain the equality

$$x^5+y^5-x^3y^2-x^2y^3=\left(x^3-y^3\right)\left(x^2-y^2\right).$$

It is clear that the final term $\left(x^3-y^3\right)\left(x^2-y^2\right)$ is always non-negative. □

Here goes a more economical solution without the brute-force computation.

Second Solution. The trick is to observe that the right hand side admits a nice decomposition:

$$a^3+b^3+c^3=\frac{a^3+b^3}{2}+\frac{b^3+c^3}{2}+\frac{c^3+a^3}{2}.$$

We then see that the inequality has the symmetric face:

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a}\geq\frac{a^3+b^3}{2}+\frac{b^3+c^3}{2}+\frac{c^3+a^3}{2}.$$

Now, the symmetry of this expression gives the right approach. We check that, for $x, y > 0$,

$$\frac{x^4+y^4}{x+y}\geq\frac{x^3+y^3}{2}.$$

However, we obtain the identity

$$2\left(x^4+y^4\right)-\left(x^3+y^3\right)(x+y)=x^4+y^4-x^3y-xy^3=\left(x^3-y^3\right)(x-y).$$

It is clear that the final term $\left(x^3-y^3\right)(x-y)$ is always non-negative. □

Delta 9. [LL 1967 POL] *Prove that, for all $a, b, c > 0$,*

$$\frac{a^8+b^8+c^8}{a^3b^3c^3}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Delta 10. [LL 1970 AUT] *Prove that, for all $a, b, c > 0$,*

$$\frac{a+b+c}{2}\geq\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}$$

Delta 11. **[SL 1995 UKR]** *Let n be an integer, $n \geq 3$. Let $a_1, \cdots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, \cdots, n$. If $s = a_1 + \cdots + a_n$, prove that*

$$\frac{{a_1}^2 + {a_2}^2 - {a_3}^2}{a_1 + a_2 + a_3} + \frac{{a_2}^2 + {a_3}^2 - {a_4}^2}{a_2 + a_3 + a_4} + \cdots + \frac{{a_n}^2 + {a_1}^2 - {a_2}^2}{a_n + a_1 + a_2} \leq 2s - 2n.$$

Delta 12. **[SL 2006]** *Let $a_1, \cdots, a_n$ be positive real numbers. Prove the inequality*

$$\frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{1 \leq i < j \leq n} a_i a_j \geq \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{a_i + a_j}$$

Epsilon 6. *Let a, b, c be positive real numbers. Prove the inequality*

$$\left(1 + a^2\right)\left(1 + b^2\right)\left(1 + c^2\right) \geq (a + b)(b + c)(c + a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Epsilon 7. (Poland 2006) Let a, b, c be positive real numbers with $ab + bc + ca = abc$. Prove that

$$\frac{a^4 + b^4}{ab(a^3 + b^3)} + \frac{b^4 + c^4}{bc(b^3 + c^3)} + \frac{c^4 + a^4}{ca(c^3 + a^3)} \geq 1.$$

Epsilon 8. (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

2.2. **Breaking Symmetry.** We now learn how to break the symmetry. Let's attack the following problem.

Example 5. *Let a, b, c be non-negative real numbers. Show the inequality*

$$a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right).$$

There are many ways to prove this inequality. In fact, it can be proved either with Schur's Inequality or with Popoviciu's Inequality. Here, we try to give another proof. One natural starting point is to apply The AM-GM Inequality to obtain the estimations

$$c^4 + 3(abc)^{\frac{4}{3}} \geq 4\left(c^4 \cdot (abc)^{\frac{4}{3}} \cdot (abc)^{\frac{4}{3}} \cdot (abc)^{\frac{4}{3}}\right)^{\frac{1}{4}} = 4abc^2$$

and

$$a^4 + b^4 \geq 2a^2b^2.$$

Adding these two inequalities, we obtain

$$a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} \geq 2a^2b^2 + 4abc^2.$$

Hence, it now remains to show that

$$2a^2b^2 + 4abc^2 \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right)$$

or equivalently

$$0 \geq 2c^2(a-b)^2,$$

which is clearly untrue in general. **It is reversed!** However, we can exploit the above idea to finsh the proof.

Proof. Using the symmetry of the inequality, we break the symmetry. Since the inequality is symmetric, we may consider the case $a, b \geq c$ only. Since The AM-GM Inequality implies the inequality $c^4 + 3(abc)^{\frac{4}{3}} \geq 4abc^2$, we obtain the estimation

$$\begin{aligned}
& a^4 + b^4 + c^4 + 3(abc)^{\frac{4}{3}} - 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) \\
\geq\ & \left(a^4 + b^4 - 2a^2b^2\right) + 4abc^2 - 2\left(b^2c^2 + c^2a^2\right) \\
=\ & \left(a^2 - b^2\right)^2 - 2c^2(a-b)^2 \\
=\ & (a-b)^2\left((a+b)^2 - 2c^2\right).
\end{aligned}$$

Since we have $a, b \geq c$, the last term is clearly non-negative. □

Epsilon 9. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Epsilon 10. (USA 1980) Prove that, for all real numbers $a, b, c \in [0,1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Epsilon 11. [AE, p. 186] Show that, for all $a, b, c \in [0,1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Epsilon 12. [SL 2006 KOR] *Let a, b, c be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Epsilon 13. *Let $f(x,y) = xy\left(x^3+y^3\right)$ for $x, y \geq 0$ with $x + y = 2$. Prove the inequality*

$$f(x,y) \leq f\left(1+\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right) = f\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right).$$

Epsilon 14. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2+b}+\sqrt{a+b^2}+\sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Epsilon 15. (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

The above examples say that, in general, **symmetric problems does not admit symmetric solutions.** We now introduce an extremely useful inequality when we make the ordering assmption.

Epsilon 16. (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n,\ y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Corollary 2.1. (The AM-HM Inequality) Let $x_1, \cdots, x_n > 0$. Then, we have

$$\frac{x_1+\cdots+x_n}{n} \geq \frac{n}{\frac{1}{x_1}+\cdots\frac{1}{x_n}}$$

or

$$\frac{1}{x_1}+\cdots\frac{1}{x_n} \geq \frac{n^2}{x_1+\cdots+x_n}.$$

The equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Since the inequality is symmetric, we may assume that $x_1 \leq \cdots \leq x_n$. We have

$$-\frac{1}{x_1} \leq \cdots \leq -\frac{1}{x_n}.$$

Chebyshev's Inequality shows that

$$x_1 \cdot \left(-\frac{1}{x_1}\right)+\cdots+x_1 \cdot \left(-\frac{1}{x_1}\right) \geq \frac{1}{n}\left(x_1+\cdots+x_n\right)\left[\left(-\frac{1}{x_1}\right)+\cdots+\left(-\frac{1}{x_1}\right)\right].$$

□

Remark 2.1. *In Chebyshev's Inequality, we do not require that the variables are positive. It also implies that if $x_1 \leq \cdots \leq x_n$ and $y_1 \geq \cdots \geq y_n$, then we have the reverse estimation*

$$\sum_{i=1}^{n} x_i y_i \leq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Epsilon 17. (United Kingdom 2002) For all $a, b, c \in (0, 1)$, show that

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

Epsilon 18. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Epsilon 19. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

We now present three different proofs of Nesbitt's Inequality:

Proposition 2.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 1. *We denote $\mathcal{L}$ the left hand side. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. Since $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$, Chebyshev's Inequality yields that*

$$\begin{aligned}
\mathcal{L} &\geq \frac{1}{3}(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \\
&= \frac{1}{3}\left(\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\right) \\
&= 3\left(1+\frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}\right) \\
&= \frac{1}{3}(3+\mathcal{L}),
\end{aligned}$$

so that $\mathcal{L} \geq \frac{3}{2}$, as desired.

Proof 2. *We now break the symmetry by a suitable normalization. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. After the substitution $x = \frac{a}{c}, y = \frac{b}{c}$, we have $x \geq y \geq 1$. It becomes*

$$\frac{\frac{a}{c}}{\frac{b}{c}+1}+\frac{\frac{b}{c}}{\frac{a}{c}+1}+\frac{1}{\frac{a}{c}+\frac{b}{c}} \geq \frac{3}{2}$$

or

$$\frac{x}{y+1}+\frac{y}{x+1} \geq \frac{3}{2}-\frac{1}{x+y}.$$

We first apply The AM-GM Inequality to deduce

$$\frac{x+1}{y+1}+\frac{y+1}{x+1} \geq 2$$

or

$$\frac{x}{y+1}+\frac{y}{x+1} \geq 2-\frac{1}{y+1}-\frac{1}{x+1}.$$

It is now enough to show that

$$2-\frac{1}{y+1}-\frac{1}{x+1} \geq \frac{3}{2}-\frac{1}{x+y}$$

or

$$\frac{1}{2}-\frac{1}{y+1} \geq \frac{1}{x+1}-\frac{1}{x+y}$$

or

$$\frac{y-1}{2(1+y)} \geq \frac{y-1}{(x+1)(x+y)}.$$

However, the last inequality clearly holds for $x \geq y \geq 1$.

Proof 3. *As in the previous proof, we may assume $a \geq b \geq 1 = c$. We present a proof of*

$$\frac{a}{b+1}+\frac{b}{a+1}+\frac{1}{a+b} \geq \frac{3}{2}.$$

Let $A = a+b$ and $B = ab$. What we want to prove is

$$\frac{a^2+b^2+a+b}{(a+1)(b+1)}+\frac{1}{a+b} \geq \frac{3}{2}$$

or

$$\frac{A^2-2B+A}{A+B+1}+\frac{1}{A} \geq \frac{3}{2}$$

or

$$2A^3 - A^2 - A + 2 \geq B(7A - 2).$$

Since $7A - 2 > 2(a + b - 1) > 0$ *and* $A^2 = (a + b)^2 \geq 4ab = 4B$, *it's enough to show that*

$$4(2A^3 - A^2 - A + 2) \geq A^2(7A - 2) \quad \Leftrightarrow \quad A^3 - 2A^2 - 4A + 8 \geq 0.$$

However, it's easy to check that $A^3 - 2A^2 - 4A + 8 = (A - 2)^2(A + 2) \geq 0.$

2.3. **Symmetrizations.** We now attack non-symmetrical inequalities by transforming them into symmetric ones.

Example 6. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

First Solution. We break the homogeneity. After the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, it becomes

$$a^2 + b^2 + c^2 \geq a + b + c.$$

We now obtain

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2 \geq (a+b+c)(abc)^{\frac{1}{3}} = a + b + c.$$

$\square$

Epsilon 20. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

Epsilon 21. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x} \leq 1.$$

Epsilon 22. *Let x, y, z be positive real numbers with $x + y + z = 3$. Show the cyclic inequality*

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

Epsilon 23. [SL 1985 CAN] *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{x^2+yz} + \frac{y^2}{y^2+zx} + \frac{z^2}{z^2+xy} \leq 2.$$

Epsilon 24. [SL 1990 THA] *Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. show that*

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Delta 13. [SL 1998 MNG] *Let $a_1, \cdots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n < 1$. Prove that*

$$\frac{a_1 \cdots a_n (1 - a_1 - \cdots - a_n)}{(a_1 + \cdots + a_n)(1 - a_1) \cdots (1 - a_n)} \leq \frac{1}{n^{n+1}}.$$

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

- P. Halmos, *I Want to be a Mathematician*

3. Geometric Inequalities

Geometry is the science of correct reasoning on incorrect figures.
- G. Pólya

3.1. **Triangle Inequalities.** Many inequalities are simplified by some suitable substitutions. We begin with a classical inequality in triangle geometry. What is the first[4] nontrivial geometric inequality?

Theorem 3.1. (Chapple 1746, Euler 1765) Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $R \geq 2r$ and the equality holds if and only if ABC is equilateral.

Proof. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$ and $S = [ABC]$.[5] We now recall the well-known identities:

$$S = \frac{abc}{4R},\ S = rs,\ S^2 = s(s-a)(s-b)(s-c).$$

Hence, the inequality $R \geq 2r$ is equivalent to

$$\frac{abc}{4S} \geq 2\frac{S}{s}$$

or

$$abc \geq 8\frac{S^2}{s}$$

or

$$abc \geq 8(s-a)(s-b)(s-c).$$

We need to prove the following. ☐

Theorem 3.2. (A. Padoa) Let a, b, c be the lengths of a triangle. Then, we have

$$abc \geq 8(s-a)(s-b)(s-c)$$

or

$$abc \geq (b+c-a)(c+a-b)(a+b-c)$$

Here, the equality holds if and only if $a = b = c$.

Proof. We exploit The Ravi Substitution. Since a, b, c are the lengths of a triangle, there are positive reals x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$. (Why?) Then, the inequality is $(y+z)(z+x)(x+y) \geq 8xyz$ for x, y, $z > 0$. However, we get

$$(y+z)(z+x)(x+y) - 8xyz = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0.$$

☐

Does the above inequality hold for arbitrary positive reals a, b, c? Yes ! It's possible to prove the inequality without the additional condition that a, b, c are the lengths of a triangle :

Theorem 3.3. *Whenever x, y, $z > 0$, we have*

$$xyz \geq (y+z-x)(z+x-y)(x+y-z).$$

Here, the equality holds if and only if $x = y = z$.

[4]The first geometric inequality is the Triangle Inequality: $AB + BC \geq AC$

[5]In this book, $[P]$ stands for the area of the polygon P.

Proof. Since the inequality is symmetric in the variables, without loss of generality, we may assume that $x \geq y \geq z$. Then, we have $x + y > z$ and $z + x > y$. If $y + z > x$, then x, y, z are the lengths of the sides of a triangle. In this case, by the previous theorem, we get the result. Now, we may assume that $y + z \leq x$. Then, it is clear that $xyz > 0 \geq (y + z - x)(z + x - y)(x + y - z)$. □

The above inequality holds when some of x, y, z are zeros:

Theorem 3.4. *Let x, y, $z \geq 0$. Then, we have $xyz \geq (y + z - x)(z + x - y)(x + y - z)$.*

Proof. Since $x, y, z \geq 0$, we can find strictly positive sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ for which

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \lim_{n\to\infty} z_n = z.$$

The above theorem says that

$$x_n y_n z_n \geq (y_n + z_n - x_n)(z_n + x_n - y_n)(x_n + y_n - z_n).$$

Now, taking the limits to both sides, we get the result.

□

We now notice that, when x, y, $z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ does not guarantee that $x = y = z$. In fact, for $x, y, z \geq 0$, the equality $xyz = (y + z - x)(z + x - y)(x + y - z)$ implies that

$$x = y = z \ \textit{ or } \ x = y, z = 0 \ \textit{ or } \ y = z, x = 0 \ \textit{ or } \ z = x, y = 0.$$

(Verify this!) It's straightforward to verify the equality

$$xyz - (y+z-x)(z+x-y)(x+y-z) = x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y).$$

Hence, it is a particular case of Schur's Inequality.

Epsilon 25. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Delta 14. *Let R and r denote the radii of the circumcircle and incircle of the right triangle ABC, resepectively. Show that*

$$R \geq (1 + \sqrt{2})r.$$

When does the equality hold ?

Delta 15. [LL 1988 ESP] *Let ABC be a triangle with inradius r and circumradius R. Show that*

$$\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} \leq \frac{5}{8} + \frac{r}{4R}.$$

In 1965, W. J. Blundon[WJB] found the best possible inequalities of the form

$$\mathcal{A}(R, r) \leq s^2 \leq \mathcal{B}(R, r),$$

where $\mathcal{A}(x, y)$ and $\mathcal{B}(x, y)$ are real quadratic forms $\alpha x^2 + \beta xy + \gamma y^2$.

Delta 16. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that*

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2.$$

Delta 17. [WJB2, RS] Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Let s be the semiperimeter of ABC. Show that

$$s \geq 2R + (3\sqrt{3} - 4)r.$$

Delta 18. *With the usual notation for a triangle, show the inequality*[6]

$$4R + r \geq \sqrt{3}s.$$

The Ravi Substitution is useful for inequalities for the lengths a, b, c of a triangle. After The Ravi Substitution, we can remove the condition that they are the lengths of the sides of a triangle.

Epsilon 26. **[IMO 1983/6 USA]** *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Delta 19. (Darij Grinberg) Let a, b, c be the lengths of a triangle. Show the inequalities

$$a^3 + b^3 + c^3 + 3abc - 2b^2a - 2c^2b - 2a^2c \geq 0,$$

and

$$3a^2b + 3b^2c + 3c^2a - 3abc - 2b^2a - 2c^2b - 2a^2c \geq 0.$$

Delta 20. **[LL 1983 UNK]** *Show that if the sides a, b, c of a triangle satisfy the equation*

$$2\left(ab^2 + bc^2 + ca^2\right) = a^2b + b^2c + c^2a + 3abc$$

then the triangle is equilateral. Show also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

Delta 21. **[IMO 1991/1 USS]** *Prove for each triangle ABC the inequality*

$$\frac{1}{4} < \frac{IA \cdot IB \cdot IC}{l_A \cdot l_B \cdot l_C} \leq \frac{8}{27},$$

where I is the incenter and l_A, l_B, l_C are the lengths of the angle bisectors of ABC.

We now discuss Weitzenböck's Inequality and related theorems.

Epsilon 27. **[IMO 1961/2 POL]** (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 28. (The Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2$$

or

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

Here is a simultaneous generalization of Weitzenböck's Inequality and Nesbitt's Inequality.

Epsilon 29. (Tsintsifas) Let p, q, r be positive real numbers and let a, b, c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Epsilon 30. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Notice that it's a generalization of Weitzenböck's Inequality. Carlitz observed that The Neuberg-Pedoe Inequality can be deduced from Aczél's Inequality.

[6] It is equivalent to The Hadwiger-Finsler Inequality.

Epsilon 31. (Aczél's Inequality) If $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$ satisfies the inequality

$$a_1{}^2 \geq a_2{}^2 + \cdots + a_n{}^2 \text{ and } b_1{}^2 \geq b_2{}^2 + \cdots + b_n{}^2,$$

then the following inequality holds.

$$a_1 b_1 - (a_2 b_2 + \cdots + a_n b_n) \geq \sqrt{(a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2))\left(b_1{}^2 - \left(b_2{}^2 + \cdots + b_n{}^2\right)\right)}$$

3.2. **Conway Substitution.** As we saw earlier, transforming geometric inequalities to algebraic ones (and vice-versa), in order to solve them, may prove to be very useful. Besides the Ravi Substitution, we remind another technique, known to the authors as the Conway Substitution Theorem.

Theorem 3.5. (Conway) Let u, v, w be three reals such that the numbers $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative. Then, there exists a triangle XYZ with sidelengths $x = YZ = \sqrt{v+w}$, $y = ZX = \sqrt{w+u}$, $z = XY = \sqrt{u+v}$. This triangle satisfies $y^2+z^2-x^2 = 2u$, $z^2+x^2-y^2 = 2v$, $x^2+y^2-z^2 = 2w$. The area T of this triangle equals $T = \frac{1}{2}\sqrt{vw+wu+uv}$. If $X = \angle ZXY, Y = \angle XYZ, Z = \angle YZX$ are the angles of this triangle, then $\cot X = \frac{u}{2T}$, $\cot Y = \frac{v}{2T}$ and $\cot Z = \frac{w}{2T}$.

Proof. Since the numbers $v+w$, $w+u$, $u+v$ are nonnegative, their square roots $\sqrt{v+w}$, $\sqrt{w+u}$, $\sqrt{u+v}$ exist, and, of course, are nonnegative as well. A straightforward computation shows that $\sqrt{w+u}+\sqrt{u+v} \geq \sqrt{v+w}$. Similarly, $\sqrt{u+v}+\sqrt{v+w} \geq \sqrt{w+u}$ and $\sqrt{v+w}+\sqrt{w+u} \geq \sqrt{u+v}$. Thus, there exists a triangle XYZ with sidelengths

$$x = YZ = \sqrt{v+w},\ y = ZX = \sqrt{w+u},\ z = XY = \sqrt{u+v}.$$

It follows that

$$y^2+z^2-x^2 = \left(\sqrt{w+u}\right)^2 + \left(\sqrt{u+v}\right)^2 - \left(\sqrt{v+w}\right)^2 = 2u.$$

Similarly, $z^2+x^2-y^2 = 2v$ and $x^2+y^2-z^2 = 2w$. According now to the fact that

$$\cot Z = \frac{x^2+y^2-z^2}{4T},$$

we deduce that so that $\cot Z = \frac{w}{2T}$, and similarly $\cot X = \frac{u}{2T}$ and $\cot Y = \frac{v}{2T}$. The well-known trigonometric identity

$$\cot Y \cdot \cot Z + \cot Z \cdot \cot X + \cot X \cdot \cot Y = 1,$$

now becomes

$$\frac{v}{2T}\cdot\frac{w}{2T}+\frac{w}{2T}\cdot\frac{u}{2T}+\frac{u}{2T}\cdot\frac{v}{2T} = 1$$

or

$$vw+wu+uv = 4T^2.$$

or

$$T = \frac{1}{2}\sqrt{4T^2} = \frac{1}{2}\sqrt{vw+wu+uv}.$$

□

Note that the positive real numbers m, n, p satisfy the above conditions, and therefore, there exists a triangle with sidelengths $m = \sqrt{n+p}$, $n = \sqrt{p+m}$, $p = \sqrt{m+n}$. However, we will further see that there are such cases when we need the version in which the numbers m, n, p are not all necessarily nonnegative.

Delta 22. (Turkey 2006) If x, y, z are positive numbers with $xy+yz+zx = 1$, show that

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 6\sqrt{3}.$$

We continue with an interesting inequality discussed on the MathLinks Forum.

Proposition 3.1. *If x, y, z are three reals such that the numbers $y+z$, $z+x$, $x+y$ and $yz+zx+xy$ are all nonnegative, then*

$$\sum \sqrt{(z+x)(x+y)} \geq x+y+z+\sqrt{3}\cdot\sqrt{yz+zx+xy}.$$

Proof. (Darij Grinberg) Applying the Conway substitution theorem to the reals x, y, z, we see that, since the numbers $y+z$, $z+x$, $x+y$ and $yz+zx+xy$ are all nonnegative, we can conclude that there exists a triangle ABC with sidelengths $a = BC = \sqrt{y+z}$, $b = CA = \sqrt{z+x}$, $c = AB = \sqrt{x+y}$ and area $S = \frac{1}{2}\sqrt{yz+zx+xy}$. Now, we have

$$\sum \sqrt{(z+x)(x+y)} = \sum \sqrt{z+x} \cdot \sqrt{x+y} = \sum b \cdot c = bc + ca + ab,$$

$$x + y + z = \frac{1}{2}\left(\left(\sqrt{y+z}\right)^2 + \left(\sqrt{z+x}\right)^2 + \left(\sqrt{x+y}\right)^2\right) = \frac{1}{2}\left(a^2 + b^2 + c^2\right),$$

and

$$\sqrt{3} \cdot \sqrt{yz+zx+xy} = 2\sqrt{3} \cdot \frac{1}{2}\sqrt{yz+zx+xy} = 2\sqrt{3} \cdot S.$$

Hence, the inequality in question becomes

$$bc + ca + ab \geq \frac{1}{2}\left(a^2 + b^2 + c^2\right) + 2\sqrt{3} \cdot S,$$

which is equivalent with

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot S + (b-c)^2 + (c-a)^2 + (a-b)^2.$$

But this is the well-known refinement of the Weintzenbock Inequality, discovered by Finsler and Hadwiger in 1937. See [FiHa]. □

Five years later, Pedoe [DP2] proved a magnificent generalization of the same Weitzenböck Inequality. In Mitrinovic, Pecaric, and Volenecs' classic *Recent Advances in Geometric Inequalities*, this generalization is referred to as the Neuberg-Pedoe Inequality. See also [DP1], [DP2], [DP3], [DP5] and [JN].

Proposition 3.2. (Neuberg-Pedoe) Let a, b, c, and x, y, z be the side lengths of two given triangles ABC, XYZ with areas S, and T, respectively. Then,

$$a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right) \geq 16ST,$$

with equality if and only if the triangles ABC and XYZ are similar.

Proof. (Darij Grinberg) First note that the inequality is homogeneous in the sidelengths x, y, z of the triangle XYZ (in fact, these sidelengths occur in the power 2 on the left hand side, and on the right hand side they occur in the power 2 as well, since the area of a triangle is quadratically dependant from its sidelengths). Hence, this inequality is invariant under any similitude transformation executed on triangle XYZ. In other words, we can move, reflect, rotate and stretch the triangle XYZ as we wish, but the inequality remains equivalent. But, of course, by applying similitude transformations to triangle XYZ, we can always achieve a situation when $Y = B$ and $Z = C$ and the point X lies in the same half-plane with respect to the line BC as the point A. Hence, in order to prove the Neuberg-Pedoe Inequality for *any* two triangles ABC and XYZ, it is enough to prove it for two triangles ABC and XYZ in this special situation.

So, assume that the triangles ABC and XYZ are in this special situation, i. e. that we have $Y = B$ and $Z = C$ and the point X lies in the same half-plane with respect to the line BC as the point A. We, thus, have to prove the inequality

$$a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right) \geq 16ST.$$

Well, by the cosine law in triangle ABX, we have

$$AX^2 = AB^2 + XB^2 - 2 \cdot AB \cdot XB \cdot \cos\angle ABX.$$

Let's figure out now what this equation means. At first, $AB = c$. Then, since $B = Y$, we have $XB = XY = z$. Finally, we have either $\angle ABX = \angle ABC - \angle XBC$ or $\angle ABX = \angle XBC - \angle ABC$ (depending on the arrangement of the points), but in both cases $\cos\angle ABX = \cos(\angle ABC - \angle XBC)$. Since $B = Y$ and $C = Z$, we can rewrite the angle $\angle XBC$ as $\angle XYZ$. Thus,

$$\cos\angle ABX = \cos(\angle ABC - \angle XYZ) = \cos\angle ABC \cos\angle XYZ + \sin\angle ABC \sin\angle XYZ.$$

By the Cosine Law in triangles ABC and XYZ, we have

$$\cos\angle ABC = \frac{c^2 + a^2 - b^2}{2ca}, \quad \text{and} \quad \cos\angle XYZ = \frac{z^2 + x^2 - y^2}{2zx}.$$

Also, since

$$\sin\angle ABC = \frac{2S}{ca}, \quad \text{and} \quad \sin\angle XYZ = \frac{2T}{zx},$$

we have that

$$\begin{aligned} & \cos\angle ABX \\ = \; & \cos\angle ABC \cos\angle XYZ + \sin\angle ABC \sin\angle XYZ \\ = \; & \frac{c^2 + a^2 - b^2}{2ca} \cdot \frac{z^2 + x^2 - y^2}{2zx} + \frac{2S}{ca} \cdot \frac{2T}{zx}. \end{aligned}$$

This makes the equation

$$AX^2 = AB^2 + XB^2 - 2 \cdot AB \cdot XB \cdot \cos\angle ABX$$

transform into

$$AX^2 = c^2 + z^2 - 2 \cdot c \cdot z \cdot \left(\frac{c^2 + a^2 - b^2}{2ca} \cdot \frac{z^2 + x^2 - y^2}{2zx} + \frac{2S}{ca} \cdot \frac{2T}{zx} \right),$$

which immediately simplifies to

$$AX^2 = c^2 + z^2 - 2\left(\frac{(c^2 + a^2 - b^2)(z^2 + x^2 - y^2)}{4ax} + \frac{4ST}{ax} \right),$$

and since $YZ = BC$,

$$AX^2 = \frac{\left(a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right)\right) - 16ST}{2ax}.$$

Thus, according to the (obvious) fact that $AX^2 \geq 0$, we conclude that

$$a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right) \geq 16ST,$$

which proves the Neuberg-Pedoe Inequality. The equality holds if and only if the points A and X coincide, i. e. if the triangles ABC and XYZ are congruent. Now, of course, since the triangle XYZ we are dealing with is not the initial triangle XYZ, but just its image under a similitude transformation, the general equality condition is that the triangles ABC and XYZ are similar (not necessarily being congruent). $\square$

Delta 23. (Bottema [BK]) Let a, b, c, and x, y, z be the side lengths of two given triangles ABC, XYZ with areas S, and T, respectively. If P is an arbitrary point in the plane of triangle ABC, then we have the inequality

$$x \cdot AP + y \cdot BP + z \cdot CP \geq \sqrt{\frac{a^2\left(y^2 + z^2 - x^2\right) + b^2\left(z^2 + x^2 - y^2\right) + c^2\left(x^2 + y^2 - z^2\right)}{2} + 8ST}.$$

Epsilon 32. *If A, B, C, X, Y, Z denote the magnitudes of the corresponding angles of triangles ABC, and XYZ, respectively, then*

$$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y \geq 2.$$

Epsilon 33. (Vasile Cârtoaje) Let a, b, c, x, y, z be nonnegative reals. Prove the inequality

$$(ay + az + bz + bx + cx + cy)^2 \geq 4\,(bc + ca + ab)\,(yz + zx + xy)\,,$$

with equality if and only if $a : x = b : y = c : z$.

Delta 24. (The Extended Tsintsifas Inequality) *Let p, q, r be positive real numbers such that the terms $q+r$, $r+p$, $p+q$ are all positive, and let a, b, c denote the sides of a triangle with area F. Then, we have*

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Epsilon 34. (Walter Janous, Crux Mathematicorum) If u, v, w, x, y, z are six reals such that the terms $y+z$, $z+x$, $x+y$, $v+w$, $w+u$, $u+v$, and $vw+wu+uv$ are all nonnegative, then

$$\frac{x}{y+z}\cdot(v+w) + \frac{y}{z+x}\cdot(w+u) + \frac{z}{x+y}\cdot(u+v) \geq \sqrt{3\,(vw+wu+uv)}.$$

Note that the Neuberg-Pedoe Inequality is a generalization (actually the better word is *parametrization*) of the Weitzenböck Inequality. How about deducing Hadwiger-Finsler's Inequality from it? Apparently this is not possible. However, the Conway Substitution Theorem will change our mind.

Lemma 3.1. *Let ABC be a triangle with side lengths a, b, c, and area S, and let u, v, w be three reals such that the numbers $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative. Then,*

$$ua^2 + vb^2 + wc^2 \geq 4\sqrt{vw+wu+uv}\cdot S.$$

Proof. According to the Conway Substitution Theorem, we can construct a triangle with sidelenghts $x = \sqrt{v+w}$, $y = \sqrt{w+u}$, $z = \sqrt{u+v}$ and area $T = \sqrt{vw+wu+uv}/2$. Let this triangle be XYZ. In this case, by the Neuberg-Pedoe Inequality, applied for the triangles ABC and XYZ, we get that

$$a^2\left(y^2+z^2-x^2\right) + b^2\left(z^2+x^2-y^2\right) + c^2\left(x^2+y^2-z^2\right) \geq 16ST.$$

By the formulas given in the Conway Substitution Theorem, this becomes equivalent with

$$a^2\cdot 2u + b^2\cdot 2v + c^2\cdot 2w \geq 16S\cdot\frac{1}{2}\sqrt{vw+wu+uv}$$

which simplifies to $ua^2 + vb^2 + wc^2 \geq 4\sqrt{vw+wu+uv}\cdot S$. □

Proposition 3.3. (Cosmin Pohoaţă) Let ABC be a triangle with side lengths a, b, c, and area S and let x, y, z be three positive real numbers. Then,

$$a^2+b^2+c^2 \geq 4\sqrt{3}S + \frac{2}{x+y+z}\left(\frac{x^2-yz}{x}\cdot a^2 + \frac{y^2-zx}{y}\cdot b^2 + \frac{z^2-xy}{z}\cdot c^2\right).$$

Proof. Let $m = xyz(x+y+z) - 2yz(x^2-yz)$, $n = xyz(x+y+z) - 2zx(y^2-zx)$, and $p = xyz(x+y+z) - 2xy(z^2-xy)$. The three terms $n+p$, $p+m$, and $m+n$ are all positive, and since

$$mn + np + pm = 3x^2y^2z^2(x+y+z)^2 \geq 0,$$

by Lemma 3.1, we get that

$$\sum_{cyc}[xyz(x+y+z) - 2yz(x^2-yz)]a^2 \geq 4xyz(x+y+z)\sqrt{3}S.$$

This rewrites as
$$\sum_{cyc}\left[(x+y+z)-2\cdot\frac{x^2-yz}{x}\right]a^2\geq 4(x+y+z)\sqrt{3}S,$$
and, thus,
$$a^2+b^2+c^2\geq 4\sqrt{3}S+\frac{2}{x+y+z}\left(\frac{x^2-yz}{x}\cdot a^2+\frac{y^2-zx}{y}\cdot b^2+\frac{z^2-xy}{z}\cdot c^2\right).$$
$\square$

Obviously, for $x=a$, $y=b$, $z=c$, and following the fact that
$$a^3+b^3+c^3-3abc=\frac{1}{2}\left(a+b+c\right)\left[(a-b)^2+(b-c)^2+(c-a)^2\right],$$
Proposition 3.3 becomes equivalent with the Hadwiger-Finsler Inequality. Note also that for $x=y=z$, Proposition 3.3 turns out to be the Weintzenbock Inequality. Therefore, by using only Conway's Substitution Theorem, we've transformed a result which strictly generalizes the Weintzebock Inequality (the Neuberg-Pedoe Inequality) into one which generalizes both the Weintzenbock Inequality and, surprisingly or not, the Hadwiger-Finsler Inequality.

3.3. **Hadwiger-Finsler Revisited.** The Hadwiger-Finsler inequality is known in literature as a refinement of Weitzenböck's Inequality. Due to its great importance and beautiful aspect, many proofs for this inequality are now known. For example, in [AE] one can find eleven proofs. Is the Hadwiger-Inequality the best we can do? The answer is indeed no. Here, we shall enlighten a few of its sharpening.

We begin with an interesting "phenomenon". Most of you might know that according to the formulas $ab + bc + ca = s^2 + r^2 + 4Rr$, and $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$, the Hadwiger-Finsler Inequality rewrites as

$$4R + r \geq s\sqrt{3},$$

where s is the semiperimeter of the triangle. However, by using this last **equivalent** form in a trickier way, we may obtain a slightly **sharper** result:

Proposition 3.4. (Cezar Lupu, Cosmin Pohoaţă) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. As announced, we start with

$$4R + r \geq s\sqrt{3}.$$

By multiplying with 2 and adding $2r(4R + r)$ to both terms, we obtain that

$$16Rr + 4r^2 \geq 2S\sqrt{3} + 2r(4R + r).$$

According now to the fact that $ab+bc+ca = s^2+r^2+4Rr$, and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, this rewrites as

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 2S\sqrt{3} + 2r(4R + r).$$

Therefore, we obtain

$$a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

□

This might seem strange, but wait until you see how does the geometric version of Schur's Inequality look like (of course, since we expect to run through another refinement of the Hadwiger-Finsler Inequality, we obviously refer to the third degree case of Schur's Inequality).

Proposition 3.5. (Cezar Lupu, Cosmin Pohoaţă [LuPo]) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3 + \frac{4(R - 2r)}{4R + r}} + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. The third degree case of Schur's Inequality says that for any three nonnegative real numbers m, n, p, we have that

$$m^3 + n^3 + p^3 + 3mnp \geq m^2(n + p) + n^2(p + m) + p^2(m + n).$$

Note that this can be rewritten as

$$2(np + pm + mn) - (m^2 + n^2 + p^2) \leq \frac{9mnp}{m + n + p},$$

and by plugging in the substitutions $x = \frac{1}{m}$, $y = \frac{1}{n}$, and $z = \frac{1}{p}$, we obtain that

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} + \frac{9xyz}{yz + zx + xy} \geq 2(x + y + z).$$

So far so good, but let's take this now geometrically. Using the Ravi Substitution (i. e.

$$x = \frac{1}{2}(b+c-a), \quad y = \frac{1}{2}(c+a-b), \quad \text{and } p = \frac{1}{2}(a+b-c),$$

where a, b, c are the sidelengths of triangle ABC), we get that the above inequality rewrites as

$$\sum_{cyc} \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{9(b+c-a)(c+a-b)(a+b-c)}{\sum(b+c-a)(c+a-b)} \geq 2(a+b+c).$$

Since $ab+bc+ca = s^2+r^2+4Rr$ and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, it follows that

$$\sum_{cyc}(b+c-a)(c+a-b) = 4r(4R+r).$$

Thus, according to Heron's area formula that

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

we obtain

$$\sum \frac{(b+c-a)(c+a-b)}{(a+b-c)} + \frac{18sr}{4R+r} \geq 4s.$$

This is now equivalent to

$$\sum_{cyc} \frac{(s-a)(s-b)}{(s-c)} + \frac{9sr}{4R+r} \geq 2s,$$

and so

$$\sum_{cyc}(s-a)^2(s-b)^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2.$$

By the identity

$$\sum_{cyc}(s-a)^2(s-b)^2 = \left(\sum_{cyc}(s-a)(s-b)\right)^2 - 2s^2r^2,$$

we have

$$\left(\sum_{cyc}(s-a)(s-b)\right)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2,$$

and since

$$\sum_{cyc}(s-a)(s-b) = r(4R+r),$$

we deduce that

$$r^2(4R+r)^2 + \frac{9s^2r^3}{4R+r} \geq 4s^2r^2.$$

This finally rewrites as

$$\left(\frac{4R+r}{s}\right)^2 + \frac{9r}{4R+r} \geq 4.$$

According again to $ab+bc+ca = s^2+r^2+4Rr$ and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$, we have

$$\left(\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\right)^2 \geq 4 - \frac{9r}{4R+r}.$$

Therefore,

$$a^2+b^2+c^2 \geq 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2.$$

$\square$

Epsilon 35. (Tran Quang Hung) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum \cos^2\frac{A}{2}-\sum \cos\frac{B}{2}\cos\frac{C}{2}\right).$$

Delta 25. *Let a, b, c be the lengths of a triangle with area S.*
(a) (Cosmin Pohoață) Prove that

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+\frac{1}{2}\left(|a-b|+|b-c|+|c-a|\right)^2.$$

(b) Show that, for all positive integers n,

$$a^{2n}+b^{2n}+c^{2n} \geq 3\left(\frac{4}{\sqrt{3}}\right)^n+(a-b)^{2n}+(b-c)^{2n}+(c-a)^{2n}.$$

3.4. **Trigonometry Rocks!** Trigonometry is an extremely powerful tool in geometry. We begin with Fagnano's theorem that among all inscribed triangles in a given acute-angled triangle, the feet of its altitudes are the vertices of the one with the least perimeter. Despite of its apparent simplicity, the problem proved itself really challenging and attractive to many mathematicians of the twentieth century. Several proofs are presented at [Fag].

Theorem 3.6. (Fagnano's Theorem) Let ABC be any triangle, with sidelengths a, b, c, and area S. If XYZ is inscribed in ABC, then

$$XY + YZ + ZX \geq \frac{8S^2}{abc}.$$

Equality holds if and only if ABC is acute-angled, and then only if XYZ is its orthic triangle.

Proof. (Finbarr Holland [FH]) Let XYZ be a triangle inscribed in ABC. Let $x = BX$, $y = CY$, and $z = AZ$. Then $0 < x < a$, $0 < y < b$, $0 < z < c$. By applying the Cosine Law in the triangle ZBX, we have

$$\begin{aligned} ZX^2 &= (c-z)^2 + x^2 - 2x(c-z)\cos B \\ &= (c-z)^2 + x^2 + 2xc - z)\cos(A+C) \\ &= (x\cos A + (c-z)\cos C)^2 + (x\sin A - (c-z)\sin C)^2. \end{aligned}$$

Hence, we have

$$ZX \geq |x\cos A + (c-z)\cos C|,$$

with equality if and only if $x\sin A = (c-z)\sin C$ or $ax + cz = c^2$, Similarly, we obtain

$$XY \geq |y\cos B + (a-x)\cos A|,$$

with equality if and only if $ax + by = a^2$. And

$$YZ \geq |z\cos C + (b-y)\cos B|,$$

with equality if and only if $by + cz = b^2$. Thus, we get

$$\begin{aligned} & XY + YZ + ZX \\ \geq\ & |y\cos B + (a-x)\cos A| + |z\cos C + (b-y)\cos B| + |x\cos A + (c-z)\cos C| \\ \geq\ & |y\cos B + (a-x)\cos A + z\cos C + (b-y)\cos B + x\cos A + (c-z)\cos C| \\ \geq\ & |a\cos A + b\cos B + c\cos C| \\ =\ & \frac{a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)}{2abc} \\ =\ & \frac{8S^2}{abc}. \end{aligned}$$

Note that we have equality here if and only if

$$ax + cz = c^2, \quad ax + by = a^2, \quad \text{and } by + cz = b^2,$$

and moreover the expressions

$$u = x\cos A + (c-z)\cos C, \quad v = y\cos B + (a-x)\cos A, \quad w = z\cos C + (b-y)\cos B,$$

are either all negative or all nonnegative. Now it is easy to very that the system of equations

$$ax + cz = c^2, \quad ax + by = a^2, \quad by + cz = b^2$$

has an unique solution given by

$$x = c\cos B, \quad y = a\cos C, \quad \text{and } z = b\cos A,$$

in which case

$$u = b\cos C, \quad v = c\cos C, \quad \text{and } w = a\cos A.$$

Thus, in this case, at most one of u, v, w can be negative. But, if one of u, v, w is zero, then one of x, y, z must be zero, which is not possible. It follows that

$$XY + YZ + ZX > \frac{8S^2}{abc},$$

unless ABC is acute-angled, and XYZ is its orthic triangle. If ABC is acute-angled, then $\frac{8S^2}{abc}$ is the perimeter of its orthic triangle, in which case we recover Fagnano's theorem. □

We continue with Morley's miracle. We first prepare two well-known trigonometric identities.

Epsilon 36. *For all $\theta \in \mathbb{R}$, we have*

$$\sin(3\theta) = 4\sin\theta \sin\left(\frac{\pi}{3} + \theta\right) \sin\left(\frac{2\pi}{3} + \theta\right).$$

Epsilon 37. *For all $A, B, C \in \mathbb{R}$ with $A + B + C = 2\pi$, we have*

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

Theorem 3.7. (Morley's Theorem) The three points of intersections of the adjacent internal angle trisectors of a triangle forms an equilateral triangle.

Proof. We want to show that the triangle $E_1E_2E_3$ is equilateral.

Let $\mathcal{R}$ denote the circumradius of $A_1A_2A_3$. Setting $\angle A_i = 3\theta_i$ for $i = 1, 2, 3$, we get $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{3}$. We now apply The Sine Law twice to deduce

$$A_1E_3 = \frac{\sin\theta_2}{\sin(\pi - \theta_1 - \theta_2)} A_1A_2 = \frac{\sin\theta_2}{\sin\left(\frac{2\pi}{3} + \theta_3\right)} \cdot 2\mathcal{R}\sin(3\theta_3) = 8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3} + \theta_3\right).$$

By symmetry, we also have

$$A_1E_2 = 8\mathcal{R}\sin\theta_3\sin\theta_2\sin\left(\frac{\pi}{3} + \theta_2\right).$$

Now, we present two different ways to complete the proof. The first method is more direct and the second one gives more information.

First Method. One of the most natural approaches to crack this is to compute the lengths of $E_1E_2E_3$. We apply The Cosine Law to obtain

$$\begin{aligned} & {E_1E_2}^2 \\ = \;& {AE_3}^2 + {AE_2}^2 - 2\cos(\angle E_3A_1E_2) \cdot AE_3 \cdot AE_1 \\ = \;& 64\mathcal{R}^2 \sin^2\theta_2 \sin^2\theta_3 \left[\sin^2\left(\frac{\pi}{3} + \theta_3\right) + \sin^2\left(\frac{\pi}{3} + \theta_2\right) - 2\cos\theta_1 \sin\left(\frac{\pi}{3} + \theta_3\right)\sin\left(\frac{\pi}{3} + \theta_2\right)\right] \end{aligned}$$

To avoid long computation, here, we employ a trick. In the view of the equality

$$(\pi - \theta_1) + \left(\frac{\pi}{6} - \theta_2\right) + \left(\frac{\pi}{6} - \theta_3\right) = \pi,$$

we have

$$\cos^2(\pi - \theta_1) + \cos^2\left(\frac{\pi}{6} - \theta_2\right) + \cos^2\left(\frac{\pi}{6} - \theta_3\right) + 2\cos(\pi - \theta_1)\cos\left(\frac{\pi}{6} - \theta_2\right)\cos\left(\frac{\pi}{6} - \theta_3\right) = 1$$

or

$$\cos^2\theta_1 + \sin^2\left(\frac{\pi}{3} + \theta_2\right) + \sin^2\left(\frac{\pi}{3} + \theta_3\right) - 2\cos\theta_1\left(\frac{\pi}{3} + \theta_2\right)\cos\left(\frac{\pi}{3} + \theta_3\right) = 1$$

or

$$\sin^2\left(\frac{\pi}{3} + \theta_2\right) + \sin^2\left(\frac{\pi}{3} + \theta_3\right) - 2\cos\theta_1\sin\left(\frac{\pi}{3} + \theta_2\right)\sin\left(\frac{\pi}{3} + \theta_3\right) = \sin^2\theta_1.$$

We therefore find that

$${E_1E_2}^2 = 64\mathcal{R}^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3$$

so that

$$E_1E_2 = 8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3$$

Remarkably, the length of E_1E_2 is symmetric in the angles! By symmetry, we therefore conclude that $E_1E_2E_3$ is an equilateral triangle with the length $8R\sin\theta_1\sin\theta_2\sin\theta_3$.

Second Method. We find the angles in the picture explicitly. Look at the triangle $E_3A_1E_2$. The equality

$$\theta_1 + \left(\frac{\pi}{3} + \theta_2\right) + \left(\frac{\pi}{3} + \theta_3\right) = \pi$$

allows us to invite a ghost triangle ABC having the angles

$$A = \theta_1,\ B = \frac{\pi}{3} + \theta_2,\ C = \frac{\pi}{3} + \theta_3.$$

Observe that two triangles BAC and $E_3A_1E_2$ are similar. Indeed, we have $\angle BAC = \angle E_3A_1E_2$ and

$$\frac{A_1E_3}{A_1E_2} = \frac{8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3}+\theta_3\right)}{8\mathcal{R}\sin\theta_3\sin\theta_2\sin\left(\frac{\pi}{3}+\theta_2\right)} = \frac{\sin\left(\frac{\pi}{3}+\theta_3\right)}{\sin\left(\frac{\pi}{3}+\theta_2\right)} = \frac{\sin C}{\sin B} = \frac{AB}{AC}.$$

It therefore follows that

$$(\angle A_1E_3E_2,\ \angle A_1E_2E_3) = \left(\frac{\pi}{3} + \theta_2,\ \frac{\pi}{3} + \theta_3\right).$$

Similarly, we also have

$$(\angle A_2E_1E_3,\ \angle A_2E_3E_1) = \left(\frac{\pi}{3} + \theta_3,\ \frac{\pi}{3} + \theta_1\right).$$

and

$$(\angle A_3E_2E_1,\ \angle A_3E_1E_2) = \left(\frac{\pi}{3} + \theta_1,\ \frac{\pi}{3} + \theta_2\right).$$

An angle computation yields

$$\begin{aligned}
\angle E_1E_2E_3 &= 2\pi - (\angle A_1E_2E_3 + \angle E_1E_2A_3 + \angle A_3E_2A_1) \\
&= 2\pi - \Big[\left(\frac{\pi}{3} + \theta_3\right) + \left(\frac{\pi}{3} + \theta_1\right) + (\pi - \theta_3 - \theta_1)\Big] \\
&= \frac{\pi}{3}.
\end{aligned}$$

Similarly, we also have $\angle E_2E_3E_1 = \frac{\pi}{3} = \angle E_3E_1E_2$. It follows that $E_1E_2E_3$ is equilateral. Furthermore, we apply The Sine Law to reach

$$\begin{aligned}
E_2E_3 &= \frac{\sin\theta_1}{\sin\left(\frac{\pi}{3}+\theta_3\right)}\ A_1E_3 \\
&= \frac{\sin\theta_1}{\sin\left(\frac{\pi}{3}+\theta_3\right)} \cdot 8\mathcal{R}\sin\theta_2\sin\theta_3\sin\left(\frac{\pi}{3}+\theta_3\right) \\
&= 8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3.
\end{aligned}$$

Hence, we find that the triangle $E_1E_2E_3$ has the length $8\mathcal{R}\sin\theta_1\sin\theta_2\sin\theta_3$. □

We pass now to another 'miracle': the Steiner-Lehmus theorem.

Theorem 3.8. (The Steiner-Lehmus Theorem) If the internal angle-bisectors of two angles of a triangle are congruent, then the triangle is isosceles.

Proof. [MH] Let BB' and CC' be the respective internal angle bisectors of angles B and C in triangle ABC, and let a, b, and c denote the sidelengths of the triangle. We set

$$\angle B = 2\beta,\quad \angle C = 2\gamma,\ u = AB',\ U = B'C,\ v = AC',\ V = C'B.$$

We shall see that the assumptions $BB' = CC'$ and $C > B$ (and hence $c > b$) lead to the contradiction that

$$\frac{b}{u} < \frac{c}{v} \ \text{ and } \ \frac{b}{u} \geq \frac{c}{v}.$$

Geometrically, this means that the line $B'C'$ intersects both rays BC and CB. On the one hand, we have

$$\frac{b}{u} - \frac{c}{v} = \frac{u+U}{u} - \frac{v+V}{v} = \frac{U}{u} - \frac{V}{v} = \frac{a}{c} - \frac{a}{b} < 0$$

or

$$\frac{b}{u} < \frac{c}{v}.$$

On the other hand, we use the identity $\sin 2\omega = 2\sin\omega\cos\omega$ to obtain

$$\begin{aligned}
\frac{b}{c}\cdot\frac{v}{u} &= \frac{\sin B}{\sin C}\cdot\frac{v}{u} \\
&= \frac{2\cos\beta\sin\beta}{2\cos\gamma\sin\gamma}\cdot\frac{v}{u} \\
&= \frac{\cos\beta}{\cos\gamma}\cdot\frac{\sin\beta}{u}\cdot\frac{v}{\sin\gamma} \\
&= \frac{\cos\beta}{\cos\gamma}\cdot\frac{\sin A}{BB'}\cdot\frac{CC'}{\sin A} \\
&= \frac{\cos\beta}{\cos\gamma}.
\end{aligned}$$

It thus follows that $\frac{b}{u} > \frac{c}{v}$. We meet a contradiction. □

The next inequality is probably the most beautiful 'modern' geometric inequality in triangle geometry.

Theorem 3.9. (The Erdős-Mordell Theorem) If from a point P inside a given triangle ABC perpendiculars PH_1, PH_2, PH_3 are drawn to its sides, then

$$PA + PB + PC \geq 2(PH_1 + PH_2 + PH_3).$$

This was conjectured by Paul Erdős in 1935, and first proved by Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's Theorem by André Avez, angular computations with similar triangles by Leon Bankoff, area inequality by V. Komornik, or using trigonometry by Mordell and Barrow.

Proof. [MB] We transform it to a trigonometric inequality. Let $h_1 = PH_1$, $h_2 = PH_2$ and $h_3 = PH_3$.

Apply the Since Law and the Cosine Law to obtain

$$\begin{aligned}
PA\sin A = H_2H_3 &= \sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)}, \\
PB\sin B = H_3H_1 &= \sqrt{{h_3}^2 + {h_1}^2 - 2h_3h_1\cos(\pi - B)}, \\
PC\sin C = H_1H_2 &= \sqrt{{h_1}^2 + {h_2}^2 - 2h_1h_2\cos(\pi - C)}.
\end{aligned}$$

So, we need to prove that

$$\sum_{\text{cyclic}} \frac{1}{\sin A}\sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)} \geq 2(h_1 + h_2 + h_3).$$

The main trouble is that the left hand side has too heavy terms with square root expressions. Our strategy is to find a lower bound without square roots. To this end, we express the terms inside the square root as the sum of two squares.

$$\begin{aligned}
{H_2H_3}^2 &= {h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A) \\
&= {h_2}^2 + {h_3}^2 - 2h_2h_3\cos(B + C) \\
&= {h_2}^2 + {h_3}^2 - 2h_2h_3(\cos B\cos C - \sin B\sin C).
\end{aligned}$$

Using $\cos^2 B + \sin^2 B = 1$ and $\cos^2 C + \sin^2 C = 1$, one finds that

$$H_2H_3{}^2 = (h_2 \sin C + h_3 \sin B)^2 + (h_2 \cos C - h_3 \cos B)^2 .$$

Since $(h_2 \cos C - h_3 \cos B)^2$ is clearly nonnegative, we get $H_2H_3 \geq h_2 \sin C + h_3 \sin B$. Hence,

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{\sqrt{h_2{}^2 + h_3{}^2 - 2h_2h_3 \cos(\pi - A)}}{\sin A} &\geq \sum_{\text{cyclic}} \frac{h_2 \sin C + h_3 \sin B}{\sin A} \\
&= \sum_{\text{cyclic}} \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \right) h_1 \\
&\geq \sum_{\text{cyclic}} 2\sqrt{\frac{\sin B}{\sin C} \cdot \frac{\sin C}{\sin B}} h_1 \\
&= 2h_1 + 2h_2 + 2h_3.
\end{aligned}$$

□

Epsilon 38. [SL 2005 KOR] *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Epsilon 39. [IMO 2001/1 KOR] *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.*

Epsilon 40. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 41. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

We close this subsection with Barrows' Inequality stronger than The Erdös-Mordell Theorem. We need the following trigonometric inequality:

Proposition 3.6. (Wolstenholme's Inequality) Let $x, y, z, \theta_1, \theta_2, \theta_3$ be real numbers with $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds:

$$x^2 + y^2 + z^2 \geq 2(yz \cos\theta_1 + zx \cos\theta_2 + xy \cos\theta_3).$$

Proof. Using $\theta_3 = \pi - (\theta_1 + \theta_2)$, we have the identity

$$\begin{aligned}
& x^2 + y^2 + z^2 - 2(yz \cos\theta_1 + zx \cos\theta_2 + xy \cos\theta_3) \\
= \quad & [\ z - (x \cos\theta_2 + y \cos\theta_1)\ ]^2 + [\ x \sin\theta_2 - y \sin\theta_1\ ]^2 .
\end{aligned}$$

□

Corollary 3.1. *Let p, q, and r be positive real numbers. Let θ_1, θ_2, and θ_3 be real numbers satisfying $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds.*

$$p \cos\theta_1 + q \cos\theta_2 + r \cos\theta_3 \leq \frac{1}{2}\left(\frac{qr}{p} + \frac{rp}{q} + \frac{pq}{r} \right).$$

Proof. Take $(x, y, z) = \left(\sqrt{\frac{qr}{p}}, \sqrt{\frac{rp}{q}}, \sqrt{\frac{pq}{r}} \right)$ and apply the above proposition. □

Delta 26. (Cosmin Pohoață) Let a, b, c be the sidelengths of a given triangle ABC with circumradius R, and let x, y, z be three arbitrary real numbers. Then, we have that

$$R\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)\geq\sqrt{xa^2+yb^2+zc^2}.$$

Epsilon 42. (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA+PB+PC\geq 2(PU+PV+PW).$$

Epsilon 43. [AK] Let $x_1,\cdots,x_4$ be positive real numbers. Let $\theta_1,\cdots,\theta_4$ be real numbers such that $\theta_1+\cdots+\theta_4=\pi$. Then, we have

$$x_1\cos\theta_1+x_2\cos\theta_2+x_3\cos\theta_3+x_4\cos\theta_4\leq\sqrt{\frac{(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)}{x_1x_2x_3x_4}}.$$

Delta 27. [RS] Let $R,r,s>0$. Show that a necessary and sufficient condition for the existence of a triangle with circumradius R, inradius r, and semiperimeter s is

$$s^4-2(2R^2+10Rr-r^2)s^2+r(4R+r)^2\leq 0.$$

3.5. **Erdős, Brocard, and Weitzenböck.** In this section, we touch Brocard geometry. We begin with a consequence of The Erdős-Mordell Theorem.

Epsilon 44. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.*

As an immediate consequence, one may consider the following symmetric situation:

Proposition 3.7. *Let ABC be a triangle. If there exists an interior point P in ABC satisfying that*

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

for some positive real number ω. Then, we have the inequality $\omega \le \frac{\pi}{6}$.

We omit the geometrical proof of the existence and the uniqueness of such point in an arbitrary triangle.(Prove it!)

Delta 28. *Let ABC be a triangle. There exists a unique interior point Ω_1, which bear the name the first Brocard point of ABC, such that*

$$\angle \Omega_1 AB = \angle \Omega_1 BC = \angle \Omega_1 CA = \omega_1$$

for some ω_1, the first Brocard angle.

By symmetry, we also include

Delta 29. *Let ABC be a triangle. There exists a unique interior point Ω_2 with*

$$\angle \Omega_2 BA = \angle \Omega_2 CB = \angle \Omega_2 AC = \omega_2$$

for some ω_2, the second Brocard angle. The point Ω_2 is called the second Brocard point of ABC.

Delta 30. *If a triangle ABC has an interior point P such that $\angle PAB = \angle PBC = \angle PCA = 30^\circ$, then it is equilateral.*

Epsilon 45. *Any triangle has the same Brocard angles.*

As a historical remark, we state that H. Brocard (1845-1922) was not the first one who discovered the Brocard points. They were also known to A. Crelle (1780-1855), C. Jacobi (1804-1851), and others some 60 years earlier. However, their results in this area were soon forgotten [RH]. Our next job is to evaluate the Brocard angle quite explicitly.

Epsilon 46. *The Brocard angle ω of the triangle ABC satisfies*

$$\cot \omega = \cot A + \cot B + \cot C.$$

Proposition 3.8. *The Brocard angle ω of the triangle with sides a, b, c and area S satisfies*

$$\cot \omega = \frac{a^2 + b^2 + c^2}{4S}.$$

Proof. We have

$$\begin{aligned} \cot A + \cot B + \cot C &= \frac{2bc \cos A}{2bc \sin A} + \frac{2ca \cos B}{2ca \sin B} + \frac{2ab \cos C}{2ab \sin C} \\ &= \frac{b^2 + c^2 - a^2}{4S} + \frac{c^2 + a^2 - b^2}{4S} + \frac{a^2 + b^2 - c^2}{4S} \\ &= \frac{a^2 + b^2 + c^2}{4S}. \end{aligned}$$

□

We revisit Weitzenböck's Inequality. It is a corollary of The Erdős-Mordell Theorem!

Proposition 3.9. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Third Proof. Letting ω denote its Brocard angle, by combining results we proved, we obtain

$$\frac{a^2 + b^2 + c^2}{4S} = \cot\omega \geq \cot\left(\frac{\pi}{6}\right) = \sqrt{3}.$$

□

We present interesting theorems from Brocard geometry.

Delta 31. [RH] Let Ω_1 and Ω_2 denote the Brocard points of a triangle ABC with the circumcenter O. Let the circumcircle of $O\Omega_1\Omega_2$, called the Brocard circle of ABC, meet the line $A\Omega_1$, $B\Omega_1$, $C\Omega_1$ at R, P, Q, respectively, again. The triangle PQR bears the name the first Brocard triangle of ABC.

(a) $O\Omega_1 = O\Omega_2$.

(b) Two triangles PQR and ABC are similar.

(c) Two triangles PQR and ABC have the same centroid.

(d) Let U, V, W denote the midpoints of QR, RP, PQ, respectively. Let U_H, V_H, W_H denote the feet of the perpendiculars from U, V, W respectively. Then, the three lines UU_H, VU_H, WW_H meet at the nine point circle of triangle ABC.

The story is not over. We establish an inequality which implies the problem **[IMO 1991/5 FRA]**.

Epsilon 47. (The Trigonometric Versions of Ceva's Theorem) For an interior point P of a triangle $A_1A_2A_3$, we write

$$\angle A_3A_1A_2 = \alpha_1,\ \angle PA_1A_2 = \vartheta_1,\ \angle PA_1A_3 = \theta_1,$$
$$\angle A_1A_2A_3 = \alpha_2,\ \angle PA_2A_3 = \vartheta_2,\ \angle PA_2A_1 = \theta_2,$$
$$\angle A_2A_3A_1 = \alpha_3,\ \angle PA_3A_1 = \vartheta_3,\ \angle PA_3A_2 = \theta_3.$$

Then, we find a hidden symmetry:

$$\frac{\sin\vartheta_1}{\sin\theta_1} \cdot \frac{\sin\vartheta_2}{\sin\theta_2} \cdot \frac{\sin\vartheta_3}{\sin\theta_3} = 1,$$

or equivalently,

$$\frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} = [\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3].$$

Epsilon 48. *Let P be an interior point of a triangle ABC. Show that*

$$\cot(\angle PAB) + \cot(\angle PBC) + \cot(\angle PCA) \geq 3\sqrt{3}.$$

Proposition 3.10. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.*

Second Solution. The above inequality implies

$$max\{\cot(\angle PAB),\ \cot(\angle PBC),\ \cot(\angle PCA)\} \geq \sqrt{3} = \cot 30°.$$

Since the cotangent function is strictly decreasing on $(0, \pi)$, we get the result. □

3.6. **From Incenter to Centroid.** We begin with an inequality regarding the incenter. In fact, the geometric inequality is equivalent to an algebraic one, Schur's Inequality!

Example 7. (Korea 1998) Let I be the incenter of a triangle ABC. Prove that

$$IA^2 + IB^2 + IC^2 \geq \frac{BC^2 + CA^2 + AB^2}{3}.$$

Proof. Let $BC = a$, $CA = b$, $AB = c$, and $s = \frac{a+b+c}{2}$. Letting r denote the inradius of $\triangle ABC$, we have

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

By The Pythagoras Theorem, the inequality is equivalent to

$$(s-a)^2 + r^2 + (s-b)^2 + r^2 + (s-c)^2 + r^2 \geq \frac{1}{3}\left(a^2 + b^2 + c^2\right).$$

or

$$(s-a)^2 + (s-b)^2 + (s-c)^2 + \frac{3(s-a)(s-b)(s-c)}{s} \geq \frac{1}{3}\left(a^2 + b^2 + c^2\right).$$

After The Ravi Substitution $x = s-a$, $y = s-b$, $z = s-c$, it becomes

$$x^2 + y^2 + z^2 + \frac{3xyz}{x+y+z} \geq \frac{(x+y)^2 + (y+z)^2 + (z+x)^2}{3}$$

or

$$3\left(x^2 + y^2 + z^2\right)(x+y+z) + 9xyz \geq (x+y+z)\left((x+y)^2 + (y+z)^2 + (z+x)^2\right)$$

or

$$9xyz \geq (x+y+z)\left(2xy + 2yz + 2zx - x^2 - y^2 - z^2\right)$$

or

$$9xyz \geq x^2y + x^2z + y^2z + y^2x + z^2x + z^2y + 6xyz - x^3 - y^3 - z^3$$

or

$$x^3 + y^3 + z^3 + 3xyz \geq x^2(y+z) + y^2(z+x) + z^2(x+y).$$

This is a particular case of Schur's Inequality. ☐

Now, one may ask more questions. Can we replace the incenter by other classical points in triangle geometry? The answer is yes. We first take the centroid.

Example 8. *Let G denote the centroid of the triangle ABC. Then, we have the geometric identity*

$$GA^2 + GB^2 + GC^2 = \frac{BC^2 + CA^2 + AB^2}{3}.$$

Proof. Let M denote the midpoint of BC. The Pappus Theorem implies that

$$\frac{GB^2 + GC^2}{2} = GM^2 + \left(\frac{BC}{2}\right)^2 = \left(\frac{GA}{2}\right)^2 + \left(\frac{BC}{2}\right)^2$$

or

$$-GA^2 + 2GB^2 + 2GC^2 = BC^2$$

Similarly, $2GA^2 - GB^2 + 2GC^2 = CA^2$ and $2GA^2 + 2GB^2 - 2GC^2 = AB^2$. Adding these three equalities, we get the identity. ☐

Before we take other classical points, we need to rethink this unexpected situation. We have an equality, instead of an inequality. According to this equality, we find that the previous inequality can be rewritten as

$$IA^2 + IB^2 + IC^2 \geq GA^2 + GB^2 + GC^2.$$

Now, it is quite reasonable to make a conjecture which states that, given a triangle ABC, the minimum value of $PA^2 + PB^2 + PC^2$ is attained when P is the centroid of $\triangle ABC$. This guess is true!

Theorem 3.10. *Let $A_1A_2A_3$ be a triangle with the centroid G. For any point P, we have*

$$PA_1{}^2 + PA_2{}^2 + PA_3{}^2 \geq GA^2 + GB^2 + GC^2.$$

Proof. Just toss the picture on the real plane $\mathbb{R}^2$ so that

$$P(p,q),\ \ A_1\left(x_1,y_1\right),\ \ A_2\left(x_2,y_2\right),\ \ A_3\left(x_3,y_3\right),\ \ G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right).$$

What we need to do is to compute

$$\begin{aligned}
& 3\left(PA_1{}^2 + PA_2{}^2 + PA_3{}^2\right) - \left(BC^2+CA^2+AB^2\right)\\
= \quad & 3\sum_{i=1}^{3}\left(p-x_i\right)^2 + \left(q-y_i\right)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3}-x_i\right)^2 + \left(\frac{y_1+y_2+y_3}{3}-y_i\right)^2\\
= \quad & 3\sum_{i=1}^{3}\left(p-x_i\right)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3}-x_i\right)^2 + 3\sum_{i=1}^{3}\left(q-y_i\right)^2 - \sum_{i=1}^{3}\left(\frac{y_1+y_2+y_3}{3}-y_i\right)^2.
\end{aligned}$$

A moment's thought shows that the quadratic polynomials are squares.

$$3\sum_{i=1}^{3}\left(p-x_i\right)^2 - \sum_{i=1}^{3}\left(\frac{x_1+x_2+x_3}{3}-x_i\right)^2 = 9\left(p-\frac{x_1+x_2+x_3}{3}\right)^2,$$

$$3\sum_{i=1}^{3}\left(q-y_i\right)^2 - \sum_{i=1}^{3}\left(\frac{y_1+y_2+y_3}{3}-y_i\right)^2 = 9\left(q-\frac{y_1+y_2+y_3}{3}\right)^2.$$

Hence, the quantity $3\left(PA_1{}^2 + PA_2{}^2 + PA_3{}^2\right) - \left(BC^2+CA^2+AB^2\right)$ is clearly non-negative. Furthermore, we notice that the above proof of the geometric inequality **discovers** a geometric identity:

$$\left(PA^2+PB^2+PC^2\right) - \left(GA^2+GB^2+GC^2\right) = 9GP^2.$$

It is clear that the equality in the above inequality holds only when $GP = 0$ or $P = G$. □

After removing the special condition that P is the incenter, we get a more general inequality, even without using a heavy machine, like Schur's Inequality. Sometimes, generalizations are more easy! Taking the point P as the circumcenter, we have

Proposition 3.11. *Let ABC be a triangle with circumradius R. Then, we have*

$$AB^2+BC^2+CA^2 \leq 9R^2.$$

Proof. Let O and G denote its circumcenter and centroid, respectively. It reads

$$9GO^2 + \left(AB^2+BC^2+CA^2\right) = 3\left(OA^2+OB^2+OC^2\right) = 9R^2.$$

□

The readers can rediscover many geometric inequalities by taking other classical points from triangle geometry.(Do it!) Here goes another inequality regarding the incenter.

Example 9. *Let I be the incenter of the triangle ABC with $BC = a$, $CA = b$ and $AB = c$. Prove that, for all points X,*

$$aXA^2 + bXB^2 + cXC^2 \geq abc.$$

First Solution. It turns out that the non-negative quantity

$$aXA^2 + bXB^2 + cXC^2 - abc$$

has a geometric meaning. This geometric inequality follows from the following geometric identity:

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + abc.$$ [7]

[7] **[SL 1988 SGP]**

There are many ways to establish this identity. To euler[8] it, we toss the picture on the real plane $\mathbb{R}^2$ with the coordinates

$$A(c\cos B, c\sin B),\quad B(0,0),\quad C(a,0).$$

Let r denote the inradius of $\triangle ABC$. Setting $s = \frac{a+b+c}{2}$, we get $I(s-b, r)$. It is well-known that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

Set $X(p,q)$. On the one hand, we obtain

$$\begin{aligned}
& aXA^2 + bXB^2 + cXC^2 \\
= \;& a\left[(p - c\cos B)^2 + (q - c\sin B)^2\right] + b\left(p^2 + q^2\right) + c\left[(p-a)^2 + q^2\right] \\
= \;& (a+b+c)p^2 - 2acp(1+\cos B) + (a+b+c)q^2 - 2acq\sin B + ac^2 + a^2c \\
= \;& 2sp^2 - 2acp\left(1 + \frac{a^2+c^2-b^2}{2ac}\right) + 2sq^2 - 2acq\frac{[\triangle ABC]}{\frac{1}{2}ac} + ac^2 + a^2c \\
= \;& 2sp^2 - p(a+c+b)\,(a+c-b) + 2sq^2 - 4q[\triangle ABC] + ac^2 + a^2c \\
= \;& 2sp^2 - p(2s)\,(2s-2b) + 2sq^2 - 4qsr + ac^2 + a^2c \\
= \;& 2sp^2 - 4s\,(s-b)\,p + 2sq^2 - 4rsq + ac^2 + a^2c.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
& (a+b+c)XI^2 + abc \\
= \;& 2s\left[(p-(s-b))^2 + (q-r)^2\right] \\
= \;& 2s\left[p^2 - 2(s-b)p + (s-b)^2 + q^2 - 2qr + r^2\right] \\
= \;& 2sp^2 - 4s\,(s-b)\,p + 2s(s-b)^2 + 2sq^2 - 4rsq + 2sr^2 + abc.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
& aXA^2 + bXB^2 + cXC^2 - (a+b+c)XI^2 - abc \\
= \;& ac^2 + a^2c - 2s(s-b)^2 - 2sr^2 - abc \\
= \;& ac(a+c) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) - abc \\
= \;& ac(a+c-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
= \;& 2ac(s-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
= \;& 2(s-b)\left[ac - s(s-b) - 2(s-a)(s-c)\right].
\end{aligned}$$

However, we compute $ac - s(s-b) - 2(s-a)(s-c) = -2s^2 + (a+b+c)s = 0$. □

Now, throw out the special condition that I is the incenter! Then, the essence appears:

Delta 32. (The Leibniz Theorem) Let $\omega_1, \omega_2, \omega_3$ be real numbers such that $\omega_1+\omega_2+\omega_3 \neq 0$. We characterize the generalized centroid $G_\omega = G_{(\omega_1,\omega_2,\omega_3)}$ by

$$\overrightarrow{XG_\omega} = \sum_{i=1}^{3} \frac{\omega_i}{\omega_1+\omega_2+\omega_3}\overrightarrow{XA_i}.$$

Then G_ω is well-defined in the sense that it doesn't depend on the choice of X. For all points P, we have

$$\sum_{i=1}^{3} \omega_i PA_i{}^2 = (\omega_1+\omega_2+\omega_3)PG_\omega{}^2 + \sum_{i=1}^{3} \frac{\omega_i\omega_{i+1}}{\omega_1+\omega_2+\omega_3}A_iA_{i+1}{}^2.$$

We show that the geometric identity $aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + abc$ is a straightforward consequence of The Leibniz Theorem.

[8] euler v. (in Mathematics) transform the geometric identity in triangle geometry to trigonometric or algebraic identity.

Second Solution. Let $BC = a, CA = b, AB = c$. With the weights (a, b, c), we have $I = G_{(a,b,c)}$. Hence,

$$\begin{aligned} aXA^2 + bXB^2 + cXC^2 &= (a+b+c)XI^2 + \frac{bc}{a+b+c}a^2 + + \frac{ca}{a+b+c}b^2 + \frac{ab}{a+b+c}c^2 \\ &= (a+b+c)XI^2 + abc. \end{aligned}$$

□

Epsilon 49. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 50. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Delta 33. [SL 1988 UNK] *The triangle ABC is acute-angled. Let L be any line in the plane of the triangle and let u, v, w be lengths of the perpendiculars from A, B, C respectively to L. Prove that*

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\triangle,$$

where $\triangle$ is the area of the triangle, and determine the lines L for which equality holds.

Delta 34. [KWL] Let G and I be the centroid and incenter of the triangle ABC with inradius r, semiperimeter s, circumradius R. Show that

$$IG^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right).$$

Inspiration is needed in geometry, just as much as in poetry. - A. Pushkin

4. Geometry Revisited

> It gives me the same pleasure when someone else proves a good theorem as when I do it myself.
>
> \- E. Landau

4.1. **Areal Co-ordinates.** In this section we aim to briefly introduce develop the theory of areal (or 'barycentric') co-ordinate methods with a view to making them accessible to a reader as a means for solving problems in plane geometry. Areal co-ordinate methods are particularly useful and important for solving problems based upon a triangle, because, unlike Cartesian co-ordinates, they exploit the natural symmetries of the triangle and many of its key points in a very beautiful and useful way.

4.1.1. *Setting up the co-ordinate system.* If we are going to solve a problem using areal co-ordinates, the first thing we must do is choose a triangle ABC, which we call the *triangle of reference*, and which plays a similar role to the axes in a cartesian co-ordinate system. Once this triangle is chosen, we can assign to each point P in the plane a unique triple (x, y, z) fixed such that $x + y + z = 1$, which we call the areal co-ordinates of P. The way these numbers are assigned can be thought of in three different ways, all of which are useful in different circumstances. We leave the proofs that these three conditions are equivalent, along with a proof of the uniqueness of areal co-ordinate representation, for the reader. The first definition we shall see is probably the most intuitive and most useful for working with. It also explains why they are known as 'areal' co-ordinates.

1st Definition: A point P internal to the triangle ABC has areal co-ordinates

$$\left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}\right).$$

If a sign convention is adopted, such that a triangle whose vertices are labelled clockwise has negative area, this definition applies for all P in the plane.

2nd Definition: If x, y, z are the masses we must place at the vertices A, B, C respectively such that the resulting system has centre of mass P, then (x, y, z) are the areal co-ordinates of P (hence the alternative name 'barycentric')

3rd Definition: If we take a system of vectors with arbitrary origin (not on the sides of triangle ABC) and let **a, b, c, p** be the position vectors of A, B, C, P respectively, then $\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ for some triple (x, y, z) such that $x + y + z = 1$. We define this triple as the areal co-ordinates of P.

There are some remarks immediately worth making:

- The vertices A, B, C of the triangle of reference have co-ordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively.
- All the co-ordinates of a point are positive if and only if the point lies within the triangle of reference, and if any of the co-ordinates are zero, the point lies on one of the sides (or extensions of the sides) of ABC.

4.1.2. *The Equation of a Line.* A line is a geometrical object such that any pair of non-parallel lines meet at one and only one point. We would therefore expect the equation of a line to be linear, such that any pair of simultaneous line equations, together with the condition $x+y+z=1$, can be solved for a unique triple (x, y, z) corresponding to the areal co-ordinates of the point of intersection of the two lines. Indeed, it follows (using the equation $x+y+z=1$ to eliminate any constant terms) that the general equation of a line is of the form

$$lx + my + nz = 0$$

where l, m, n are constants and not all zero. Clearly there exists a *unique* line (up to multiplication by a constant) containing any two given points $P(x_p, y_p, z_p), Q(x_q, y_q, z_q)$. This line can be written explicitly as

$$(y_p z_q - y_q z_p)x + (z_p x_q - z_q x_p)y + (x_p y_q - x_q y_p)z = 0$$

This equation is perhaps more neatly expressed in the determinant[9] form:

$$\text{Det}\begin{pmatrix} x & x_p & x_q \\ y & y_p & y_q \\ z & z_p & z_q \end{pmatrix} = 0.$$

While the above form is useful, it is often quicker to just spot the line automatically. For example try to spot the equation of the line BC, containing the points $B(0, 1, 0)$ and $C(0, 0, 1)$, without using the above equation.

Of particular interest (and simplicity) are *Cevian* lines, which pass through the vertices of the triangle of reference. We define a **Cevian through A** as a line whose equation is of the form $my = nz$. Clearly any line containing A must have this form, because setting $y = z = 0, x = 1$ any equation with a nonzero x coefficient would not vanish. It is easy to see that any point on this line therefore has form $(x, y, z) = (1 - mt - nt, nt, mt)$ where t is a parameter. In particular, it will intersect the side BC with equation $x = 0$ at the point $U(0, \frac{n}{m+n}, \frac{m}{m+n})$. Note that from definition 1 (or 3) of areal co-ordinates, this implies that the ratio $\frac{BU}{UC} = \frac{[ABU]}{[AUC]} = \frac{m}{n}$.

4.1.3. *Example: Ceva's Theorem.* We are now in a position to start using areal co-ordinates to prove useful theorems. In this section we shall state and prove (one direction of) an important result of Euclidean geometry known as Ceva's Theorem. The author recommends a keen reader only reads the statement of Ceva's theorem initially and tries to prove it for themselves using the ideas introduced above, before reading the proof given.

[9]The Determinant of a 3×3 Matrix. Matrix determinants play an important role in areal co-ordinate methods. We define the **determinant** of a 3 by 3 square matrix A as

$$\text{Det}(A) = \text{Det}\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - bx_c z) + a_z(b_x c_y - b_y c_x).$$

This can be thought of as (as the above equation suggests) multiplying each element of the first column by the determinants of 2x2 matrices formed in the 2nd and 3rd columns and the rows not containing the element of the first column. Alternatively, if you think of the matrix as wrapping around (so b_x is in some sense directly beneath b_z in the above matrix) you can simply take the sum of the products of diagonals running from top-left to bottom-right and subtract from it the sum of the products of diagonals running from bottom-left to top-right (so think of the above RHS as $(a_x b_y c_z + a_y b_z c_x + a_z b_x c_y) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z)$). In any case, it is worth making sure you are able to quickly evaluate these determinants if you are to be successful with areal co-ordinates.

Theorem 4.1. (Ceva's Theorem) Let ABC be a triangle and let L, M, N be points on the sides BC, CA, AB respectively. Then the cevians AL, BM, CN are concurrent at a point P if and only if

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$$

Proof. Suppose first that the cevians are concurrent at a point P, and let P have areal co-ordinates (p, q, r). Then AL has equation $qz = ry$ (following the discussion of Cevian lines above), so $L\left(0, \frac{q}{q+r}, \frac{r}{q+r}\right)$, which implies $\frac{BL}{LC} = \frac{r}{q}$. Similarly, $\frac{CM}{MA} = \frac{p}{r}, \frac{AN}{NB} = \frac{q}{p}$. Taking their product we get $\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1$, proving one direction of the theorem. We leave the converse to the reader. □

The above proof was very typical of many areal co-ordinate proofs. We only had to go through the details for one of the three cevians, and then could say 'similarly' and obtain ratios for the other two by symmetry. This is one of the great advantages of the areal co-ordinate system in solving problems where such symmetries do exist (particularly problems symmetric in a triangle ABC: such that relabelling the triangle vertices would result in the same problem).

4.1.4. *Areas and Parallel Lines.* One might expect there to be an elegant formula for the area of a triangle in areal co-ordinates, given they are a system constructed on areas. Indeed, there is. If PQR is an arbitrary triangle with $P(x_p, y_p, z_p), Q(x_q, y_q, z_q), R(x_r, y_r, z_r)$ then

$$\frac{[PQR]}{[ABC]} = \text{Det}\begin{pmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ z_p & z_q & z_r \end{pmatrix}$$

An astute reader might notice that this seems like a plausible formula, because if P, Q, R are collinear, it tells us that the triangle PQR has area zero, by the line formula already mentioned. It should be noted that the area comes out as negative if the vertices PQR are labelled in the opposite direction to ABC.

It is now fairly obvious what the general equation for a line parallel to a given line passing through two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ should be, because the area of the triangle formed by any point on such a line and these two points must be constant, having a constant base and constant height. Therefore this line has equation

$$\text{Det}\begin{pmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{pmatrix} = k = k(x + y + z),$$

where $k \in \mathbb{R}$ is a constant.

Delta 35. (United Kingdom 2007) Given a triangle ABC and an arbitrary point P internal to it, let the line through P parallel to BC meet AC at M, and similarly let the lines through P parallel to CA,AB meet AB,BC at N,L respectively. Show that

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} \leq \frac{1}{8}$$

Delta 36. (Nikolaos Dergiades) Let DEF be the medial triangle of ABC, and P a point with cevian triangle XYZ (with respect to ABC). Find P such that the lines DX, EY, FZ are parallel to the internal bisectors of angles A, B, C, respectively.

4.1.5. *To infinity and beyond.* Before we start looking at some more definite specific useful tools (like the positions of various interesting points in the triangle), we round off the general theory with a device that, with practice, greatly simplifies areal manipulations. Until now we have been acting subject to the constraint that $x+y+z=1$. In reality, if we are just intersecting lines with lines or lines with conics, and not trying to calculate any ratios, it is legitimate to ignore this constraint and to just consider the points (x,y,z) and (kx,ky,kz) as being the same point for all $k \neq 0$. This is because areal co-ordinates are a special case of a more general class of co-ordinates called **projective homogeneous co-ordinates**[10], where here the projective line at infinity is taken to be the line $x+y+z=0$. This system only works if one makes all equations homogeneous (of the same degree in x,y,z), so, for example, $x+y=1$ and $x^2+y=z$ are not homogeneous, whereas $x+y-z=0$ and $a^2yz+b^2zx+c^2xy=0$ are homogeneous. We can therefore, once all our line and conic equations are happily in this form, no longer insist on $x+y+z=1$, meaning points like the incentre $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c})$ can just be written (a,b,c). Such represenataions are called *unnormalised areal co-ordinates* and usually provide a significant advantage for the practical purposes of doing manipulations. However, if any ratios or areas are to be calculated, it is imperative that the co-ordinates are *normalised* again to make $x+y+z=1$. This process is easy: just apply the map

$$(x,y,z) \mapsto \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$$

4.1.6. *Significant areal points and formulae in the triangle.* We have seen that the vertices are given by $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, and the sides by $x=0$,$y=0$,$z=0$. In the section on the equation of a line we examined the equation of a cevian, and this theory can, together with other knowledge of the triangle, be used to give areal expressions for familiar points in Euclidean triangle geometry. We invite the reader to prove some of the facts below as exercises.

- Triangle centroid: $G(1,1,1)$.[11]
- Centre of the inscribed circle: $I(a,b,c)$.[12]
- Centres of escribed circles: $I_a(-a,b,c)$, $I_b(a,-b,c)$, $I_c(a,b,-c)$.
- Symmedian point: $K(a^2,b^2,c^2)$.
- Circumcentre: $O(\sin 2A, \sin 2B, \sin 2C)$.
- Orthocentre: $H(\tan A, \tan B, \tan C)$.
- The isogonal conjugate of $P(x,y,z)$: $P^*\left(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}\right)$.
- The isotomic conjugate of $P(x,y,z)$: $P^t\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$.

It should be noted that the rather nasty trigonometric forms of O and H mean that they should be approached using areals with caution, preferably only if the calculations will be relatively simple.

Delta 37. *Let D,E be the feet of the altitudes from A and B respectively, and P,Q the meets of the angle bisectors AI,BI with BC,CA respectively. Show that D,I,E are collinear if and only if P,O,Q are.*

[10]The author regrets that, in the interests of concision, he is unable to deal with these co-ordinates in this document, but strongly recommends Christopher Bradley's *The Algebra of Geometry*, published by Highperception, as a good modern reference also with a more detailed account of areals and a plethora of applications of the methods touched on in this document. Even better, though only for projectives and lacking in the wealth of fascinating modern examples, is E.A.Maxwell's *The methods of plane projective geometry based on the use of general homogeneous coordinates*, recommended to the present author by the author of the first book.

[11]The midpoints of the sides BC, CA, AB are given by $(0,1,1)$,$(1,0,1)$ and $(1,1,0)$ respectively.

[12]Hint: use the angle bisector theorem.

4.1.7. *Distances and circles.* We finally quickly outline some slightly more advanced theory, which is occasionally quite useful in some problems, We show how to manipulate conics (with an emphasis on circles) in areal co-ordinates, and how to find the distance between two points in areal co-ordinates. These are placed in the same section because the formulae look quite similar and the underlying theory is quite closely related. Derivations can be found in [Bra1].

Firstly, the general equation of a conic in areal co-ordinates is, since a conic is a general equation of the second degree, and areals are a homogeneous system, given by

$$px^2 + qy^2 + rz^2 + 2dyz + 2ezx + 2fxy = 0$$

Since multiplication by a nonzero constant gives the same equation, we have five independent degrees of freedom, and so may choose the coefficients uniquely (up to multiplication by a constant) in such a way as to ensure five given points lie on such a conic.

In Euclidean geometry, the conic we most often have to work with is the circle. The most important circle in areal co-ordinates is the circumcircle of the reference triangle, which has the equation (with a, b, c equal to BC, CA, AB respectively)

$$a^2yz + b^2zx + c^2xy = 0$$

In fact, sharing two infinite points[13] with the above, a general circle is just a variation on this theme, being of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0$$

We can, given three points, solve the above equation for u, v, w substituting in the three desired points to obtain the equation for the unique circle passing through them.

Now, the areal distance formula looks very similar to the circumcircle equation. If we have a pair of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, which must be normalised, we may define the displacement $PQ : (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (u, v, w)$, and it is this we shall measure the distance of. So the distance of a displacement $PQ(u, v, w), u + v + w = 0$ is given by

$$PQ^2 = -a^2vw - b^2wu - c^2uv$$

Since $u+v+w = 0$ this is, despite the negative signs, always positive unless $u = v = w = 0$.

Delta 38. *Use the vector definition of areal co-ordinates to prove the areal distance formula and the circumcircle formula.*

4.1.8. *Miscellaneous Exercises.* Here we attach a selection of problems compiled by Tim Hennock, largely from UK IMO activities in 2007 and 2008. None of them are trivial, and some are quite difficult. Good luck!

Delta 39. (UK Pre-IMO training 2007) Let ABC be a triangle. Let D, E, F be the reflections of A, B, C in BC, AC, AB respectively. Show that D, E, F are collinear if and only if $OH = 2R$.

Delta 40. (Balkan MO 2005) Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60°$

[13] All circles have two (imaginary) points in common on the line at infinity. It follows that if a conic is a circle, its behaviour at the line at infinity $x + y + z = 0$ must be the same as that of the circumcircle, hence the equation given.

Delta 41. (United Kingdom 2007) Triangle ABC has circumcentre O and centroid M. The lines OM and AM are perpendicular. Let AM meet the circumcircle of ABC again at A'. Lines CA' and AB intersect at D and BA' and AC intersect at E. Prove that the circumcentre of triangle ADE lies on the circumcircle of ABC.

Delta 42. [IMO 2007/4] *In triangle ABC the bisector of $\angle BCA$ intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.*

Delta 43. (RMM 2008) Let ABC be an equilateral triangle. P is a variable point internal to the triangle, and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P such that a, b and c can be the side lengths of a non-degenerate triangle.

Delta 44. [SL 2006] *Let ABC be a triangle such that $\angle C < \angle A < \frac{\pi}{2}$. Let D be on AC such that $BD = BA$. The incircle of ABC touches AB at K and AC at L. Let J be the incentre of triangle BCD. Prove that KL bisects AJ.*

Delta 45. (United Kingdom 2007) The excircle of a triangle ABC touches the side AB and the extensions of the sides BC and CA at points M, N and P, respectively, and the other excircle touches the side AC and the extensions of the sides AB and BC at points S, Q and R, respectively. If X is the intersection point of the lines PN and RQ, and Y the intersection point of RS and MN, prove that the points X, A and Y are collinear.

Delta 46. (Sharygin GMO 2008) Let ABC be a triangle and let the excircle opposite A be tangent to the side BC at A_1. N is the Nagel point of ABC, and P is the point on AA_1 such that $AP = NA_1$. Prove that P lies on the incircle of ABC.

Delta 47. (United Kingdom 2007) Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of ABC touches the sides BC, CA, AB at the points D, E, F, respectively. Let AD intersect I at D and P. Let Q be the intersection of the lines EF and the line passing through P and perpendicular to AD, and let X, Y be intersections of the line AQ and DE, DF, respectively. Show that the point A is the midpoint of XY.

Delta 48. (Sharygin GMO 2008) Given a triangle ABC. Point A_1 is chosen on the ray BA so that the segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that the segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 are parallel.

4.2. **Concurrencies around Ceva's Theorem.** In this section, we shall present some corollaries and applications of Ceva's theorem.

Theorem 4.2. *Let $\triangle ABC$ be a given triangle and let A_1, B_1, C_1 be three points on lying on its sides BC, CA and AB, respectively. Then, the three lines AA_1 ,BB_1, CC_1 concur if and only if*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

Proof. We shall resume to proving only the direct implication. After reading the following proof, you will understand why. Denote by P the intersection of the lines AA_1, BB_1, CC_1. The parallel to BC through P meets CA at B_a and AB at C_a. The parallel to CA through P meets AB at C_b and BC at A_b. The parallel to AB through P meets BC at A_c and CA at B_c. As segments on parallels, we get $\frac{C_1A}{C_1B} = \frac{PB_c}{PA_c}$. On the other hand, we get

$$\frac{B_cP}{AB} = \frac{PB_1}{BB_1} \text{ and } \frac{PA_c}{AB} = \frac{PA_1}{AA_1}.$$

It follows that

$$\frac{B_cP}{AB} : \frac{PA_c}{AB} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}$$

so that

$$\frac{B_cP}{PA_c} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}.$$

Consequently, we obtain

$$\frac{C_1A}{C_1B} = \frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}.$$

Similarly, we deduce that $\frac{A_1B}{A_1C} = \frac{PC_1}{CC_1} : \frac{PB_1}{BB_1}$ and $\frac{B_1C}{B_1A} = \frac{PA_1}{AA_1} : \frac{PC_1}{CC_1}$. Now

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \left(\frac{PC_1}{CC_1} : \frac{PB_1}{BB_1}\right) \cdot \left(\frac{PA_1}{AA_1} : \frac{PC_1}{CC_1}\right) \cdot \left(\frac{PB_1}{BB_1} : \frac{PA_1}{AA_1}\right) = 1,$$

which proves Ceva's theorem. □

Corollary 4.1. (The Trigonometric Version of Ceva's Theorem) In the configuration described above, the lines AA_1, BB_1, CC_1 are concurrent if and only if

$$\frac{\sin A_1AB}{\sin A_1AC} \cdot \frac{\sin C_1CA}{\sin C_1CB} \cdot \frac{\sin B_1BC}{\sin B_1BA} = 1.$$

Proof. By the Sine Law, applied in the triangles A_1AB and A_1AC, we have

$$\frac{A_1B}{\sin A_1AB} = \frac{AB}{\sin AA_1B}, \quad \text{and} \quad \frac{A_1C}{\sin A_1AC} = \frac{AC}{\sin AA_1C}.$$

Hence,

$$\frac{A_1B}{A_1C} = \frac{AB}{AC} \cdot \frac{\sin A_1AB}{\sin A_1AC}.$$

Similarly, $\frac{B_1C}{B_1A} = \frac{BC}{AB} \cdot \frac{\sin B_1BC}{\sin B_1BA}$ and $\frac{C_1A}{C_1B} = \frac{AC}{BC} \cdot \frac{\sin C_1CA}{\sin C_1CB}$. Thus, we conclude that

$$\begin{aligned} & \frac{\sin A_1AB}{\sin A_1AC} \cdot \frac{\sin C_1CA}{\sin C_1CB} \cdot \frac{\sin B_1BC}{\sin B_1BA} \\ = \quad & \left(\frac{A_1B}{A_1C} \cdot \frac{AC}{AB}\right) \cdot \left(\frac{C_1A}{C_1B} \cdot \frac{BC}{AC}\right) \cdot \left(\frac{B_1C}{B_1A} \cdot \frac{AB}{BC}\right) \\ = \quad & 1. \end{aligned}$$

□

We begin now with a result, which most of you might know it as *Jacobi's theorem.*

Proposition 4.1. (Jacobi's Theorem) Let ABC be a triangle, and let X, Y, Z be three points in its plane such that $\angle YAC = \angle BAZ$, $\angle ZBA = \angle CBX$ and $\angle XCB = \angle ACY$. Then, the lines AX, BY, CZ are concurrent.

Proof. We use directed angles taken modulo 180°. Denote by A, B, C, x, y, z the magnitudes of the angles $\angle CAB$, $\angle ABC$, $\angle BCA$, $\angle YAC$, $\angle ZBA$, and $\angle XCB$, respectively. Since the lines AX, BX, CX are (obviously) concurrent (at X), the trigonometric version of Ceva's theorem yields

$$\frac{\sin CAX}{\sin XAB} \cdot \frac{\sin ABX}{\sin XBC} \cdot \frac{\sin BCX}{\sin XCA} = 1.$$

We now notice that

$$\angle ABX = \angle ABC + \angle CBX = B + y, \quad \angle XBC = -\angle CBX = -y,$$

$$\angle BCX = -\angle XCB = -z, \quad \angle XCA = \angle XCB + \angle BCA = z + C.$$

Hence, we get

$$\frac{\sin CAX}{\sin XAB} \cdot \frac{\sin (B+y)}{\sin (-y)} \cdot \frac{\sin (-z)}{\sin (C+z)} = 1.$$

Similarly, we can find

$$\frac{\sin ABY}{\sin YBC} \cdot \frac{\sin (C+z)}{\sin (-z)} \cdot \frac{\sin (-x)}{\sin (A+x)} = 1,$$

$$\frac{\sin BCZ}{\sin ZCA} \cdot \frac{\sin(A+x)}{\sin(-x)} \cdot \frac{\sin(-y)}{\sin(B+y)} = 1.$$

Multiplying all these three equations and canceling the same terms, we get

$$\frac{\sin CAX}{\sin XAB} \cdot \frac{\sin ABY}{\sin YBC} \cdot \frac{\sin BCZ}{\sin ZCA} = 1.$$

According to the trigonometric version of Ceva's theorem, the lines AX, BY, CZ are concurrent. □

We will see that Jacobi's theorem has many interesting applications. We start with the well-known Karyia theorem.

Theorem 4.3. (Kariya's Theorem) Let I be the incenter of a given triangle ABC, and let D, E, F be the points where the incircle of ABC touches the sides BC, CA, AB. Now, let X, Y, Z be three points on the lines ID, IE, IF such that the directed segments IX, IY, IZ are congruent. Then, the lines AX, BY, CZ are concurrent.

Proof. (Darij Grinberg) Being the points of tangency of the incircle of triangle ABC with the sides AB and BC, the points F and D are symmetric to each other with respect to the angle bisector of the angle ABC, i. e. with respect to the line BI. Thus, the triangles BFI and BDI are inversely congruent. Now, the points Z and X are corresponding points in these two inversely congruent triangles, since they lie on the (corresponding) sides IF and ID of these two triangles and satisfy $IZ = IX$. Corresponding points in inversely congruent triangles form oppositely equal angles, i.e. $\angle ZBF = -\angle XBD$. In other words, $\angle ZBA = \angle CBX$. Similarly, we have that $\angle XCB = \angle ACY$ and $\angle YAC = \angle BAZ$. Note that the points X, Y, Z satisfy the condition from Jacobi's theorem, and therefore, we conclude that the lines AX, BY, CZ are concurrent. □

Another such corollary is the Kiepert theorem, which generalizes the existence of the Fermat points.

Delta 49. (Kiepert's Theorem) Let ABC be a triangle, and let BXC, CYA, AZB be three directly similar isosceles triangles erected on its sides BC, CA, and AB, respectively. Then, the lines AX, BY, CZ concur at one point.

Delta 50. (Floor van Lamoen) Let A', B', C' be three points in the plane of a triangle ABC such that $\angle B'AC = \angle BAC'$, $\angle C'BA = \angle CBA'$ and $\angle A'CB = \angle ACB'$. Let X, Y, Z be the feet of the perpendiculars from the points A', B', C' to the lines BC, CA, AB. Then, the lines AX, BY, CZ are concurrent.

Delta 51. (Cosmin Pohoață) Let ABC be a given triangle in plane. From each of its vertices we draw two arbitrary isogonals. Then, these six isogonals determine a hexagon with concurrent diagonals.

Epsilon 51. (USA 2003) Let ABC be a triangle. A circle passing through A and B intersects the segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Delta 52. (Romanian jBMO 2007 Team Selection Test) Let ABC be a right triangle with $\angle A = 90°$, and let D be a point lying on the side AC. Denote by E reflection of A into the line BD, and by F the intersection point of CE with the perpendicular in D to the line BC. Prove that AF, DE and BC are concurrent.

Delta 53. Denote by AA_1, BB_1, CC_1 the altitudes of an acute triangle ABC, where A_1, B_1, C_1 lie on the sides BC, CA, and AB, respectively. A circle passing through A_1 and B_1 touches the arc AB of its circumcircle at C_2. The points A_2, B_2 are defined similarly.

1. (Tuymaada Olympiad 2007) Prove that the lines AA_2, BB_2, CC_2 are concurrent.
2. (Cosmin Pohoață, MathLinks Contest 2008, Round 1) Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent on the Euler line of ABC.

4.3. **Tossing onto Complex Plane.** Here, we discuss some applications of complex numbers to geometric inequality. Every complex number corresponds to a unique point in the complex plane. The standard symbol for the set of all complex numbers is $\mathbb{C}$, and we also refer to the complex plane as $\mathbb{C}$. We can identify the points in the real plane $\mathbb{R}^2$ as numbers in $\mathbb{C}$. The main tool is the following fundamental inequality.

Theorem 4.4. (Triangle Inequality) If $z_1, \cdots, z_n \in \mathbb{C}$, then $|z_1| + \cdots + |z_n| \geq |z_1 + \cdots + z_n|$.

Proof. Induction on n. □

Theorem 4.5. (Ptolemy's Inequality) For any points A, B, C, D in the plane, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Proof. Let a, b, c and 0 be complex numbers that correspond to A, B, C, D in the complex plane $\mathbb{C}$. It then becomes

$$|a - b| \cdot |c| + |b - c| \cdot |a| \geq |a - c| \cdot |b|.$$

Applying the Triangle Inequality to the identity $(a - b)c + (b - c)a = (a - c)b$, we get the result. □

Remark 4.1. *Investigate the equality case in Ptolemy's Inequality.*

Delta 54. [SL 1997 RUS] *Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that*

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does the equality occur?

Epsilon 52. [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $BC \cdot PB \cdot PC + AB \cdot PA \cdot PB + CA \cdot PC \cdot PA \geq BC \cdot CA \cdot AB$,
(2) $PA^3 \cdot BC + PB^3 \cdot CA + PC^3 \cdot AB \geq 3PG \cdot BC \cdot CA \cdot AB$.

Delta 55. *Let H denote the orthocenter of an acute triangle ABC. Prove the geometric identity*

$$BC \cdot HB \cdot HC + AB \cdot HA \cdot HB + CA \cdot HC \cdot HA = BC \cdot CA \cdot AB.$$

Epsilon 53. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Epsilon 54. [SL 2002 KOR] *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $AB + AC \geq 4DE$.*

4.4. **Generalize Ptolemy's Theorem!** The story begins with three trigonometric proofs of Ptolemy's Theorem.

Theorem 4.6. (Ptolemy's Theorem) Let $ABCD$ be a convex quadrilateral. If $ABCD$ is cyclic, then we have

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

First Proof. Set $AB = a$, $BC = b$, $CD = c$, $DA = d$. One natural approach is to compute $BD = x$ and $AC = y$ in terms of a, b, c and d. We apply the Cosine Law to obtain

$$x^2 = a^2 + d^2 - 2ad \cos A$$

and

$$x^2 = b^2 + c^2 - 2bc \cos D = b^2 + c^2 + 2bc \cos A.$$

Equating two equations, we meet

$$a^2 + d^2 - 2ad \cos A = b^2 + c^2 + 2bc \cos A$$

or

$$\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}.$$

It follows that

$$x^2 = a^2 + d^2 - 2ad \cos A = a^2 + d^2 - 2ad \left(\frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} \right) = \frac{(ac + bd)(ab + cd)}{ad + bc}.$$

Similarly, we also obtain

$$y^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Multiplying these two, we obtain $x^2 y^2 = (ac + bd)^2$ or $xy = ac + bd$, as desired. □

Second Proof. (Hojoo Lee) As in the classical proof via the inversion, we rewrite it as

$$\frac{AB}{DA \cdot DB} + \frac{BC}{DB \cdot DC} = \frac{AC}{DA \cdot DC}.$$

We now trigonometrize each term. Letting $\mathcal{R}$ denote the circumradius of $ABCD$ and noticing that $\sin(\angle ADB) = \sin(\angle DBA + \angle DAB)$ in triangle DAB, we obtain

$$\begin{aligned}
\frac{AB}{DA \cdot DB} &= \frac{2\mathcal{R} \sin(\angle ADB)}{2\mathcal{R} \sin(\angle DBA) \cdot 2\mathcal{R} \sin(\angle DAB)} \\
&= \frac{\sin \angle DBA \cos \angle DAB + \cos \angle DBA \sin \angle DAB}{2\mathcal{R} \sin(\angle DBA) \sin(\angle DAB)} \\
&= \frac{1}{2\mathcal{R}} \left(\cot \angle DAB + \cot \angle DBA \right).
\end{aligned}$$

Likewise, we have

$$\frac{BC}{DB \cdot DC} = \frac{1}{2\mathcal{R}} \left(\cot \angle DBC + \cot \angle DCB \right)$$

and

$$\frac{AC}{DA \cdot DC} = \frac{1}{2\mathcal{R}} \left(\cot \angle DAC + \cot \angle DCA \right).$$

Hence, the geometric identity in Ptolemy's Theorem is equivalent to the cotangent identity

$$\left(\cot \angle DAB + \cot \angle DBA \right) + \left(\cot \angle DBC + \cot \angle DCB \right) = \left(\cot \angle DAC + \cot \angle DCA \right).$$

However, since the convex quadrilateral $ABCD$ admits a circumcircle, it is clear that

$$\angle DAB + \angle DCB = \pi, \ \angle DBA = \angle DCA, \ \angle DBC = \angle DAC$$

so that

$$\cot \angle DAB + \cot \angle DCB = 0, \ \cot \angle DBA = \cot \angle DCA, \ \cot \angle DBC = \cot \angle DAC.$$

□

Third Proof. We exploit the Sine Law to convert the geometric identity to the trigonometric identity. Let $\mathcal{R}$ denote the circumradius of $ABCD$. We set

$$\angle AOB = 2\theta_1,\ \angle BOC = 2\theta_2,\ \angle COD = 2\theta_3,\ \angle DOA = 2\theta_4,$$

where O is the center of the circumcircle of $ABCD$. It's clear that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$. It follows that $AB = 2\mathcal{R}\sin\theta_1$, $BC = 2\mathcal{R}\sin\theta_2$, $CD = 2\mathcal{R}\sin\theta_3$, $DA = 2\mathcal{R}\sin\theta_4$, $AC = 2\mathcal{R}\sin(\theta_1+\theta_2)$, $AB = 2\mathcal{R}\sin(\theta_2+\theta_3)$. Our job is to establish

$$AB\cdot CD + BC\cdot DA = AC\cdot BD$$

or

$$(2\mathcal{R}\sin\theta_1)(2\mathcal{R}\sin\theta_3) + (2\mathcal{R}\sin\theta_2)(2\mathcal{R}\sin\theta_4) = (2\mathcal{R}\sin(\theta_1+\theta_2))(2\mathcal{R}\sin(\theta_2+\theta_3))$$

or equivalently

$$\sin\theta_1\sin\theta_3 + \sin\theta_2\sin\theta_4 = \sin(\theta_1+\theta_2)\sin(\theta_2+\theta_3).$$

We use the well-known identity $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$ to rewrite it as

$$\begin{aligned} & \frac{\cos(\theta_1-\theta_3)-\cos(\theta_1+\theta_3)}{2} + \frac{\cos(\theta_2-\theta_4)-\cos(\theta_2+\theta_4)}{2} \\ = \ & \frac{\cos(\theta_1-\theta_3)-\cos(\theta_1+2\theta_2+\theta_3)}{2} \end{aligned}$$

or equivalently

$$-\cos(\theta_1+\theta_3) + \cos(\theta_2-\theta_4) - \cos(\theta_2+\theta_4) = -\cos(\theta_1+2\theta_2+\theta_3).$$

Since $\theta_1+\theta_2+\theta_3+\theta_4 = \pi$ or since $\cos(\theta_1+\theta_3)+\cos(\theta_2+\theta_4) = 0$, it is equivalent to

$$\cos(\theta_2-\theta_4) = -\cos(\theta_1+2\theta_2+\theta_3).$$

However, we employ $\theta_1+\theta_2+\theta_3 = \pi-\theta_4$ to deduce

$$\cos(\theta_1+2\theta_2+\theta_3) = \cos(\theta_2+\pi-\theta_4) = -\cos(\theta_2-\theta_4).$$

□

When the second author of this weblication was a high school student, one day, he was trying to device a coordinate proof of Ptolemy's Theorem. However, we immediately realize that the direct approach using only the distance formula is hopeless. The geometric identity reads, in coordinates,

$$\begin{aligned} & \sqrt{\left\{(x_1-x_2)^2+(y_1-y_2)^2\right\}\left\{(x_3-x_4)^2+(y_3-y_4)^2\right\}} \\ & +\sqrt{\left\{(x_2-x_3)^2+(y_2-y_3)^2\right\}\left\{(x_4-x_1)^2+(y_4-y_1)^2\right\}} \\ & = \sqrt{\left\{(x_1-x_3)^2+(y_1-y_3)^2\right\}\left\{(x_2-x_4)^2+(y_2-y_4)^2\right\}}. \end{aligned}$$

What he realized was that the key point is to find a natural coordinate condition. First, forget about the destination $AB\cdot CD + BC\cdot DA = AC\cdot BD$ and, instead, find out what the initial condition that $ABCD$ is cyclic says in coordinates.

Lemma 4.1. *Let $ABCD$ be a convex quadrilateral. We toss $ABCD$ on the real plane $\mathbb{R}^2$ with the coordinates $A(a_1,a_2)$, $B(b_1,b_2)$, $C(c_1,c_2)$, $D(d_1,d_2)$. Then, the necessary and sufficient condition that $ABCD$ is cyclic is that the following equality holds.*

$$\begin{aligned} & \frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2} \\ = \ & \frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2} \end{aligned}$$

Proof. The quadrilateral $ABCD$ is cyclic if and only if $\angle BAC = \angle BDC$, or equivalently $\cot(\angle BAC) = \cot(\angle BDC)$. It is equivalent to

$$\frac{\cos(\angle BAC)}{\sin(\angle BAC)} = \frac{\cos(\angle BDC)}{\sin(\angle BDC)}$$

or

$$\frac{\frac{BA^2+AC^2-CB^2}{2BA\cdot AC}}{\frac{2[ABC]}{BA\cdot AC}} = \frac{\frac{BD^2+DC^2-CB^2}{2BD\cdot DC}}{\frac{2[DBC]}{BD\cdot DC}}$$

or

$$\frac{BA^2+AC^2-CB^2}{2[ABC]} = \frac{BD^2+DC^2-CB^2}{2[DBC]}$$

or in coordinates,

$$\begin{aligned} & \frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2} \\ = \quad & \frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2}. \end{aligned}$$

□

The coordinate condition and its proof is natural. However, something weird happens here. It does not look like being cyclic in the coordinates . Indeed, when $ABCD$ is a cyclic quadrilateral, we notice that the same quadrilateral $BCDA$, $CDAB$, $DABC$ are also trivially cyclic. It turns out that the coordinate condition indeed admits a certain symmetry. Now, it is time to consider the destination

$$AB\cdot CD + BC\cdot DA = AC\cdot BD.$$

As we see above, the direct application of the distance formula gives us a monster identity with square roots. What we need is a reformulation without square roots. We recall that Ptolemy's Theorem is trivialized by the inversive geometry. As in the proof via the inversion, we rewrite it in the symmetric form

$$\frac{AB}{DA\cdot DB} + \frac{BC}{DB\cdot DC} = \frac{AC}{DA\cdot DC}.$$

Now, we reach the key step. Let $\mathcal{R}$ denote the circumcircle of $ABCD$. The formulas

$$[DAB] = \frac{AB\cdot DA\cdot DB}{4\mathcal{R}},\ [DBC] = \frac{BC\cdot DB\cdot DC}{4\mathcal{R}},\ [DCA] = \frac{CA\cdot DC\cdot DA}{4\mathcal{R}}$$

allows us to realize that it is equivalent to the geometric identity.

$$\frac{[DAB]}{DA^2\cdot DB^2} + \frac{[DBC]}{DB^2\cdot DC^2} = \frac{[DAC]}{DA^2\cdot DC^2}$$

or

$$DC^2[DAB] + DA^2[DBC] = DB^2[DAC].$$

Summarizing up the result, we have

Lemma 4.2. *Let $ABCD$ be a convex and cyclic quadrilateral. Then the following two geometric identities are equivalent.*

$$\begin{aligned} &(1)\quad AB\cdot CD + BC\cdot DA = AC\cdot BD. \\ &(2)\quad DC^2[DAB] + DA^2[DBC] = DB^2[DAC]. \end{aligned}$$

It is awesome. Why? It is because we can express the second condition in the coordinates without the horrible square root! After a long, very long computation by hand, we can check that

$$\begin{aligned} & \frac{{a_1}^2+{a_2}^2-(a_1b_1+a_2b_2+a_1c_1+a_2c_2-b_1c_1-b_2c_2)}{b_1a_2+a_1c_2+c_1b_2-a_1b_2-c_1a_2-b_1c_2} \\ = \quad & \frac{{d_1}^2+{d_2}^2-(d_1b_1+d_2b_2+d_1a_1+d_2a_2-b_1a_1-b_2a_2)}{b_1d_2+d_1a_2+a_1b_2-d_1b_2-a_1d_2-b_1a_2}. \end{aligned}$$

indeed implies the coordinate condition of the reformulation (2). The brute-force computation will be simplified if we take $D(d_1, d_2) = (0, 0)$. However, it is not the end of the story. Actually, he found a symmetry in the coordinate computation. It leads him to rediscover The Feuerbach-Luchterhand Theorem, which generalize Ptolemy's Theorem.

Theorem 4.7. (The Feuerbach-Luchterhand Theorem) Let $ABCD$ be a convex and cyclic quadrilateral. For any point O in the plane, we have

$$OA^2 \cdot BC \cdot CD \cdot DB - OB^2 \cdot CD \cdot DA \cdot AB + OC^2 \cdot DA \cdot AB \cdot BD - OD^2 \cdot AB \cdot BC \cdot CD = 0.$$

Proof. We toss the picture on the real plane $\mathbb{R}^2$ with the coordinates

$$O(0,0),\ A(a_1, a_2),\ B(b_1, b_2),\ C(c_1, c_2),\ D(d_1, d_2),$$

Letting $\mathcal{R}$ denote the circumcircle of $ABCD$, it can be rewritten as

$$OA^2 \cdot \frac{[BCD]}{4R} - OB^2 \cdot \frac{[CDA]}{4R} + OC^2 \cdot \frac{[DAB]}{4R} - OD^2 \cdot \frac{[DBC]}{4R} = 0$$

or

$$OA^2 \cdot [BCD] - OB^2 \cdot [CDA] + OC^2 \cdot [DAB] - OD^2 \cdot [ABC] = 0.$$

We can rewrite this in the coordinates without square roots. Now, after long computation, we can check, by hand, that it is equivalent to the coordinate condition that $ABCD$ is cyclic:

$$\begin{aligned} & \frac{{a_1}^2 + {a_2}^2 - (a_1 b_1 + a_2 b_2 + a_1 c_1 + a_2 c_2 - b_1 c_1 - b_2 c_2)}{b_1 a_2 + a_1 c_2 + c_1 b_2 - a_1 b_2 - c_1 a_2 - b_1 c_2} \\ = & \frac{{d_1}^2 + {d_2}^2 - (d_1 b_1 + d_2 b_2 + d_1 a_1 + d_2 a_2 - b_1 a_1 - b_2 a_2)}{b_1 d_2 + d_1 a_2 + a_1 b_2 - d_1 b_2 - a_1 d_2 - b_1 a_2}. \end{aligned}$$

□

The end? Not yet. It turns out that after throwing out the essential condition that the quadrilateral is cyclic, we can extend the theorem to arbitrary quadrilaterals!

Theorem 4.8. (Hojoo Lee) For an arbitrary point P in the plane of the convex quadrilateral $A_1A_2A_3A_4$, we obtain

$$\begin{aligned} & {PA_1}^2[\triangle A_2A_3A_4] - {PA_2}^2[\triangle A_3A_4A_1] + {PA_3}^2[\triangle A_4A_1A_2] - {PA_4}^2[\triangle A_1A_2A_3] \\ = & \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3}[\triangle A_2A_3A_4] - \overrightarrow{A_4A_2} \cdot \overrightarrow{A_4A_3}[\triangle A_1A_2A_3]. \end{aligned}$$

After removing the convexity of $A_1A_2A_3A_4$, we get the same result regarding the signed area of triangle.

Outline of Proof. We toss the figure on the real plane $\mathbb{R}^2$ and write $P(0,0)$ and $A_i = (x_i, y_i)$, where $1 \leq i \leq 4$. Our task is to check that two matrices

$$L = \begin{pmatrix} {PA_1}^2 & {PA_2}^2 & {PA_3}^2 & {PA_4}^2 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$R = \begin{pmatrix} \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3} & 0 & 0 & \overrightarrow{A_4A_2} \cdot \overrightarrow{A_4A_3} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

have the same determinant. We now invite the readers to find a neat proof, of course without the brute force expansion of the determinants! □

Is it a re-discovery again? Is this the end of the story? No. There is no end of generalizations in Mathematics. The lesson we want to deliver here is simple: Even the brute-force coordinate proofs offer good motivations. There is no bad proof. We now present some applications of the The Feuerbach-Luchterhand Theorem.

Corollary 4.2. *Let $ABCD$ be a rectangle. For any point P, we have*

$$PA^2 - PB^2 + PC^2 - PD^2 = 0.$$

Now, let's see what happens if we apply The Feuerbach-Luchterhand Theorem to a geometric situation from the triangle geometry. Let ABC be a triangle with the incenter I and the circumcenter O. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$. Let R and r denote the circumradius and inradius, respectively. Let P and Q denote the feet of perpendiculars from I to the sides CA and CB, respectively. Since $\angle IPC = 90^\circ = \angle IQC$, we find that $IQCP$ is cyclic.

We then apply The Feuerbach-Luchterhand Theorem to the pair $(O, IQCP)$ to deduce the geometric identity

$$0 = OI^2 QC \cdot CP \cdot PQ - OQ^2 CP \cdot PI \cdot IC + OC^2 PI \cdot IQ \cdot QP - OP^2 IQ \cdot QC \cdot CI.$$

What does it mean? We observe that, in the isosceles triangles COA and BOC,

$$OP^2 = R^2 - AP \cdot PC = R^2 - (s-a)(s-c),$$
$$OQ^2 = R^2 - BQ \cdot QC = R^2 - (s-b)(s-c).$$

Now, it becomes

$$\begin{aligned} 0 &= OI^2(s-c)^2 \left(CI\frac{c}{2R}\right) - \left[R^2 - (s-b)(s-c)\right](s-c)r \cdot IC \\ &+ R^2r^2\left(CI\frac{c}{2R}\right) - \left[R^2 - (s-a)(s-c)\right] r(s-c) \cdot CI \end{aligned}$$

or

$$\begin{aligned} 0 &= OI^2(s-c)^2 \cdot \frac{c}{2R} - \left[R^2 - (s-b)(s-c)\right](s-c)r \\ &+ R^2r^2 \cdot \frac{c}{2R} - \left[R^2 - (s-a)(s-c)\right] r(s-c). \end{aligned}$$

or

$$OI^2(s-c)^2 \cdot \frac{c}{2R} = -\frac{Rr^2c}{2} + \left[2R^2 - c(s-c)\right](s-c)r$$

or

$$\frac{OI^2}{R} = -\frac{Rr^2}{(s-c)^2} + \frac{4R^2r}{c(s-c)} - 2r = R\left[\frac{4Rr(s-c) - r^2c}{(s-c)^2c}\right] - 2r.$$

Now, we apply Ptolemy's Theorem and The Pythagoras Theorem to deduce

$$2r(s-c) = CP \cdot IQ + PI \cdot IQ = PQ \cdot CI = CI^2\frac{c}{2R} = \left[(s-c)^2 + r^2\right]\frac{c}{2R}$$

or

$$4Rr(s-c) = c\left((s-c)^2 + r^2\right)$$

or

$$4Rr(s-c) - r^2c = (s-c)^2c$$

or

$$\frac{4Rr(s-c) - r^2c}{(s-c)^2c} = 1.$$

It therefore follows that

$$OI^2 = R^2 - 2rR.$$

It is the theorem proved first by L. Euler. There are lots of way to establish this. Device your own proofs! Find other corollaries of The Feuerbach-Luchterhand Theorem. Another possible generalization of Ptolemy's relation is Casey's theorem:

Theorem 4.9. (Casey's Theorem) Given four circles $\mathcal{C}_i$, $i = 1,\ 2,\ 3,\ 4$, let t_{ij} be the length of a common tangent between $\mathcal{C}_i$ and $\mathcal{C}_j$. The four circles are tangent to a fifth circle (or line) if and only if for an appropriate choice of signs, we have that

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

The most common proof for this result is by making use of inversion. See [RJ]. We shall omit it here. We now work on the Feuerbach's celebrated theorem (actually its first version).

Theorem 4.10. (Feuerbach's Theorem) The incircle and nine-point circle of a triangle are tangent to one another.

Why *first* version? Of course, most of you might know that the nine-point circle is also tangent to the three excircles of the triangle. Most of the geometry textbooks include this last remark in the theorem's statement as well, but this is mostly for sake of completeness, since the proof is similar with the incenter case.

Proof. Let the sides BC, CA, AB of triangle ABC have midpoints D, E, F respectively, and let Γ be the incircle of the triangle. Let a, b, c be the sidelengths of ABC, and let s be its semiperimeter. We now consider the 4-tuple of circles $(D,\ E,\ F,\ \Gamma)$. Here is what we find:

$$t_{DE} = \frac{c}{2},\quad t_{DF} = \frac{b}{2},\quad t_{EF} = \frac{a}{2},$$

$$t_{D\Gamma} = \left|\frac{a}{2} - (s-b)\right| = \left|\frac{b-c}{2}\right|,$$

$$t_{E\Gamma} = \left|\frac{b}{2} - (s-c)\right| = \left|\frac{a-c}{2}\right|,$$

$$t_{F\Gamma} = \left|\frac{c}{2} - (s-a)\right| = \left|\frac{b-a}{2}\right|.$$

We need to check whether, for some combination of $+,\ -$ signs, we have

$$\pm(c(b-a) \pm a(b-c) \pm b(a-c) = 0.$$

But this is immediate! According to Casey's theorem there exists a circle what touches each of D, E, F and Γ. Since the circle passing through D, E, F is the ninepoint circle of the triangle, it follows that Γ and the nine-point circle are tangent to each other. $\square$

We shall see now an interesting particular case of Thebault's theorem.

Proposition 4.2. (IMO Longlist 1991, proposed by India) Circles Γ_1 and Γ_2 are externally tangent at a point I, and both are enclosed by and tangent to a third circle Γ. One common tangent to Γ_1 and Γ_2 meets Γ in B and C, while the common tangent at I meets Γ in A on the same side of BC as I. Then, we have that I is the incenter of triangle ABC.

Proof. Let X, Y be the tangency points of BC with the circles Γ_1, and Γ_2, respectively, and let x, y be the lengths of the tangents from B and C to Γ_1 and Γ_2. Denote by D the intersection of AI with the line BC, and let $z = AI$, $u = ID$. According to Casey's theorem, applied for the two 4-tuples of circles $(A,\ \Gamma_1,\ B,\ C)$ and $(A,\ \Gamma_2,\ C,\ B)$, we obtain $az + bx = c(2u + y)$ and $az + cy = b(2u + x)$. Subtracting the second equation from the first, we obtain that $bx - cy = u(c - b)$, and therefore $\frac{x+u}{y+u} = \frac{c}{b}$, that is, $\frac{BD}{DC} = \frac{AB}{AC}$, which implies that AI bisects $\angle BAC$, and that $BD = \frac{ac}{b+c}$. Adding the two equations mentioned before, we finally obtain that $az = u(b + c)$, which rewrites as $\frac{AI}{ID} = \frac{AB}{BD}$. This implies that BI bisects $\angle ABC$. Thus, I is the incenter of triangle ABC. $\square$

But can we prove Thebault's theorem using Casey? S. Gueron [SG] says yes!

Delta 56. (Thebault [VT]) Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the incenter of triangle ABC, and let P be the center of the circle which touches DC, DA, and (internally) the circumcircle of ABC, and let Q be the center of the circle which touches DB, DA, and (internally) the circumcircle of ABC. Then, the points P, I, Q are collinear.

Delta 57. (Jean-Pierre Ehrmann and Cosmin Pohoață, MathLinks Contest 2008) Let P be an arbitrary point on the side BC of a given triangle ABC with circumcircle Γ. Let $\mathcal{T}_A^b$ be the circle tangent to AP, PB, and internally to Γ, and let $\mathcal{T}_A^c$ be the circle tangent to AP, PC, and internally to Γ. Then, the circles $\mathcal{T}_A^b$ and $\mathcal{T}_A^c$ are congruent if and only if AP passes through the Nagel point of triangle ABC.

Delta 58. (Lev Emelyanov [LE]) Let P be a point in the interior of a given triangle ABC. Denote by A_1, B_1, C_1 the intersections of AP, BP, CP with the sidelines BC, CA, and AB, respectively (in other words, the triangle $A_1B_1C_1$ is the cevian triangle of P with respect to ABC). Construct the three circles (O_1), (O_2) and (O_3) outside the triangle which are tangent to the sides of ABC at A_1, B_1, C_1, and also tangent to the circumcircle of ABC. Then, the circle tangent externally to these three circles is also tangent to teh incircle of triangle ABC.

It is impossible to be a mathematician without being a poet in soul. - S. Kovalevskaya

A long time ago an older and well-known number theorist made some disparaging remarks about Paul Erdős's work. You admire Erdős's contributions to mathematics as much as I do, and I felt annoyed when the older mathematician flatly and definitively stated that all of Erdős's work could be "reduced" to a few tricks which Erdős repeatedly relied on in his proofs. What the number theorist did not realize is that other mathematicians, even the very best, also rely on a few tricks which they use over and over. **Take Hilbert.** The second volume of Hilbert's collected papers contains Hilbert's papers in invariant theory. I have made a point of reading some of these papers with care. It is sad to note that some of Hilbert's beautiful results have been completely forgotten. But on reading the proofs of Hilbert's striking and deep theorems in invariant theory, it was surprising to verify that Hilbert's proofs relied on the same few tricks. **Even Hilbert had only a few tricks!**

- G-C Rota, *Ten Lessons I Wish I Had Been Taught*

5.1. **'T'rigonometric Substitutions.** If you are faced with an integral that contains square root expressions such as

$$\int \sqrt{1-x^2}\,dx, \quad \int \sqrt{1+y^2}\,dy, \quad \int \sqrt{z^2-1}\,dz$$

then trigonometric substitutions such as $x = \sin t$, $y = \tan t$, $z = \sec t$ are very useful. We will learn that making a suitable trigonometric substitution simplifies the given inequality.

Epsilon 55. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 56. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

Epsilon 57. (Korea 1998) Let x, y, z be the positive reals with $x + y + z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Since the function $f(t) = \frac{1}{\sqrt{1+t^2}}$ is not concave on $\mathbb{R}^+$, we cannot apply Jensen's Inequality directly. However, the function $f(\tan\theta)$ is concave on $\left(0, \frac{\pi}{2}\right)$!

Proposition 5.1. *In any acute triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Proof. Since $\cos x$ is concave on $\left(0, \frac{\pi}{2}\right)$, it's a direct consequence of Jensen's Inequality. ☐

We note that the function $\cos x$ is not concave on $(0, \pi)$. In fact, it's convex on $\left(\frac{\pi}{2}, \pi\right)$. One may think that the inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$ doesn't hold for any triangles. However, it's known that it holds for all triangles.

Proposition 5.2. *In any triangle ABC, we have*

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

First Proof. It follows from $\pi - C = A + B$ that

$$\cos C = -\cos(A+B) = -\cos A\cos B + \sin A\sin B$$

or

$$3 - 2(\cos A + \cos B + \cos C) = (\sin A - \sin B)^2 + (\cos A + \cos B - 1)^2 \geq 0.$$

□

Second Proof. Let $BC = a$, $CA = b$, $AB = c$. Use The Cosine Law to rewrite the given inequality in the terms of a, b, c :

$$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} \leq \frac{3}{2}.$$

Clearing denominators, this becomes

$$3abc \geq a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2),$$

which is equivalent to $abc \geq (b+c-a)(c+a-b)(a+b-c)$. □

We remind that the geometric inequality $R \geq 2r$ is equivalent to the *algebraic* inequality $abc \geq (b+c-a)(c+a-b)(a+b-c)$. We now find that, in the proof of the above theorem, $abc \geq (b+c-a)(c+a-b)(a+b-c)$ is equivalent to the trigonometric inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$. One may ask that

> in any triangles ABC, is there a natural relation between $\cos A + \cos B + \cos C$ and $\frac{R}{r}$, where R and r are the radii of the circumcircle and incircle of ABC?

Theorem 5.1. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have*

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Proof. Use the algebraic identity

$$a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2) = 2abc + (b+c-a)(c+a-b)(a+b-c).$$

We leave the details for the readers. □

Delta 59. (China 2004) Let ABC be a triangle with $BC = a, CA = b, AB = c$. Prove that, for all $x \geq 0$,

$$a^x \cos A + b^x \cos B + c^x \cos C \leq \frac{1}{2}\left(a^x + b^x + c^x\right).$$

Delta 60. *(a) Let p, q, r be the positive real numbers such that $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there exists an acute triangle ABC such that $p = \cos A$, $q = \cos B$, $r = \cos C$.*
(b) Let $p, q, r \geq 0$ with $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there are $A, B, C \in \left[0, \frac{\pi}{2}\right]$ with $p = \cos A$, $q = \cos B$, $r = \cos C$, and $A + B + C = \pi$.

Epsilon 58. (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.

Life is good for only two things, discovering mathematics and teaching mathematics.
- S. Poisson

5.2. **'A'lgebraic Substitutions.** We know that some inequalities in triangle geometry can be treated by the Ravi substitution and trigonometric substitutions. We can also transform the given inequalities into easier ones through some clever algebraic substitutions.

Epsilon 59. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 60. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Epsilon 61. (Korea 1998) Let x, y, z be the positive reals with $x+y+z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

We now prove a classical theorem in various ways.

Proposition 5.3. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 4. *After the substitution $x = b+c,\ y = c+a,\ z = a+b$, it becomes*

$$\sum_{\text{cyclic}}\frac{y+z-x}{2x}\geq\frac{3}{2}\quad\text{or}\quad\sum_{\text{cyclic}}\frac{y+z}{x}\geq 6,$$

which follows from The AM-GM Inequality as following:

$$\sum_{\text{cyclic}}\frac{y+z}{x}=\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6.$$

Proof 5. *We make the substitution*

$$x=\frac{a}{b+c},\ y=\frac{b}{c+a},\ z=\frac{c}{a+b}.$$

It follows that

$$\sum_{\text{cyclic}}f(x)=\sum_{\text{cyclic}}\frac{a}{a+b+c}=1,$$

where $f(t)=\frac{t}{1+t}$. Since f is concave on $(0,\infty)$, Jensen's Inequality shows that

$$f\left(\frac{1}{2}\right)=\frac{1}{3}=\frac{1}{3}\sum_{\text{cyclic}}f(x)\leq f\left(\frac{x+y+z}{3}\right)$$

Since f is monotone increasing, it implies that

$$\frac{1}{2}\leq\frac{x+y+z}{3}$$

or

$$\sum_{\text{cyclic}}\frac{a}{b+c}=x+y+z\geq\frac{3}{2}.$$

Proof 6. *As in the previous proof, it suffices to show that*

$$T:=\frac{x+y+z}{3}\geq\frac{1}{2},$$

where we have

$$\sum_{\text{cyclic}}\frac{x}{1+x}=1$$

or equivalently,

$$1 = 2xyz + xy + yz + zx.$$

We apply The AM-GM Inequality to deduce

$$1 = 2xyz + xy + yz + zx \leq 2T^3 + 3T^2$$

It follows that

$$2T^3 + 3T^2 - 1 \geq 0$$

so that

$$(2T-1)(T+1)^2 \geq 0$$

or

$$T \geq \frac{1}{2}.$$

Epsilon 62. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Epsilon 63. *Let a, b, c be positive real numbers satisfying $a + b + c = 1$. Show that*

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{\sqrt{abc}}{c+ab} \leq 1 + \frac{3\sqrt{3}}{4}.$$

Epsilon 64. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

Delta 61. [SL 1993 USA] *Prove that*

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 65. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Epsilon 66. (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + 1.$$

Delta 62. [IMO 1969 USS] *Under the conditions $x_1, x_2 > 0$, $x_1y_1 > {z_1}^2$, and $x_2y_2 > {z_2}^2$, prove the inequality*

$$\frac{8}{(x_1+x_2)(y_1+y_2) - (z_1+z_2)^2} \leq \frac{1}{x_1y_1 - {z_1}^2} + \frac{1}{x_2y_2 - {z_2}^2}.$$

Epsilon 67. [SL 2001] *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1+{x_1}^2} + \frac{x_2}{1+{x_1}^2+{x_2}^2} + \cdots + \frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2} < \sqrt{n}.$$

Delta 63. [LL 1987 FRA] *Given n real numbers $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < 1$, prove that*

$$\left(1 - {t_n}^2\right)\left(\frac{t_1}{(1-{t_1}^2)^2} + \frac{{t_2}^2}{(1-{t_2}^3)^2} + \cdots + \frac{{t_n}^n}{(1-{t_n}^{n+1})^2}\right) < 1.$$

5.3. **'E'stablishing New Bounds.** The following examples give a nice description of the title of this subsection.

Example 10. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

Second Solution. We first use the auxiliary inequality $t^2 \geq 2t-1$ to deduce

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2} \geq 2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1.$$

It now remains to check that

$$2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1 \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

or equivalently

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3.$$

However, The AM-GM Inequality shows that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3\left(\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}\right)^{\frac{1}{3}}=3.$$

□

Proposition 5.4. (**Nesbitt**) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 7. *From $\left(\frac{a}{b+c}-\frac{1}{2}\right)^2 \geq 0$, we deduce that*

$$\frac{a}{b+c} \geq \frac{1}{4}\cdot\frac{\frac{8a}{b+c}-1}{\frac{a}{b+c}+1}=\frac{8a-b-c}{4(a+b+c)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{a}{b+c} \geq \sum_{\text{cyclic}}\frac{8a-b-c}{4(a+b+c)}=\frac{3}{2}.$$

Proof 8. *We claim that*

$$\frac{a}{b+c} \geq \frac{3a^{\frac{3}{2}}}{2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)} \quad \textit{or} \quad 2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right) \geq 3a^{\frac{1}{2}}(b+c).$$

The AM-GM inequality gives $a^{\frac{3}{2}}+b^{\frac{3}{2}}+b^{\frac{3}{2}} \geq 3a^{\frac{1}{2}}b$ and $a^{\frac{3}{2}}+c^{\frac{3}{2}}+c^{\frac{3}{2}} \geq 3a^{\frac{1}{2}}c$. Adding these two inequalities yields $2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right) \geq 3a^{\frac{1}{2}}(b+c)$, as desired. Therefore, we have

$$\sum_{\text{cyclic}}\frac{a}{b+c} \geq \frac{3}{2}\sum_{\text{cyclic}}\frac{a^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}}=\frac{3}{2}.$$

Epsilon 68. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Some cyclic inequalities can be established by finding some clever bounds. Suppose that we want to establish that

$$\sum_{\text{cyclic}} F(x,y,z) \geq C$$

for some given constant $C \in \mathbb{R}$. Whenever we have a function G such that, for all $x, y, z > 0$,

$$F(x,y,z) \geq G(x,y,z)$$

and

$$\sum_{\text{cyclic}} G(x,y,z) = C,$$

we then deduce that

$$\sum_{\text{cyclic}} F(x,y,z) \geq \sum_{\text{cyclic}} G(x,y,z) = C.$$

For instance, if a function F satisfies the inequality

$$F(x,y,z) \geq \frac{x}{x+y+z}$$

for all $x, y, z > 0$, then F obeys the inequality

$$\sum_{\text{cyclic}} F(x,y,z) \geq 1.$$

Epsilon 69. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Epsilon 70. [IMO 2005/3 KOR] *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Epsilon 71. (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Epsilon 72. (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

Epsilon 73. (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Delta 64. [SL 1996 SVN] *Let a, b, and c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1.$$

Delta 65. [SL 1971 YUG] *Prove the inequality*

$$\frac{a_1+a_3}{a_1+a_2} + \frac{a_2+a_4}{a_2+a_3} + \frac{a_3+a_1}{a_3+a_4} + \frac{a_4+a_2}{a_4+a_1} \geq 4$$

where $a_1, a_2, a_3, a_4 > 0$.

There is a simple way to find new bounds for given differentiable functions. We begin to show that every supporting lines are tangent lines in the following sense.

Proposition 5.5. (The Characterization of Supporting Lines) Let f be a real valued function. Let $m, n \in \mathbb{R}$. Suppose that

(1) $f(\alpha) = m\alpha + n$ for some $\alpha \in \mathbb{R}$,
(2) $f(x) \geq mx + n$ for all x in some interval (ϵ_1, ϵ_2) including α, and
(3) f is differentiable at α.

Then, the supporting line $y = mx + n$ of f is the tangent line of f at $x = \alpha$.

Proof. Let us define a function $F : (\epsilon_1, \epsilon_2) \longrightarrow \mathbb{R}$ by $F(x) = f(x) - mx - n$ for all $x \in (\epsilon_1, \epsilon_2)$. Then, F is differentiable at α and we obtain $F'(\alpha) = f'(\alpha) - m$. By the assumption (1) and (2), we see that F has a local minimum at α. So, the first derivative theorem for local extreme values implies that $0 = F'(\alpha) = f'(\alpha) - m$ so that $m = f'(\alpha)$ and that $n = f(\alpha) - m\alpha = f(\alpha) - f'(\alpha)\alpha$. It follows that $y = mx + n = f'(\alpha)(x - \alpha) + f(\alpha)$. $\square$

Proposition 5.6. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 9. *We may normalize to $a + b + c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}} f(a) \geq \frac{3}{2}$$

or

$$\frac{f(a) + f(b) + f(c)}{3} \geq f\left(\frac{1}{3}\right),$$

where where $f(x) = \frac{x}{1-x}$. The equation of the tangent line of f at $x = \frac{1}{3}$ is given by $y = \frac{9x-1}{4}$. We claim that the inequality

$$f(x) \geq \frac{9x-1}{4}$$

holds for all $x \in (0, 1)$. However, it immediately follows from the equality

$$f(x) - \frac{9x-1}{4} = \frac{(3x-1)^2}{4(1-x)}.$$

Now, we conclude that

$$\sum_{\text{cyclic}} \frac{a}{1-a} \geq \sum_{\text{cyclic}} \frac{9a-1}{4} = \frac{9}{4} \sum_{\text{cyclic}} a - \frac{3}{4} = \frac{3}{2}.$$

The above argument can be generalized. If a function f has a supporting line at some point on the graph of f, then f satisfies Jensen's Inequality in the following sense.

Theorem 5.2. (Supporting Line Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose that $\alpha \in [a, b]$ and $m \in \mathbb{R}$ satisfy

$$f(x) \geq m(x - \alpha) + f(\alpha)$$

for all $x \in [a, b]$. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. Then, the following inequality holds

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\alpha)$$

for all $x_1, \cdots, x_n \in [a, b]$ such that $\alpha = \omega_1 x_1 + \cdots + \omega_n x_n$. In particular, we obtain

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{s}{n}\right),$$

where $x_1, \cdots, x_n \in [a, b]$ with $x_1 + \cdots + x_n = s$ for some $s \in [na, nb]$.

Proof.

$$\begin{aligned}
& \omega_1 f(x_1) + \cdots + \omega_n f(x_n) \\
\geq\ & \omega_1[m(x_1 - \alpha) + f(\alpha)] + \cdots + \omega_1[m(x_n - \alpha) + f(\alpha)] \\
=\ & f(\alpha).
\end{aligned}$$

□

We can apply the supporting line inequality to deduce Jensen's inequality for differentiable functions.

Lemma 5.1. *Let $f : (a, b) \longrightarrow \mathbb{R}$ be a convex function which is differentiable twice on (a, b). Let $y = l_\alpha(x)$ be the tangent line at $\alpha \in (a, b)$. Then, $f(x) \geq l_\alpha(x)$ for all $x \in (a, b)$. So, the convex function f admits the supporting lines.*

Proof. Let $\alpha \in (a, b)$. We want to show that the tangent line $y = l_\alpha(x) = f'(\alpha)(x - \alpha) + f(\alpha)$ is the supporting line of f at $x = \alpha$ such that $f(x) \geq l_\alpha(x)$ for all $x \in (a, b)$. However, by Taylor's Theorem, we can find a real number θ_x between α and x such that

$$f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{f''(\theta_x)}{2}(x - \alpha)^2 \geq f(\alpha) + f'(\alpha)(x - \alpha).$$

□

Theorem 5.3. (The Weighted Jensen's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous convex function which is differentiable twice on (a, b). Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a, b]$,

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

First Proof. By the continuity of f, we may assume that $x_1, \cdots, x_n \in (a, b)$. Now, let $\mu = \omega_1\, x_1 + \cdots + \omega_n\, x_n$. Then, $\mu \in (a, b)$. By the above lemma, f has the tangent line $y = l_\mu(x) = f'(\mu)(x - \mu) + f(\mu)$ at $x = \mu$ satisfying $f(x) \geq l_\mu(x)$ for all $x \in (a, b)$. Hence, the supporting line inequality shows that

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1 f(\mu) + \cdots + \omega_n f(\mu) = f(\mu) = f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

□

Non-convex functions can be convex locally and have supporting lines at some points. This means that the supporting line inequality is a powerful tool because we can also produce Jensen-type inequalities for non-convex functions.

Epsilon 74. (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1 + x^2} + \frac{1}{1 + y^2} + \frac{1}{1 + z^2} \leq \frac{27}{10}.$$

Epsilon 75. (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b + c - a)^2}{(b + c)^2 + a^2} + \frac{(c + a - b)^2}{(c + a)^2 + b^2} + \frac{(a + b - c)^2}{(a + b)^2 + c^2} \geq \frac{3}{5}.$$

Any good idea can be stated in fifty words or less. - S. Ulam

Mathematicians do not study objects, but relations between objects.

- H. Poincaré

6.1. **Homogenizations.** Many inequality problems come with constraints such as $ab = 1$, $xyz = 1$, $x + y + z = 1$. A non-homogeneous symmetric inequality can be transformed into a homogeneous one. Then we apply two powerful theorems: Schur's Inequality and Muirhead's Theorem. We begin with a simple example.

Example 11. (Hungary, 1996) Let a and b be positive real numbers with $a+b = 1$. Prove that

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{1}{3}.$$

Solution. Using the condition $a+b = 1$, we can reduce the given inequality to homogeneous one:

$$\frac{1}{3} \leq \frac{a^2}{(a+b)(a+(a+b))} + \frac{b^2}{(a+b)(b+(a+b))}$$

or

$$a^2b + ab^2 \leq a^3 + b^3,$$

which follows from

$$(a^3 + b^3) - (a^2b + ab^2) = (a-b)^2(a+b) \geq 0.$$

The equality holds if and only if $a = b = \frac{1}{2}$. □

Theorem 6.1. *Let a_1, a_2, b_1, b_2 be positive real numbers such that $a_1 + a_2 = b_1 + b_2$ and $max(a_1, a_2) \geq max(b_1, b_2)$. Let x and y be nonnegative real numbers. Then, we have*

$$x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2} + x^{b_2}y^{b_1}.$$

Proof. Without loss of generality, we can assume that $a_1 \geq a_2, b_1 \geq b_2, a_1 \geq b_1$. If x or y is zero, then it clearly holds. So, we assume that both x and y are nonzero. It follows from $a_1 + a_2 = b_1 + b_2$ that $a_1 - a_2 = (b_1 - a_2) + (b_2 - a_2)$. It's easy to check

$$\begin{aligned}
& x^{a_1}y^{a_2} + x^{a_2}y^{a_1} - x^{b_1}y^{b_2} - x^{b_2}y^{b_1} \\
= \; & x^{a_2}y^{a_2}\left(x^{a_1-a_2} + y^{a_1-a_2} - x^{b_1-a_2}y^{b_2-a_2} - x^{b_2-a_2}y^{b_1-a_2}\right) \\
= \; & x^{a_2}y^{a_2}\left(x^{b_1-a_2} - y^{b_1-a_2}\right)\left(x^{b_2-a_2} - y^{b_2-a_2}\right) \\
= \; & \frac{1}{x^{a_2}y^{a_2}}\left(x^{b_1} - y^{b_1}\right)\left(x^{b_2} - y^{b_2}\right) \geq 0.
\end{aligned}$$

□

Remark 6.1. *When does the equality hold in the above theorem?*

We now introduce two summation notations. Let $\mathcal{P}(x, y, z)$ be a three variables function of x, y, z. Let us define

$$\sum_{\text{cyclic}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y)$$

and

$$\sum_{\text{sym}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(x, z, y) + \mathcal{P}(y, x, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y) + \mathcal{P}(z, y, x).$$

Here, we have some examples:

$$\sum_{\text{cyclic}} x^3y = x^3y + y^3z + z^3x, \quad \sum_{\text{sym}} x^3 = 2(x^3 + y^3 + z^3),$$

$$\sum_{\text{sym}} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y, \quad \sum_{\text{sym}} xyz = 6xyz$$

Example 12. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

Third Solution. We break the homogeneity. After the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, it becomes

$$a^2 + b^2 + c^2 \geq a + b + c.$$

Using the constraint $abc = 1$, we now impose the homogeneity to this as follows:

$$a^2 + b^2 + c^2 \geq (abc)^{\frac{1}{3}} (a + b + c).$$

After setting $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it then becomes

$$x^6 + y^6 + z^6 \geq x^4yz + xy^4z + xyz^4.$$

We now deduce

$$\sum_{\text{cyclic}} x^6 = \sum_{\text{cyclic}} \frac{x^6 + y^6}{2} \geq \sum_{\text{cyclic}} \frac{x^4y^2 + x^2y^4}{2} = \sum_{\text{cyclic}} x^4 \left(\frac{y^2 + z^2}{2} \right) \geq \sum_{\text{cyclic}} x^4yz.$$

□

Epsilon 76. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x+y+z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Epsilon 77. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

6.2. **Schur and Muirhead.**

Theorem 6.2. (Schur's Inequality) Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. $\square$

Remark 6.2. *When does the equality hold in Schur's Inequality?*

Delta 66. *Disprove the following proposition: for all $a, b, c, d \geq 0$ and $r > 0$, we have*

$$a^r(a-b)(a-c)(a-d)+b^r(b-c)(b-d)(b-a)+c^r(c-a)(c-c)(a-d)+d^r(d-a)(d-b)(d-c) \geq 0.$$

Delta 67. [LL 1971 HUN] *Let a, b, c, d, e be real numbers. Prove the expression*

$$\begin{aligned} & (a-b)(a-c)(a-d)(a-e)+(b-a)(b-c)(b-d)(b-e) \\ + \quad & (c-a)(c-b)(c-d)(c-e)+(d-a)(d-b)(d-c)(a-e) \\ + \quad & (e-a)(e-b)(e-c)(e-d) \end{aligned}$$

is nonnegative.

The following special case of Schur's Inequality is useful:

$$\sum_{\text{cyclic}} x(x-y)(x-z) \geq 0 \;\Leftrightarrow\; 3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y \;\Leftrightarrow\; \sum_{\text{sym}} xyz + \sum_{\text{sym}} x^3 \geq 2\sum_{\text{sym}} x^2y.$$

Epsilon 78. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Epsilon 79. *Let $t \in (0, 3]$. For all $a, b, c \geq 0$, we have*

$$(3-t) + t(abc)^{\frac{2}{t}} + \sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

Epsilon 80. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 81. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Epsilon 82. (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Delta 68. [TZ, p.142] Prove that for any acute triangle ABC,

$$\cot^3 A + \cot^3 B + \cot^3 C + 6\cot A\cot B\cot C \geq \cot A + \cot B + \cot C.$$

Delta 69. [IN, p.103] Let a, b, c be the lengths of a triangle. Prove that

$$a^2b + a^2c + b^2c + b^2a + c^2a + c^2b > a^3 + b^3 + c^3 + 2abc.$$

Delta 70. (Surányi's Inequality) Show that, for all $x_1, \cdots, x_n \geq 0$,

$$(n-1)\left({x_1}^n + \cdots {x_n}^n\right) + nx_1\cdots x_n \geq (x_1 + \cdots + x_n)\left({x_1}^{n-1} + \cdots {x_n}^{n-1}\right).$$

Epsilon 83. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be non-negative real numbers such that

$$a_1 \geq a_2 \geq a_3,\ b_1 \geq b_2 \geq b_3,\ a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2,\ a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

(In this case, we say that the vector $\mathrm{a} = (a_1, a_2, a_3)$ majorizes the vector $\mathrm{b} = (b_1, b_2, b_3)$ and write $\mathrm{a} \succ \mathrm{b}$.) For all positive real numbers x, y, z, we have

$$\sum_{\mathrm{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\mathrm{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Remark 6.3. *The equality holds if and only if $x = y = z$. However, if we allow $x = 0$ or $y = 0$ or $z = 0$, then one may easily check that the equality holds (after assuming $a_1, a_2, a_3 > 0$ and $b_1, b_2, b_3 > 0$) if and only if*

$$x = y = z \ \text{ or } \ x = y,\ z = 0 \ \text{ or } \ y = z,\ x = 0 \ \text{ or } \ z = x,\ y = 0.$$

We can apply Muirhead's Theorem to establish Nesbitt's Inequality.

Proposition 6.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 10. *Clearing the denominators of the inequality, it becomes*

$$2 \sum_{\mathrm{cyclic}} a(a+b)(a+c) \geq 3(a+b)(b+c)(c+a)$$

or

$$\sum_{\mathrm{sym}} a^3 \geq \sum_{\mathrm{sym}} a^2 b.$$

Epsilon 84. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Epsilon 85. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Epsilon 86. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Epsilon 87. [SC] If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that

$$\frac{r_a r_b}{m_a m_b} + \frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} \geq 3.$$

We now offer a criterion for the homogeneous symmetric polynomial inequalities with degree 3. It is a direct consequence of Schur's Inequality and Muirhead's Theorem.

Epsilon 88. *Let $\mathcal{P}(u, v, w) \in \mathbb{R}[u, v, w]$ be a homogeneous symmetric polynomial with degree 3. Then the following two statements are equivalent.*

(a) $\mathcal{P}(1,1,1),\ \mathcal{P}(1,1,0),\ \mathcal{P}(1,0,0) \geq 0$.
(b) $\mathcal{P}(x,y,z) \geq 0$ *for all* $x, y, z \geq 0$.

Example 13. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution. Using $x+y+z=1$, we convert the given inequality to the equivalent form:

$$0 \leq (xy+yz+zx)(x+y+z) - 2xyz \leq \frac{7}{27}(x+y+z)^3.$$

Let us define $\mathcal{L}(u,v,w), \mathcal{R}(u,v,w) \in \mathbb{R}[u,v,w]$ by

$$\mathcal{L}(u,v,w) = (uv+vw+wu)(u+v+w) - 2uvw,$$

$$\mathcal{R}(u,v,w) = \frac{7}{27}(u+v+w)^3 - (uv+vw+wu)(u+v+w) + 2uvw.$$

However, one may easily check that

$$\mathcal{L}(1,1,1) = 7,\ \mathcal{L}(1,1,0) = 2,\ \mathcal{L}(1,0,0) = 0,$$

$$\mathcal{R}(1,1,1) = 0,\ \mathcal{R}(1,1,0) = \frac{2}{27},\ \mathcal{R}(1,0,0) = \frac{7}{27}.$$

□

In other words, we don't need to employ Schur's Inequality and Muirhead's Theorem to get a straightforward result.

Delta 71. (M. S. Klamkin) Determine the maximum and minimum values of

$$x^2+y^2+z^2+\lambda xyz$$

where $x+y+z=1$, $x,y,z \geq 0$, and λ is a given constant.

Delta 72. (W. Janous) Let $x,y,z \geq 0$ with $x+y+z=1$. For fixed real numbers $a \geq 0$ and b, determine the maximum $c = c(a,b)$ such that

$$a + bxyz \geq c(xy+yz+zx).$$

As a corollary of the above criterion, we obtain the following proposition for homogeneous symmetric polynomial inequalities for the triangles :

Theorem 6.3. (K. B. Stolarsky) Let $\mathcal{P}(u,v,w)$ be a real symmetric form of degree 3. If we have

$$\mathcal{P}(1,1,1),\ \mathcal{P}(1,1,0),\ \mathcal{P}(2,1,1) \geq 0,$$

then $\mathcal{P}(a,b,c) \geq 0$, where a,b,c are the lengths of the sides of a triangle.

Proof. Employ The Ravi Substitution together with the above crieterion. We leave the details for the readers. For an alternative proof, see [KS]. □

Delta 73. (China 2007) Let a,b,c be the lengths of a triangle with $a+b+c=3$. Determine the minimum value of

$$a^2+b^2+c^2+\frac{4abc}{3}.$$

As noted in [KS], applying Stolarsky's Crieterion, we obtain various cubic inequalities in triangle geometry.

Example 14. *Let a,b,c be the lengths of the sides of a triangle. Let s be the semiperimeter of the triangle. Then, the following inequalities holds.*

(a) $4(ab+bc+ca) > (a+b+c)^2 \geq 3(ab+bc+ca)$
(b) [DM] $a^2+b^2+c^2 \geq \frac{36}{35}\left(s^2+\frac{abc}{s}\right)$
(c) [AP] $abc \geq 8(s-a)(s-b)(s-c)$
(d) [EC] $8abc \geq (a+b)(b+c)(c+a)$
(e) [AP] $8(a^3+b^3+c^3) \geq 3(a+b)(b+c)(c+a)$
(f) [MC] $2(a+b+c)(a^2+b^2+c^2) \geq 3(a^3+b^3+c^3+3abc)$
(g) $\frac{3}{2}abc \geq a^2(s-a)+b^2(s-b)+c^2(s-c) > abc$
(h) $bc(b+c)+ca(c+a)+ab(a+b) \geq 48(s-a)(s-b)(s-c)$
(i) $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} \geq \frac{9}{s}$
(j) [AN, MP] $2 > \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$

(k) $\frac{9}{2} > \frac{s+a}{b+c} + \frac{s+b}{c+a} + \frac{s+c}{a+b} \geq \frac{15}{4}$
(l) [SR1] $5[ab(a+b) + bc(b+c) + ca(c+a)] - 3abc \geq (a+b+c)^3$

Proof. We only check the left hand side inequality in (j). One may easily check that it is equivalent to the cubic inequality $\mathcal{T}(a,b,c) \geq 0$, where

$$\mathcal{T}(a,b,c) = 2(a+b)(b+c)(c+a) - (a+b)(b+c)(c+a)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Since $\mathcal{T}(1,1,1) = 4$, $\mathcal{T}(1,1,0) = 0$, and $\mathcal{T}(2,1,1) = 6$, the result follows from Stolarsky's Criterion. For alternative proofs, see [BDJMV]. $\square$

6.3. **Normalizations.** In the previous subsections, we transformed non-homogeneous inequalities into homogeneous ones. On the other hand, homogeneous inequalities also can be normalized in various ways. We offer two alternative solutions of the problem 8 by normalizations :

Epsilon 89. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 90. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Epsilon 91. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)\,(ab^2+bc^2+ca^2)}\geq abc+\sqrt[3]{(a^3+abc)\,(b^3+abc)\,(c^3+abc)}$$

Epsilon 92. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$. (a) Determine the least constant C such that the inequality*

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2)\leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

Delta 74. [SL 1991 POL] *Let n be a given integer with $n \geq 2$. Find the maximum value of*

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i+x_j),$$

where $x_1, \cdots, x_n \geq 0$ and $x_1 + \cdots + x_n = 1$.

We close this subsection with another proofs of Nesbitt's Inequality.

Proposition 6.2. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 11. *We may normalize to $a + b + c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}}\frac{a}{b+c}=\sum_{\text{cyclic}} f(a)\geq\frac{3}{2},\ \ \textit{where}\ \ f(x)=\frac{x}{1-x}.$$

Since f is convex on $(0, 1)$, Jensen's Inequality shows that

$$\frac{1}{3}\sum_{\text{cyclic}} f(a)\geq f\left(\frac{a+b+c}{3}\right)=f\left(\frac{1}{3}\right)=\frac{1}{2}\quad\textit{or}\quad\sum_{\text{cyclic}} f(a)\geq\frac{3}{2}.$$

Proof 12. (Cao Minh Quang) Assume that $a + b + c = 1$. Note that $ab + bc + ca \leq \frac{1}{3}(a + b + c)^2 = \frac{1}{3}$. More strongly, we establish that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 3-\frac{9}{2}(ab+bc+ca)$$

or

$$\left(\frac{a}{b+c}+\frac{9a(b+c)}{4}\right)+\left(\frac{b}{c+a}+\frac{9b(c+a)}{4}\right)+\left(\frac{c}{a+b}+\frac{9c(a+b)}{4}\right)\geq 3.$$

The AM-GM inequality shows that

$$\sum_{\text{cyclic}}\frac{a}{b+c}+\frac{9a(b+c)}{4}\geq\sum_{\text{cyclic}} 2\sqrt{\frac{a}{b+c}\cdot\frac{9a(b+c)}{4}}=\sum_{\text{cyclic}} 3a=3.$$

6.4. **Cauchy-Schwarz and Hölder.** We begin with the following famous theorem:

Theorem 6.4. (The Cauchy-Schwarz Inequality) Whenever $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$, we have

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1b_1 + \cdots + a_nb_n)^2.$$

First Proof. Let $A = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$ and $B = \sqrt{{b_1}^2 + \cdots + {b_n}^2}$. In the case when $A = 0$, we get $a_1 = \cdots = a_n = 0$. Thus, the given inequality clearly holds. From now on, we assume that $A, B > 0$. Since the inequality is homogeneous, we may normalize to

$$1 = {a_1}^2 + \cdots + {a_n}^2 = {b_1}^2 + \cdots + {b_n}^2.$$

We now need to to show that

$$|a_1b_1 + \cdots + a_nb_n| \leq 1.$$

Indeed, we deduce

$$|a_1b_1 + \cdots + a_nb_n| \leq |a_1b_1| + \cdots + |a_nb_n| \leq \frac{{a_1}^2 + {b_1}^2}{2} + \cdots + \frac{{a_n}^2 + {b_n}^2}{2} = 1.$$

$\square$

Second Proof. It immediately follows from The Lagrange Identity:

$$\left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right) - \left(\sum_{i=1}^{n} a_ib_i\right)^2 = \sum_{1\leq i<j\leq n} (a_ib_j - a_jb_i)^2.$$

$\square$

Delta 75. [IMO 2003/5 IRL] *Let n be a positive integer and let $x_1 \leq \cdots \leq x_n$ be real numbers. Prove that*

$$\left(\sum_{1\leq i,j\leq n} |x_i - x_j|\right)^2 \leq \frac{2\left(n^2-1\right)}{3} \sum_{1\leq i,j\leq n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \cdots, x_n$ is an arithmetic progression.

Delta 76. (Darij Grinberg) Suppose that $0 < a_1 \leq \cdots \leq a_n$ and $0 < b_1 \leq \cdots \leq b_n$ be real numbers. Show that

$$\frac{1}{4}\left(\sum_{k=1}^{n} a_k\right)^2\left(\sum_{k=1}^{n} b_k\right)^2 > \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) - \left(\sum_{k=1}^{n} a_kb_k\right)^2$$

Delta 77. [LL 1971 AUT] *Let a, b, c be positive real numbers, $0 < a \leq b \leq c$. Prove that for any $x, y, z > 0$ the following inequality holds:*

$$\frac{(a+c)^2}{4ac}(x+y+z)^2 \geq (ax+by+cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).$$

Delta 78. [LL 1987 AUS] *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive real numbers. Prove that*

$$(a_1b_2 + a_1b_3 + a_2b_1 + a_2b_3 + a_3b_1 + a_3b_2)^2 \geq 4\left(a_1a_2 + a_2a_3 + a_3a_1\right)\left(b_1b_2 + b_2b_3 + b_3b_1\right)$$

and show that the two sides of the inequality are equal if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Delta 79. [PF] Let $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$. Suppose that $x \in [0, 1]$. Show that

$$\left(\sum_{i=1}^{n} {a_i}^2 + 2x\sum_{i<j} a_ia_j\right)\left(\sum_{i=1}^{n} {b_i}^2 + 2x\sum_{i<j} b_ib_j\right) \geq \left(\sum_{i=1}^{n} a_ib_i + x\sum_{i\leq j} a_ib_j\right)^2.$$

Delta 80. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\begin{cases} (1)\ \sqrt{(a_1+\cdots+a_n)(b_1+\cdots+b_n)} \geq \sqrt{a_1b_1}+\cdots+\sqrt{a_nb_n}, \\ (2)\ \frac{{a_1}^2}{b_1}+\cdots+\frac{{a_n}^2}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{b_1+\cdots+b_n}, \\ (3)\ \frac{a_1}{{b_1}^2}+\cdots+\frac{a_n}{{b_n}^2} \geq \frac{1}{a_1+\cdots+a_n}\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)^2, \\ (4)\ \frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{a_1b_1+\cdots+a_nb_n}. \end{cases}$$

Delta 81. [SL 1993 USA] *Prove that*

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 93. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1+\cdots+a_n = b_1+\cdots+b_n$. Show that

$$\frac{{a_1}^2}{a_1+b_1}+\cdots+\frac{{a_n}^2}{a_n+b_n} \geq \frac{a_1+\cdots+a_n}{2}.$$

Epsilon 94. *Let $a, b \geq 0$ with $a+b=1$. Prove that*

$$\sqrt{a^2+b}+\sqrt{a+b^2}+\sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Epsilon 95. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

We now apply The Cauchy-Schwarz Inequality to prove Nesbitt's Inequality.

Proposition 6.3. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 13. *Applying The Cauchy-Schwarz Inequality, we have*

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 3^2.$$

It follows that

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b} \geq \frac{9}{2}$$

or

$$3+\sum_{\text{cyclic}} \frac{a}{b+c} \geq \frac{9}{2}.$$

Proof 14. *The Cauchy-Schwarz Inequality yields*

$$\sum_{\text{cyclic}} \frac{a}{b+c} \sum_{\text{cyclic}} a(b+c) \geq \left(\sum_{\text{cyclic}} a\right)^2$$

or

$$\sum_{\text{cyclic}} \frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}.$$

Epsilon 96. (Gazeta Matematicã) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Epsilon 97. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

Epsilon 98. (Andrei Ciupan) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c \geq ab+bc+ca$.

We now illustrate normalization techniques to establish classical theorems. Using the same idea in the proof of The Cauchy-Schwarz Inequality, we find a natural generalization :

Theorem 6.5. *Let $a_{ij}(i,j=1,\cdots,n)$ be positive real numbers. Then, we have*

$$({a_{11}}^n+\cdots+{a_{1n}}^n)\cdots({a_{n1}}^n+\cdots+{a_{nn}}^n)\geq(a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn})^n.$$

Proof. The inequality is homogeneous. We make the normalizations:

$$({a_{i1}}^n+\cdots+{a_{in}}^n)^{\frac{1}{n}}=1$$

or

$${a_{i1}}^n+\cdots+{a_{in}}^n=1,$$

for all $i=1,\cdots,n$. Then, the inequality takes the form

$$a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn}\leq 1$$

or

$$\sum_{i=1}^{n}a_{i1}\cdots a_{in}\leq 1.$$

Hence, it suffices to show that, for all $i=1,\cdots,n$,

$$a_{i1}\cdots a_{in}\leq\frac{1}{n}$$

where ${a_{i1}}^n+\cdots+{a_{in}}^n=1$. To finish the proof, it remains to show the following homogeneous inequality. □

Theorem 6.6. (The AM-GM Inequality) Let $a_1,\cdots,a_n$ be positive real numbers. Then, we have

$$\frac{a_1+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}.$$

Proof. Since it's homogeneous, we may rescale $a_1,\cdots,a_n$ so that $a_1\cdots a_n=1$. [14] We want to show that

$$a_1\cdots a_n=1 \implies a_1+\cdots+a_n\geq n.$$

The proof is by induction on n. If $n=1$, it's trivial. If $n=2$, then we get $a_1+a_2-2=a_1+a_2-2\sqrt{a_1a_2}=(\sqrt{a_1}-\sqrt{a_2})^2\geq 0$. Now, we assume that it holds for some positive integer $n\geq 2$. And let a_1, $\cdots$, a_{n+1} be positive numbers such that $a_1\cdots a_na_{n+1}$=1. We may assume that $a_1\geq 1\geq a_2$. (Why?) It follows that $a_1a_2+1-a_1-a_2=(a_1-1)(a_2-1)\leq 0$ so that $a_1a_2+1\leq a_1+a_2$. Since $(a_1a_2)a_3\cdots a_n=1$, by the induction hypothesis, we have

$$a_1a_2+a_3+\cdots+a_{n+1}\geq n.$$

It follows that $a_1+a_2-1+a_3+\cdots+a_{n+1}\geq n$. □

[14] Set $x_i=\frac{a_i}{(a_1\cdots a_n)^{\frac{1}{n}}}$ $(i=1,\cdots,n)$. Then, we get $x_1\cdots x_n=1$ and it becomes $x_1+\cdots+x_n\geq n$.

We now make simple observation. Let $a, b > 0$ and $m, n \in \mathbf{N}$. Take $x_1 = \cdots = x_m = a$ and $x_{m+1} = \cdots = x_{x_{m+n}} = b$. Applying the AM-GM inequality to $x_1, \cdots, x_{m+n} > 0$, we obtain

$$\frac{ma + nb}{m + n} \geq (a^m b^n)^{\frac{1}{m+n}} \quad \text{or} \quad \frac{m}{m+n}a + \frac{n}{m+n}b \geq a^{\frac{m}{m+n}} b^{\frac{n}{m+n}}.$$

Hence, for all positive rational numbers ω_1 and ω_2 with $\omega_1 + \omega_2 = 1$, we get

$$\omega_1\, a + \omega_2\, b \geq a^{\,\omega_1} b^{\,\omega_2}.$$

We now immediately have

Theorem 6.7. *Let ω_1, $\omega_2 > 0$ with $\omega_1 + \omega_2 = 1$. For all x, $y > 0$, we have*

$$\omega_1\, x + \omega_2\, y \geq x^{\,\omega_1} y^{\,\omega_2}.$$

Proof. We can choose a sequence $a_1, a_2, a_3, \cdots \in (0, 1)$ of rational numbers such that

$$\lim_{n \to \infty} a_n = \omega_1.$$

Set $b_i = 1 - a_i$, where $i \in \mathbb{N}$. Then, $b_1, b_2, b_3, \cdots \in (0, 1)$ is a sequence of rational numbers with

$$\lim_{n \to \infty} b_n = \omega_2.$$

From the previous observation, we have $a_n\, x + b_n\, y \geq x^{a_n} y^{b_n}$. By taking the limits to both sides, we get the result. □

We can extend the above arguments to the n-variables.

Theorem 6.8. (The Weighted AM-GM Inequality) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\,\omega_1} \cdots {x_n}^{\,\omega_n}.$$

Since we now get the weighted version of The AM-GM Inequality, we establish weighted version of The Cauchy-Schwarz Inequality.

Epsilon 99. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

My brain is open. - P. Erdős

7. Convexity and Its Applications

> The art of doing mathematics consists in finding that special case which contains all the germs of generality.
>
> - D. Hilbert

7.1. **Jensen's Inequality.** In the previous section, we deduced the weighted AM-GM inequality from The AM-GM Inequality. We use the same idea to study the following functional inequalities.

Epsilon 100. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.

(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.

(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots +\ y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a, b]$.

(4) For all $k \in \{0, 1, 2, \cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots +\ y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a, b]$.

(5) We have $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a, b]$.

(6) We have $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

Definition 7.1. *A real valued function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be convex if the inequality*

$$\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$$

holds for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

The above proposition says that

Corollary 7.1. (Jensen's Inequality) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous convex function, then for all $x_1, \cdots, x_n \in [a, b]$, we have

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + \cdots +\ x_n}{n}\right).$$

Delta 82. [SL 1998 AUS] *Let $r_1, \cdots, r_n$ be real numbers greater than or equal to 1. Prove that*

$$\frac{1}{r_1+1} + \cdots + \frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 \cdots r_n} + 1}.$$

Corollary 7.2. (The Weighted Jensen's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous convex function. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a, b]$, we have

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

In fact, we can almost drop the continuity of f. As an exercise, show that every convex function on $[a,b]$ is continuous on (a,b). Hence, every convex function on $\mathbb{R}$ is continuous on $\mathbb{R}$.

Corollary 7.3. (The Convexity Criterion I) If a continuous function $f : [a,b] \longrightarrow \mathbb{R}$ satisfies the midpoint convexity

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$$

for all $x, y \in [a,b]$, then the function f is convex on $[a,b]$.

Delta 83. (The Convexity Criterion II) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function which are differentiable twice in (a,b). Show that (1) $f''(x) \geq 0$ for all $x \in (a,b)$ if and only if (2) f is convex on (a,b).

We now present an inductive proof of The Weighted Jensen's Inequality. It turns out that we can completely drop the continuity of f.

Third Proof. It clearly holds for $n = 1, 2$. We now assume that it holds for some $n \in \mathbb{N}$. Let $x_1, \cdots, x_n, x_{n+1} \in [a,b]$ and $\omega_1, \cdots, \omega_{n+1} > 0$ with $\omega_1 + \cdots + \omega_{n+1} = 1$. Since we have the equality

$$\frac{\omega_1}{1-\omega_{n+1}} + \cdots + \frac{\omega_n}{1-\omega_{n+1}} = 1,$$

by the induction hypothesis, we obtain

$$\begin{aligned}
& \omega_1 f(x_1) + \cdots + \omega_{n+1} f(x_{n+1}) \\
= \; & (1-\omega_{n+1}) \left(\frac{\omega_1}{1-\omega_{n+1}} f(x_1) + \cdots + \frac{\omega_n}{1-\omega_{n+1}} f(x_n) \right) + \omega_{n+1} f(x_{n+1}) \\
\geq \; & (1-\omega_{n+1}) f\left(\frac{\omega_1}{1-\omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1-\omega_{n+1}} x_n \right) + \omega_{n+1} f(x_{n+1}) \\
\geq \; & f\left((1-\omega_{n+1}) \left[\frac{\omega_1}{1-\omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1-\omega_{n+1}} x_n \right] + \omega_{n+1} x_{n+1} \right) \\
= \; & f(\omega_1 x_1 + \cdots + \omega_{n+1} x_{n+1}).
\end{aligned}$$

$\square$

7.2. **Power Mean Inequality.** The notion of convexity is one of the most important concepts in analysis. Jensen's Inequality is the most powerful tool in theory of inequalities. We begin with a convexity proof of The Weighted AM-GM Inequality.

Theorem 7.1. (The Weighted AM-GM Inequality) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Proof. It is a straightforward consequence of the concavity of $\ln x$. Indeed, The Weighted Jensen's Inequality shows that

$$\ln(\omega_1\, x_1 + \cdots + \omega_n\, x_n) \geq \omega_1 \ln(x_1) + \cdots + \omega_n \ln(x_n) = \ln({x_1}^{\omega_1} \cdots {x_n}^{\omega_n}).$$

□

The Power Mean Inequality can be proved by exploiting Jensen's inequality in two ways. We begin with two simple lemmas.

Lemma 7.1. *Let a, b, and c be positive real numbers. Let*

$$f(x) = \ln\left(\frac{a^x + b^x + c^x}{3}\right)$$

for all $x \in \mathbb{R}$. Then, we obtain $f'(0) = \ln\ \sqrt[3]{abc}$.

Proof. We compute

$$f'(x) = \frac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x + b^x + c^x}.$$

It follows that

$$f'(0) = \frac{\ln a + \ln b + \ln c}{3} = \ln\ \sqrt[3]{abc}.$$

□

Lemma 7.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Suppose that f is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. Then, the function f is monotone increasing on $\mathbb{R}$.*

Proof. We first show that f is monotone increasing on $[0, \infty)$. By the hypothesis, it remains to show that $f(x) \geq f(0)$ for all $x > 0$. For all $\epsilon \in (0, x)$, we have $f(x) \geq f(\epsilon)$. Since f is continuous at 0, we obtain

$$f(x) \geq \lim_{\epsilon \to 0^+} f(\epsilon) = f(0).$$

Similarly, we find that f is monotone increasing on $(-\infty, 0]$. We now show that f is monotone increasing on $\mathbb{R}$. Let x and y be real numbers with $x > y$. We want to show that $f(x) \geq f(y)$. In case $0 \notin (x, y)$, we get the result by the hypothesis. In case $x \geq 0 \geq y$, it follows that $f(x) \geq f(0) \geq f(y)$. □

Theorem 7.2. (Power Mean inequality for Three Variables) Let a, b, and c be positive real numbers. We define a function $M_{(a,b,c)} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$M_{(a,b,c)}(0) = \sqrt[3]{abc}\,, \quad M_{(a,b,c)}(r) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(a,b,c)}$ is a monotone increasing continuous function.

First Proof. Write $M(r) = M_{(a,b,c)}(r)$. We first establish that the function M is continuous. Since M is continuous at r for all $r \neq 0$, it's enough to show that

$$\lim_{r\to 0} M(r) = \sqrt[3]{abc}.$$

Let $f(x) = \ln\left(\frac{a^x+b^x+c^x}{3}\right)$, where $x \in \mathbb{R}$. Since $f(0) = 0$, the above lemma implies that

$$\lim_{r\to 0}\frac{f(r)}{r} = \lim_{r\to 0}\frac{f(r)-f(0)}{r-0} = f'(0) = \ln\sqrt[3]{abc}\,.$$

Since e^x is a continuous function, this means that

$$\lim_{r\to 0} M(r) = \lim_{r\to 0} e^{\frac{f(r)}{r}} = e^{\ln\sqrt[3]{abc}} = \sqrt[3]{abc}.$$

Now, we show that the function M is monotone increasing. It will be enough to establish that M is monotone increasing on $(0,\infty)$ and monotone increasing on $(-\infty, 0)$. We first show that M is monotone increasing on $(0,\infty)$. Let $x \geq y > 0$. We want to show that

$$\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} \geq \left(\frac{a^y+b^y+c^y}{3}\right)^{\frac{1}{y}}.$$

After the substitution $u = a^y$, $v = a^y$, $w = a^z$, it becomes

$$\left(\frac{u^{\frac{x}{y}}+v^{\frac{x}{y}}+w^{\frac{x}{y}}}{3}\right)^{\frac{1}{x}} \geq \left(\frac{u+v+w}{3}\right)^{\frac{1}{y}}.$$

Since it is homogeneous, we may normalize to $u + v + w = 3$. We are now required to show that

$$\frac{G(u)+G(v)+G(w)}{3} \geq 1,$$

where $G(t) = t^{\frac{x}{y}}$, where $t > 0$. Since $\frac{x}{y} \geq 1$, we find that G is convex. Jensen's inequality shows that

$$\frac{G(u)+G(v)+G(w)}{3} \geq G\left(\frac{u+v+w}{3}\right) = G(1) = 1.$$

Similarly, we may deduce that M is monotone increasing on $(-\infty, 0)$. ☐

We've learned that the convexity of $f(x) = x^\lambda \quad (\lambda \geq 1)$ implies the monotonicity of the power means. Now, we shall show that the convexity of $x\ln x$ also implies The Power Mean Inequality.

Second Proof of the Monotonicity. Write $f(x) = M_{(a,b,c)}(x)$. We use the increasing function theorem. It's enough to show that $f'(x) \geq 0$ for all $x \neq 0$. Let $x \in \mathbb{R} - \{0\}$. We compute

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln f(x)) = -\frac{1}{x^2}\ln\left(\frac{a^x+b^x+c^x}{3}\right) + \frac{1}{x}\,\frac{\frac{1}{3}\left(a^x\ln a + b^x\ln b + c^x\ln c\right)}{\frac{1}{3}(a^x+b^x+c^x)}$$

or

$$\frac{x^2 f'(x)}{f(x)} = -\ln\left(\frac{a^x+b^x+c^x}{3}\right) + \frac{a^x\ln a^x + b^x\ln b^x + c^x\ln c^x}{a^x+b^x+c^x}.$$

To establish $f'(x) \geq 0$, we now need to establish that

$$a^x\ln a^x + b^x\ln b^x + c^x\ln c^x \geq (a^x+b^x+c^x)\ln\left(\frac{a^x+b^x+c^x}{3}\right).$$

Let us introduce a function $f : (0,\infty) \longrightarrow \mathbf{R}$ by $f(t) = t\ln t$, where $t > 0$. After the substitution $p = a^x$, $q = a^y$, $r = a^z$, it becomes

$$f(p) + f(q) + f(r) \geq 3f\left(\frac{p+q+r}{3}\right).$$

Since f is convex on $(0,\infty)$, it follows immediately from Jensen's Inequality. ☐

In particular, we deduce The RMS-AM-GM-HM Inequality for three variables.

Corollary 7.4. *For all positive real numbers a, b, and c, we have*

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Proof. The Power Mean Inequality implies that

$$M_{(a,b,c)}(2) \geq M_{(a,b,c)}(1) \geq M_{(a,b,c)}(0) \geq M_{(a,b,c)}(-1).$$

□

Delta 84. [SL 2004 THA] *Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality*

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

Delta 85. [SL 1998 RUS] *Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Delta 86. [LL 1992 POL] *For positive real numbers a, b, c, define*

$$A = \frac{a+b+c}{3},\ \ G = \sqrt[3]{abc},\ \ H = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Prove that

$$\left(\frac{A}{G}\right)^3 \geq \frac{1}{4}+\frac{3}{4}\cdot\frac{A}{H}.$$

Using the convexity of $x\ln x$ or the convexity of x^λ $(\lambda \geq 1)$, we can also establish the monotonicity of the power means for n positive real numbers.

Theorem 7.3. (The Power Mean Inequality) Let $x_1, \cdots, x_n$ be positive real numbers. The power mean of order r is defined by

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1\cdots x_n}\,,\ \ M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r+\cdots+{x_n}^r}{n}\right)^{\frac{1}{r}}\quad (r \neq 0).$$

Then, the function $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Corollary 7.5. (The Geometric Mean as a Limit) Let $x_1, \cdots, x_n > 0$. Then,

$$\sqrt[n]{x_1\cdots x_n} = \lim_{r\to 0}\left(\frac{{x_1}^r+\cdots+{x_n}^r}{n}\right)^{\frac{1}{r}}.$$

Theorem 7.4. (The RMS-AM-GM-HM Inequality) For all $x_1, \cdots, x_n > 0$, we have

$$\sqrt{\frac{{x_1}^2+\cdots+{x_n}^2}{n}} \geq \frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1\cdots x_n} \geq \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}.$$

Delta 87. [SL 2004 IRL] *Let $a_1, \cdots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, \cdots, A_n$ the sequence of arithmetic means defined by*

$$A_k = \frac{a_1+\cdots+a_k}{k},\ k = 1, \cdots, n.$$

Let G_n be the geometric mean of $A_1, \cdots, A_n$. Prove the inequality

$$n+1 \geq \sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}$$

and establish the cases of equality.

7.3. **Hardy-Littlewood-Pólya Inequality.** We first meet a famous inequality established by the Romanian mathematician T. Popoviciu.

Theorem 7.5. (Popoviciu's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function. For all $x, y, z \in [a, b]$, we have

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right).$$

Proof. We break the symmetry. Since the inequality is symmetric, we may assume that $x \leq y \leq z$.

Case 1. $y \geq \frac{x+y+z}{3}$: The key idea is to make the following geometric observation:

$$\frac{z+x}{2},\ \frac{x+y}{2} \in \left[x,\ \frac{x+y+z}{3}\right].$$

It guarantees the existence of two positive weights $\lambda_1, \lambda_2 \in [0, 1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\lambda_1)\,x + \lambda_1 \frac{x+y+z}{3}, \\ \frac{x+y}{2} = (1-\lambda_2)\,x + \lambda_2 \frac{x+y+z}{3}, \\ \lambda_1 + \lambda_2 = \frac{3}{2}. \end{cases}$$

Now, Jensen's inequality shows that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \\ \leq\ & (1-\lambda_2)\,f(x) + \lambda_2 f\left(\frac{x+y+z}{3}\right) + \frac{f(y)+f(z)}{2} + (1-\lambda_1)\,f(x) + \lambda_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\left(f(x)+f(y)+f(z)\right) + \frac{3}{2} f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

The proof of the second case uses the same idea.

Case 2. $y \leq \frac{x+y+z}{3}$: We make the following geometric observation:

$$\frac{z+x}{2},\ \frac{y+z}{2} \in \left[\frac{x+y+z}{3}, z\right].$$

It guarantees the existence of two positive weights $\mu_1, \mu_2 \in [0, 1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\mu_1)\,z + \mu_1 \frac{x+y+z}{3}, \\ \frac{y+z}{2} = (1-\mu_2)\,z + \mu_2 \frac{x+y+z}{3}, \\ \mu_1 + \mu_2 = \frac{3}{2}. \end{cases}$$

Jensen's inequality implies that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \\ \leq\ & \frac{f(x)+f(y)}{2} + (1-\mu_2)\,f(z) + \mu_2 f\left(\frac{x+y+z}{3}\right) + (1-\mu_1)\,f(z) + \mu_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\left(f(x)+f(y)+f(z)\right) + \frac{3}{2} f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

□

Epsilon 101. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Extending the proof of Popoviciu's Inequality, we can establish a majorization inequality.

Definition 7.2. *We say that a vector* $\mathrm{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *majorizes a vector* $\mathrm{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$ *if we have*

(1) $x_1 \geq \cdots \geq x_n$, $y_1 \geq \cdots \geq y_n$,
(2) $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$ *for all* $1 \leq k \leq n-1$,
(3) $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

In this case, we write $\mathrm{x} \succ \mathrm{y}$.

Theorem 7.6. (The Hardy-Littlewood-Pólya Inequality) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a,b]$. Then, we obtain

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Epsilon 102. *Let ABC be an acute triangle. Show that*

$$\cos A + \cos B + \cos C \geq 1.$$

Epsilon 103. *Let ABC be a triangle. Show that*

$$\tan^2\left(\frac{A}{4}\right) + \tan^2\left(\frac{B}{4}\right) + \tan^2\left(\frac{C}{4}\right) \leq 1.$$

Epsilon 104. *Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.*

Epsilon 105. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$. (a) Determine the least constant C such that the inequality*

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

It's not that I'm so smart, it's just that I stay with problems longer. - A. Einstein

8. Epsilons

God has a transfinite book with all the theorems and their best proofs

- P. Erdős

Epsilon 1. [NS] Let a and b be positive integers such that

$$a^k \mid b^{k+1}$$

for all positive integers k. Show that b is divisible by a.

Solution. Let p be a prime. Our job is to establish the equality

$$ord_p\left(b\right) \geq ord_p\left(a\right).$$

According to the condition that b^{k+1} is divisibly by a^k, we find that the inequality

$$(k+1)ord_p\left(b\right) = ord_p\left(b^{k+1}\right) \geq ord_p\left(a^k\right) = k\ ord_p\left(a\right)$$

or

$$\frac{ord_p\left(b\right)}{ord_p\left(a\right)} \geq \frac{k}{k+1}$$

holds for all positive integers k. Letting $k \to \infty$, we have the estimation

$$\frac{ord_p\left(b\right)}{ord_p\left(a\right)} \geq 1.$$

$\square$

Epsilon 2. [IMO 1972/3 UNK] *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Solution. We want to show that $\mathcal{L} = (2m)!(2n)!$ is divisible by $\mathcal{R} = m!n!(m+n)!$ Let p be a prime. Our job is to establish the inequality

$$ord_p\left(\mathcal{L}\right) \geq ord_p\left(\mathcal{R}\right).$$

or

$$\sum_{k=1}^{\infty}\left(\left\lfloor\frac{2m}{p^k}\right\rfloor + \left\lfloor\frac{2n}{p^k}\right\rfloor\right) \geq \sum_{k=1}^{\infty}\left(\left\lfloor\frac{m}{p^k}\right\rfloor + \left\lfloor\frac{n}{p^k}\right\rfloor + \left\lfloor\frac{m+n}{p^k}\right\rfloor\right).$$

It is an easy job to check the auxiliary inequality

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$$

holds for all real numbers x and y. □

Epsilon 3. *Let* $n \in \mathbb{N}$. *Show that* $\mathcal{L}_n := lcm(1, 2, \cdots, 2n)$ *is divisible by* $\mathcal{R}_n := \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$.

Solution. Let p be a prime. We want to show the inequality

$$ord_p(\mathcal{K}_n) \le ord_p(\mathcal{L}_n).$$

Now, we first compute $ord_p(\mathcal{K}_n)$.

$$ord_p(\mathcal{K}_n) = ord_p\left(\frac{(2n)!}{(n!)^2}\right) = ord_p((2n)!) - 2ord_p(n!) = \sum_{k=1}^{\infty}\left(\left\lfloor\frac{2n}{p^k}\right\rfloor - 2\left\lfloor\frac{n}{p^k}\right\rfloor\right).$$

The key observation is that both $\frac{2n}{p^k}$ and $\frac{n}{p^k}$ vanish for all sufficiently large integer k. Let N denote the the largest integer $N \ge 0$ such that $p^N \le 2n$. The maximality of the exponent $N = ord_p(\mathcal{L}_n)$ guarantees that, whenever $k > N$, both $\frac{2n}{p^k}$ and $\frac{n}{p^k}$ are smaller than 1, so that the term $\lfloor\frac{2n}{p^k}\rfloor - 2\lfloor\frac{n}{p^k}\rfloor$ vanishes. It follows that

$$ord_p(\mathcal{K}_n) = \sum_{k=1}^{N}\left(\left\lfloor\frac{2n}{p^k}\right\rfloor - 2\left\lfloor\frac{n}{p^k}\right\rfloor\right).$$

Since $\lfloor 2x\rfloor - 2\lfloor x\rfloor$ is either 0 or 1 for all $x \in \mathbb{R}$, this gives the estimation

$$ord_p(\mathcal{K}_n) \le \sum_{k=1}^{N} 1 = N.$$

However, since $\mathcal{L}_n$ is the least common multiple of $1, \cdots, 2n$, we see that $ord_p(\mathcal{L}_n)$ is the largest integer $N \ge 0$ such that $p^N \le 2n$ $\square$

Epsilon 4. *Let* $f : \mathbb{N} \to \mathbb{R}^+$ *be a function satisfying the conditions:*
(a) $f(mn) = f(m)f(n)$ *for all positive integers* m *and* n*, and*
(b) $f(n+1) \geq f(n)$ *for all positive integers* n*.*
Then, there is a constant $\alpha \in \mathbb{R}$ *such that* $f(n) = n^\alpha$ *for all* $n \in \mathbb{N}$*.*

Proof. We have $f(1) = 1$. Our job is to show that the function

$$\frac{\ln f(n)}{\ln n}$$

is constant when $n > 1$. Assume to the contrary that

$$\frac{\ln f(m)}{\ln m} > \frac{\ln f(n)}{\ln n}$$

for some positive integers $m, n > 1$. Writing $f(m) = m^x$ and $f(n) = n^y$, we have $x > y$ or

$$\frac{\ln n}{\ln m} > \frac{\ln n}{\ln m} \cdot \frac{y}{x}$$

So, we can pick a positive rational number $\frac{A}{B}$, where $A, B \in \mathbb{N}$, so that

$$\frac{\ln n}{\ln m} > \frac{A}{B} > \frac{\ln n}{\ln m} \cdot \frac{y}{x}.$$

Hence, $m^A < n^B$ and $m^{Ax} > n^{By}$. One the one hand, since f is monotone increasing, the first inequality $m^A < n^B$ means that $f\left(m^A\right) \leq f\left(n^B\right)$. On the other hand, since $f\left(m^A\right) = f(m)^A = m^{Ax}$ and $f\left(n^B\right) = f(n)^B = n^{By}$, the second inequality $m^{Ax} > n^{By}$ means that

$$f\left(m^A\right) = m^{Ax} > n^{By} = f\left(n^B\right)$$

This is a contradiction. $\square$

Epsilon 5. (Putnam 1963/A2) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for all relatively prime m and n. Then, f is the identity function on $\mathbb{N}$.

Proof. Since f is strictly increasing, we find that $f(n+1) \geq f(n) + 1$ for all positive integers n. It follows that $f(n+k) \geq f(n) + k$ for all positive integers n and k. We now determine $p = f(3)$. On the one hand, we obtain

$$f(18) \geq f(15) + 3 \geq f(3)f(5) + 3 \geq f(3)(f(3) + 2) + 3 = p^2 + 2p + 3.$$

On the other hand, we obtain

$$f(18) = f(2)f(9) \leq 2(f(10)-1) = 2f(2)f(5)-2 \leq 4(f(6)-1)-2 = 4f(2)f(3)-6 = 8p-6.$$

Combining these two, we deduce $p^2 + 2p + 3 \leq 8p - 6$ or $(p-3)^2 \leq 0$. So, we have $f(3) = p = 3$.

We now prove that $f\left(2^l + 1\right) = 2^l + 1$ for all positive integers l. Since $f(3) = 3$, it clearly holds for $l = 1$. Assuming that $f\left(2^l + 1\right) = 2^l + 1$ for some positive integer l, we obtain

$$f\left(2^{l+1} + 2\right) = f(2)f\left(2^l + 1\right) = 2\left(2^l + 1\right) = 2^{l+1} + 2.$$

Since f is strictly increasing, this means that $f\left(2^l + k\right) = 2^l + k$ for all $k \in \{1, \cdots, 2^l + 2\}$. In particular, we get $f\left(2^{l+1} + 1\right) = 2^{l+1} + 1$, as desired.

Now, we find that $f(n) = n$ for all positive integers n. It clearly holds for $n = 1, 2$. Let l be a fixed positive integer. We have $f\left(2^l + 1\right) = 2^l + 1$ and $f\left(2^{l+1} + 1\right) = 2^{l+1} + 1$. Since f is strictly increasing, this means that $f\left(2^l + k\right) = 2^l + k$ for all $k \in \{1, \cdots, 2^l + 1\}$. Since it holds for all positive integers l, we conclude that $f(n) = n$ for all $n \geq 3$. This completes the proof. $\square$

Epsilon 6. *Let a, b, c be positive real numbers. Prove the inequality*

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right) \geq (a+b)(b+c)(c+a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Solution. The inequality has the *symmetric* face:

$$\left(1+a^2\right)\left(1+b^2\right) \cdot \left(1+b^2\right)\left(1+c^2\right) \cdot \left(1+c^2\right)\left(1+a^2\right) \geq (a+b)^2(b+c)^2(c+a)^2.$$

Now, the symmetry of this expression gives the *right* approach. We check that, for $x, y > 0$,

$$\left(1+x^2\right)\left(1+y^2\right) \geq (x+y)^2$$

with the equality condition $xy = 1$. However, it immediately follows from the identity

$$\left(1+x^2\right)\left(1+y^2\right) - (x+y)^2 = (1-xy)^2.$$

It is easy to check that the equality in the original inequalty occurs only when $a = b = c = 1$. $\square$

Epsilon 7. (Poland 2006) Let a, b, c be positive real numbers with $ab+bc+ca = abc$. Prove that

$$\frac{a^4+b^4}{ab(a^3+b^3)} + \frac{b^4+c^4}{bc(b^3+c^3)} + \frac{c^4+a^4}{ca(c^3+a^3)} \geq 1.$$

Solution. We first notice that the constraint can be written as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

It is now enough to establish the auxiliary inequality

$$\frac{x^4+y^4}{xy(x^3+y^3)} \geq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

or

$$2\left(x^4+y^4\right) \geq \left(x^3+y^3\right)(x+y),$$

where $x, y > 0$. However, we obtain

$$2\left(x^4+y^4\right) - \left(x^3+y^3\right)(x+y) = x^4+y^4-x^3y-xy^3 = \left(x^3-y^3\right)(x-y) \geq 0.$$

$\square$

Epsilon 8. (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Proof. The left hand side admits the following decomposition

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2}+\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2}+\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2}.$$

We now use the inequality $\frac{\sqrt{x}+\sqrt{y}}{2} \leq \sqrt{\frac{x+y}{2}}$ to deduce

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2} \leq \sqrt{a},$$

$$\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2} \leq \sqrt{b},$$

$$\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2} \leq \sqrt{c}.$$

Adding these three inequalities, we get the result. $\square$

Epsilon 9. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Proof. Since the inequality is symmetric in the three variables, we may assume that $a \leq b \leq c$. We obtain

$$\frac{a}{b+c}\leq\frac{a}{a+b},\quad \frac{b}{c+a}\leq\frac{b}{a+b},\quad \frac{c}{a+b}<1.$$

Adding these three inequalities, we get the result. □

Epsilon 10. (USA 1980) Prove that, for all positive real numbers $a, b, c \in [0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $0 \leq a \leq b \leq c \leq 1$. Our first step is to bring the estimation

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \leq \frac{a}{a+b+1} + \frac{b}{a+b+1} + \frac{c}{a+b+1} \leq \frac{a+b+c}{a+b+1}.$$

It now remains to check that

$$\frac{a+b+c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

or

$$(1-a)(1-b)(1-c) \leq \frac{1-c}{a+b+1}$$

or

$$(1-a)(1-b)(a+b+1) \leq 1.$$

We indeed obtain the estimation

$$(1-a)(1-b)(a+b+1) \leq (1-a)(1-b)(1+a)(1+b) = \left(1-a^2\right)\left(1-b^2\right) \leq 1.$$

$\square$

Epsilon 11. [AE, p. 186] Show that, for all $a, b, c \in [0,1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \le 2.$$

Proof. Since the inequality is symmetric in the three variables, we may begin with the assumption $0 \le a \ge b \ge c \le 1$. We first give term-by-term estimation:

$$\frac{a}{1+bc} \le \frac{a}{1+ab}, \ \frac{b}{1+ca} \le \frac{b}{1+ab}, \ \frac{c}{1+ab} \le \frac{1}{1+ab}.$$

Summing up these three, we reach

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \le \frac{a+b+1}{1+ab}.$$

We now want to show the inequality

$$\frac{a+b+1}{1+ab} \le 2$$

or

$$a+b+1 \le 2+2ab$$

or

$$a+b \le 1+2ab.$$

However, it is immediate that $1+2ab-a-b = ab+(1-a)(1-b)$ is clearly non-negative. $\square$

Epsilon 12. [SL 2006 KOR] *Let a, b, c be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 2.$$

It is clear that the denominators are positive. So, the first inequality is equivalent to

$$\sqrt{a}+\sqrt{b} \geq \sqrt{a+b-c}+\sqrt{c}.$$

or

$$\left(\sqrt{a}+\sqrt{b}\right)^2 \geq \left(\sqrt{a+b-c}+\sqrt{c}\right)^2$$

or

$$\sqrt{ab} \geq \sqrt{c(a+b-c)}$$

or

$$ab \geq c(a+b-c),$$

which immediately follows from $(a-c)(b-c) \geq 0$. Now, we prove the second inequality. Setting $p=\sqrt{a}+\sqrt{b}$ and $q=\sqrt{a}-\sqrt{b}$, we obtain $a-b=pq$ and $p \geq 2\sqrt{c}$. It now becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q} \leq 2.$$

We now apply The Cauchy-Schwartz Inequality to deduce

$$\begin{aligned}
\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 &\leq \left(\frac{c-pq}{\sqrt{c}-q}+\frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q}+\frac{1}{\sqrt{c}+q}\right) \\
&= \frac{2\left(c\sqrt{c}-pq^2\right)}{c-q^2}\cdot\frac{2\sqrt{c}}{c-q^2} \\
&= 4\,\frac{c^2-\sqrt{c}pq^2}{(c-q^2)^2} \\
&\leq 4\,\frac{c^2-2cq^2}{(c-q^2)^2} \\
&\leq 4\,\frac{c^2-2cq^2+q^4}{(c-q^2)^2} \\
&\leq 4.
\end{aligned}$$

We find that the equality holds if and only if $a=b=c$. □

Epsilon 13. *Let* $f(x,y) = xy\left(x^3+y^3\right)$ *for* $x, y \geq 0$ *with* $x+y=2$*. Prove the inequality*

$$f(x,y) \leq f\left(1+\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right) = f\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right).$$

First Solution. We write $(x,y) = (1+\epsilon, 1-\epsilon)$ for some $\epsilon \in (-1,1)$. It follows that

$$\begin{aligned} f(x,y) &= (1+\epsilon)(1-\epsilon)\left((1+\epsilon)^3+(1-\epsilon)^3\right) \\ &= \left(1-\epsilon^2\right)\left(6\epsilon^2+2\right) \\ &= -6\left(\epsilon^2-\frac{1}{3}\right)^2+\frac{8}{3} \\ &\leq \frac{8}{3} \\ &= f\left(1\pm\frac{1}{\sqrt{3}}, 1\mp\frac{1}{\sqrt{3}}\right). \end{aligned}$$

□

Second Solution. The AM-GM Inequality gives

$$f(x,y) = xy(x+y)\left((x+y)^2-3xy\right) = 2xy(4-4xy) \leq \frac{2}{3}\left(\frac{3xy+(4-3xy)}{2}\right)^2 = \frac{8}{3}.$$

□

Epsilon 14. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

First Solution. We may begin with the assumption $a \geq \frac{1}{2} \geq b$. The AM-GM Inequality yields

$$2 + b \geq 1 + (1 + ab) \geq 2\sqrt{1 + ab}$$

with the equality $b = 0$. We next show that

$$3 + a \geq 4\sqrt{a^2 - a + 1}$$

or

$$(3 + a)^2 \geq 16\left(a^2 - a + 1\right)$$

or

$$(15a - 7)(1 - a) \geq 0.$$

Since we have $a \in \left[\frac{1}{2}, 1\right]$, the inequality clearly holds with the equality $a = 1$. Since we have

$$a^2 + b = a^2 - a + 1 = a + (1 - a)^2 = a + b^2$$

we conclude that

$$2\sqrt{a^2 + b} + 2\sqrt{a + b^2} + 2\sqrt{1 + ab} \leq 3 + a + (2 + b) = 6.$$

□

Epsilon 15. (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \le \frac{3\sqrt{3}}{2}.$$

Solution. We observe that the sine function is *not* cocave on $[0, 3\pi]$ and that it is negative on $(\pi, 2\pi)$. Since the inequality is symmetric in the three variables, we may assume that $A \le B \le C$. Observe that $A + B + C = \pi$ and that $3A, 3B, 3C \in [0, 3\pi]$. It is clear that $A \le \frac{\pi}{3} \le C$.

We see that either $3B \in [2\pi, 3\pi)$ or $3C \in (0, \pi)$ is impossible. In the case when $3B \in [\pi, 2\pi)$, we obtain the estimation

$$\sin 3A + \sin 3B + \sin 3C \le 1 + 0 + 1 = 2 < \frac{3\sqrt{3}}{2}.$$

So, we may assume that $3B \in (0, \pi)$. Similarly, in the case when $3C \in [\pi, 2\pi]$, we obtain

$$\sin 3A + \sin 3B + \sin 3C \le 1 + 1 + 0 = 2 < \frac{3\sqrt{3}}{2}.$$

Hence, we also assume $3C \in (2\pi, 3\pi)$. Now, our assmptions become $A \le B < \frac{1}{3}\pi$ and $\frac{2}{3}\pi < C$. After the substitution $\theta = C - \frac{2}{3}\pi$, the trigonometric inequality becomes

$$\sin 3A + \sin 3B + \sin 3\theta \le \frac{3\sqrt{3}}{2}.$$

Since $3A, 3B, 3\theta \in (0, \pi)$ and since the sine function is concave on $[0, \pi]$, Jensen's Inequality gives

$$\sin 3A + \sin 3B + \sin 3\theta \le 3\sin\left(\frac{3A + 3B + 3\theta}{3}\right) = 3\sin\left(\frac{3A + 3B + 3C - 2\pi}{3}\right) = 3\sin\left(\frac{\pi}{3}\right).$$

Under the assumption $A \le B \le C$, the equality occurs only when $(A, B, C) = \left(\frac{1}{9}\pi, \frac{1}{9}\pi, \frac{7}{9}\pi\right)$. $\square$

Epsilon 16. (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n, \; y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right).$$

Proof. We observe that two sequences are similarly ordered in the sense that

$$(x_i - x_j)(y_i - y_j) \geq 0$$

for all $1 \leq i, j \leq n$. Now, the given inequality is an immediate consequence of the identity

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \frac{1}{n} \left(\sum_{i=1}^{n} y_i \right) = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} (x_i - x_j)(y_i - y_j).$$

□

Epsilon 17. (United Kingdom 2002) For all $a, b, c \in (0, 1)$, show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

First Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. Then, we have $\frac{1}{1-a} \geq \frac{1}{1-b} \geq \frac{1}{1-c}$. By Chebyshev's Inequality, The AM-HM Inequality and The AM-GM Inequality, we obtain

$$\begin{aligned}
\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} &\geq \frac{1}{3}(a+b+c)\left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}\right) \\
&\geq \frac{1}{3}(a+b+c)\left(\frac{9}{(1-a)+(1-b)+(1-c)}\right) \\
&= \frac{1}{3}\left(\frac{a+b+c}{3-(a+b+c)}\right) \\
&\geq \frac{1}{3} \cdot \frac{3\sqrt[3]{abc}}{3-3\sqrt[3]{abc}}
\end{aligned}$$

$\square$

Epsilon 18. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

First Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Since the inequality is symmetric in the three variables, we may assume that $x \geq y \geq z$. Observe that $x^2 \geq y^2 \geq z^2$ and $\frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y}$. Chebyshev's Inequality and The AM-HM Inequality offer the estimation

$$\begin{aligned}
\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) \\
&\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{9}{(y+z)+(z+x)+(x+y)}\right) \\
&= \frac{3}{2} \cdot \frac{x^2+y^2+z^2}{x+y+z}.
\end{aligned}$$

Finally, we have $x^2 + y^2 + z^2 \geq \frac{1}{3}(x+y+z)^2 \geq (x+y+z)\sqrt[3]{xyz} = x+y+z$. $\square$

Epsilon 19. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

First Solution. [MEK1] We assume that $x \geq y \geq z \geq 0$ and $y > 0$ (not excluding $z = 0$). Let F denote the left hand side of the inequality. We define

$$\begin{aligned}
A &= (2x + 2y - z)(x - z)(y - z) + z(x + y)^2,\\
B &= \tfrac{1}{4}x(x + y - 2z)(11x + 11y + 2z),\\
C &= (x + y)(x + z)(y + z),\\
D &= (x + y + z)(x + y - 2z) + x(y - z) + y(z - x) + (x - y)^2,\\
E &= \tfrac{1}{4}(x + y)z(x + y + 2z)^2(x + y - 2z)^2.
\end{aligned}$$

It can be verified that

$$C^2(4F - 9) = (x - y)^2[(x + y)(A + B + C) + \frac{1}{2}(x + z)(y + z)D] + E.$$

The right hand side is clearly nonnegative. It becomes an equality only for $x = y = z$ and for $x = y > 0$, $z = 0$. $\square$

Epsilon 20. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1+b_1} + \cdots + \frac{{a_n}^2}{a_n+b_n} \geq \frac{a_1+\cdots+a_n}{2}.$$

First Solution. The key observation is the following identity:

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i+b_i} = \frac{1}{2}\sum_{i=1}^{n} \frac{{a_i}^2+{b_i}^2}{a_i+b_i},$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i+b_i} = \sum_{i=1}^{n} \frac{{b_i}^2}{a_i+b_i},$$

which immediately follows from

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i+b_i} - \sum_{i=1}^{n} \frac{{b_i}^2}{a_i+b_i} = \sum_{i=1}^{n} \frac{{a_i}^2-{b_i}^2}{a_i+b_i} = \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0.$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{1}{2}\sum_{i=1}^{n} \frac{{a_i}^2+{b_i}^2}{a_i+b_i} \geq \frac{a_1+\cdots+a_n+b_1+\cdots+b_n}{4}.$$

It now remains to check the the auxiliary inequality

$$\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2},$$

where $a, b > 0$. Indeed, we have $2\left(a^2+b^2\right) - (a+b)^2 = (a-b)^2 \geq 0$. $\square$

Epsilon 21. *Let x, y, z be positive real numbers. Show the cyclic inequality*
$$\frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x} \leq 1.$$

Solution. We first break the homogeneity. The original inequality can be rewritten as
$$\frac{1}{2+\frac{y}{x}} + \frac{1}{2+\frac{z}{y}} + \frac{1}{2+\frac{x}{z}} \leq 1$$
The key idea is to employ the substitution
$$a = \frac{y}{x},\ b = \frac{z}{y},\ c = \frac{x}{z}.$$
It follows that $abc = 1$. It now admits the symmetry in the variables:
$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1$$
Clearing denominators, it becomes
$$(2+a)(2+b) + (2+b)(2+c) + (2+c)(2+a) \leq (2+a)(2+b)(2+c)$$
or
$$12 + 4(a+b+c) + ab + bc + ca \leq 8 + 4(a+b+c) + 2(ab+bc+ca) + 1$$
or
$$3 \leq ab + bc + ca.$$
Applying The AM-GM Inequality, we obtain $ab + bc + ca \geq 3\,(abc)^{\frac{1}{3}} = 3.$ $\square$

Epsilon 22. *Let x, y, z be positive real numbers with $x + y + z = 3$. Show the cyclic inequality*

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

Proof. We begin with the observation

$$\begin{aligned}
&\frac{x^3-y^3}{x^2+xy+y^2} + \frac{y^3-z^3}{y^2+yz+z^2} + \frac{z^3-x^3}{z^2+zx+x^2} \\
&\qquad = (x-y)+(y-z)+(z-x) \\
&\qquad = 0
\end{aligned}$$

or

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} = \frac{y^3}{x^2+xy+y^2} + \frac{z^3}{y^2+yz+z^2} + \frac{x^3}{z^2+zx+x^2}.$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2} \geq 2.$$

It now remains to check the the auxiliary inequality

$$\frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b}{3},$$

where $a, b > 0$. Indeed, we obtain the equality

$$3\left(a^3+b^3\right) - (a+b)\left(a^2+ab+b^2\right) = 2(a+b)(a-b)^2.$$

We now conclude that

$$\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2} \geq \frac{x+y}{3} + \frac{y+z}{3} + \frac{z+x}{3} = 2.$$

□

Epsilon 23. [SL 1985 CAN] *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{x^2+yz}+\frac{y^2}{y^2+zx}+\frac{z^2}{z^2+xy}\le 2.$$

First Solution. We first break the homogeneity. The original inequality can be rewritten as

$$\frac{1}{1+\frac{yz}{x^2}}+\frac{1}{1+\frac{zx}{y^2}}+\frac{1}{1+\frac{xy}{z^2}}\le 2$$

The key idea is to employ the substitution

$$a=\frac{yz}{x^2},\ b=\frac{zx}{y^2},\ c=\frac{z^2}{xy}.$$

It then follows that $abc = 1$. It now admits the symmetry in the variables:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\le 2$$

Since it is symmetric in the three variables, we may break the symmetry. Let's assume $a \le b, c$. Since it is obvious that $\frac{1}{1+a} < 1$, it is enough to check the estimation

$$\frac{1}{1+b}+\frac{1}{1+c}\le 1$$

or equivalently

$$\frac{2+b+c}{1+b+c+bc}\le 1$$

or equivalently

$$bc \ge 1.$$

However, it follows from $abc = 1$ and from $a \le b, c$ that $a \le 1$ and so that $bc \ge 1$. $\square$

Epsilon 24. [SL 1990 THA] *Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. show that*

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solution. Since the constraint $ab+bc+cd+da = 1$ is *not* symmetric in the variables, we *cannot* consider the case when $a \geq b \geq c \geq d$ only. We first make the observation that

$$a^2+b^2+c^2+d^2 = \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+d^2}{2} + \frac{d^2+a^2}{2} \geq ab+bc+cd+da = 1.$$

Our strategy is to establish the following result. It is *symmetric*.

Let $a, b, c, d \geq 0$ with $a^2+b^2+c^2+d^2 \geq 1$. Then, we obtain

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

We now exploit the symmetry! Since everything is symmetric in the variables, we may assume that $a \geq b \geq c \geq d$. Two applications of Chebyshev's Inequality and one application of The AM-GM Inequality yield

$$\begin{aligned}
& \frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \\
\geq\ & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\left(\frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} + \frac{1}{a+b+c}\right) \\
\geq\ & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\frac{4^2}{(b+c+d)+(c+d+a)+(d+a+b)+(a+b+c)} \\
\geq\ & \frac{1}{4^2}\left(a^2+b^2+c^2+d^2\right)(a+b+c+d)\frac{4^2}{3(a+b+c+d)} \\
=\ & \frac{1}{3}.
\end{aligned}$$

□

Epsilon 25. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

First Solution. Since $abc = 1$, we can make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for some positive real numbers x, y, z.[15] Then, it becomes a well-known symmetric inequality:

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1$$

or

$$xyz \geq (y + z - x)(z + x - y)(x + y - z).$$

□

[15] For example, take $x = 1$, $y = \frac{1}{a}$, $z = \frac{1}{ab}$.

Epsilon 26. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

First Solution. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

However, an application of The Cauchy-Schwarz Inequality gives

$$(y+z+x)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\geq(x+y+z)^2.$$

□

Epsilon 27. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

First Solution. Write $a = y + z$, $b = z + x$, $c = x + y$ for $x, y, z > 0$. It's equivalent to

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 48(x + y + z)xyz,$$

which can be obtained as following :

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 16(yz + zx + xy)^2 \geq 16 \cdot 3(xy \cdot yz + yz \cdot zx + xy \cdot yz).$$

Here, we used the well-known inequalities $p^2+q^2 \geq 2pq$ and $(p+q+r)^2 \geq 3(pq+qr+rp)$. $\square$

Epsilon 28. (Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds.

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

First Proof. After the substitution $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z > 0$, it becomes

$$xy + yz + zx \geq \sqrt{3xyz(x + y + z)},$$

which follows from the identity

$$(xy + yz + zx)^2 - 3xyz(x + y + z) = \frac{(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2}{2}.$$

□

Second Proof. We now present a convexity proof. It is easy to deduce

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F}.$$

Since the function $\tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, Jensen's Inequality implies that

$$\frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F} \geq 3\tan\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sqrt{3}.$$

□

Epsilon 29. (Tsintsifas) Let p, q, r be positive real numbers and let a, b, c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Proof. (V. Pambuccian) By Hadwiger-Finsler Inequality, it suffices to show that

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq \frac{1}{2}\left(a+b+c\right)^2 - \left(a^2+b^2+c^2\right)$$

or

$$\left(\frac{p+q+r}{q+r}\right)a^2 + \left(\frac{p+q+r}{r+p}\right)b^2 + \left(\frac{p+q+r}{p+q}\right)c^2 \geq \frac{1}{2}\left(a+b+c\right)^2$$

or

$$\left((q+r)+(r+p)+(p+q)\right)\left(\frac{1}{q+r}a^2 + \frac{1}{r+p}b^2 + \frac{1}{p+q}c^2\right) \geq \left(a+b+c\right)^2.$$

However, this is a straightforward consequence of The Cauchy-Schwarz Inequality. □

Epsilon 30. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

First Proof. ([LC1], Carlitz) We begin with the following lemma.

Lemma 8.1. *We have*

$$a_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) + b_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + c_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) > 0.$$

Proof. Observe that it's equivalent to

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) > 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2).$$

From Heron's Formula, we find that, for $i = 1, 2$,

$$16F_i{}^2 = (a_i{}^2+b_i{}^2+c_i{}^2)^2 - 2(a_i{}^4+b_i{}^4+c_i{}^4) > 0 \quad \text{or} \quad a_i{}^2+b_i{}^2+c_i{}^2 > \sqrt{2(a_i{}^4 + b_i{}^4 + c_i{}^4)}\,.$$

The Cauchy-Schwarz Inequality implies that

$$(a_1{}^2+b_1{}^2+c_1{}^2)(a_2{}^2+b_2{}^2+c_2{}^2) > 2\sqrt{(a_1{}^4 + b_1{}^4 + c_1{}^4)(a_2{}^4 + b_2{}^4 + c_2{}^4)} \geq 2(a_1{}^2a_2{}^2+b_1{}^2b_2{}^2+c_1{}^2c_2{}^2).$$

$\square$

By the lemma, we obtain

$$L = a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) > 0,$$

Hence, we need to show that

$$L^2 - (16F_1{}^2)(16F_2{}^2) \geq 0.$$

One may easily check the following identity

$$L^2 - (16F_1{}^2)(16F_2{}^2) = -4(UV + VW + WU),$$

where

$$U = b_1{}^2c_2{}^2 - b_2{}^2c_1{}^2,\ V = c_1{}^2a_2{}^2 - c_2{}^2a_1{}^2 \ \text{ and } \ W = a_1{}^2b_2{}^2 - a_2{}^2b_1{}^2.$$

Using the identity

$$a_1{}^2U + b_1{}^2V + c_1{}^2W = 0 \ \text{ or } \ W = -\frac{a_1{}^2}{c_1{}^2}U - \frac{b_1{}^2}{c_1{}^2}V,$$

one may also deduce that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{4a_1{}^2b_1{}^2 - (c_1{}^2 - a_1{}^2 - b_1{}^2)^2}{4a_1{}^2c_1{}^2}V^2.$$

It follows that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{16F_1{}^2}{4a_1{}^2c_1{}^2}V^2 \leq 0.$$

$\square$

Second Proof. ([LC2], Carlitz) We rewrite it in terms of $a_1, b_1, c_1, a_2, b_2, c_2$:

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) - 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2)$$

$$\geq \sqrt{\left((a_1{}^2 + b_1{}^2 + c_1{}^2)^2 - 2(a_1{}^4 + b_1{}^4 + c_1{}^4)\right)\left((a_2{}^2 + b_2{}^2 + c_2{}^2)^2 - 2(a_2{}^4 + b_2{}^4 + c_2{}^4)\right)}.$$

We employ the following substitutions

$$x_1 = a_1{}^2 + b_1{}^2 + c_1{}^2, x_2 = \sqrt{2}\,a_1{}^2, x_3 = \sqrt{2}\,b_1{}^2, x_4 = \sqrt{2}\,c_1{}^2,$$
$$y_1 = a_2{}^2 + b_2{}^2 + c_2{}^2, y_2 = \sqrt{2}\,a_2{}^2, y_3 = \sqrt{2}\,b_2{}^2, y_4 = \sqrt{2}\,c_2{}^2.$$

We now observe

$$x_1{}^2 > x_2{}^2 + y_3{}^2 + x_4{}^2 \quad \text{and} \quad y_1{}^2 > y_2{}^2 + y_3{}^2 + y_4{}^2.$$

We now apply Aczél's inequality to get the inequality

$$x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \geq \sqrt{(x_1{}^2 - (x_2{}^2 + y_3{}^2 + x_4{}^2))\,(y_1{}^2 - (y_2{}^2 + y_3{}^2 + y_4{}^2))}.$$

□

Epsilon 31. (Aczél's Inequality) If $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$ satisfies the inequality

$$a_1{}^2 \geq a_2{}^2 + \cdots + a_n{}^2 \text{ and } b_1{}^2 \geq b_2{}^2 + \cdots + b_n{}^2,$$

then the following inequality holds.

$$a_1b_1 - (a_2b_2 + \cdots + a_nb_n) \geq \sqrt{(a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2))\left(b_1{}^2 - \left(b_2{}^2 + \cdots + b_n{}^2\right)\right)}$$

Proof. [MV] The Cauchy-Schwarz Inequality shows that

$$a_1b_1 \geq \sqrt{(a_2{}^2 + \cdots + a_n{}^2)(b_2{}^2 + \cdots + b_n{}^2)} \geq a_2b_2 + \cdots + a_nb_n.$$

Then, the above inequality is equivalent to

$$(a_1b_1 - (a_2b_2 + \cdots + a_nb_n))^2 \geq \left(a_1{}^2 - \left(a_2{}^2 + \cdots + a_n{}^2\right)\right)\left(b_1{}^2 - \left(b_2{}^2 + \cdots + b_n{}^2\right)\right).$$

In case $a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2) = 0$, it's trivial. Hence, we now assume that $a_1{}^2 - (a_2{}^2 + \cdots + a_n{}^2) > 0$. The main trick is to think of the following quadratic polynomial

$$\mathcal{P}(x) = (a_1x - b_1)^2 - \sum_{i=2}^{n}(a_ix - b_i)^2 = \left(a_1{}^2 - \sum_{i=2}^{n} a_i{}^2\right)x^2 + 2\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)x + \left(b_1{}^2 - \sum_{i=2}^{n} b_i{}^2\right).$$

We now observe that

$$\mathcal{P}\left(\frac{b_1}{a_1}\right) = -\sum_{i=2}^{n}\left(a_i\left(\frac{b_1}{a_1}\right) - b_i\right)^2.$$

Since $\mathcal{P}\left(\frac{b_1}{a_1}\right) \leq 0$ and since the coefficient of x^2 in the quadratic polynomial P is positive, $\mathcal{P}$ should have at least one real root. Therefore, $\mathcal{P}$ has nonnegative discriminant. It follows that

$$\left(2\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)\right)^2 - 4\left(a_1{}^2 - \sum_{i=2}^{n} a_i{}^2\right)\left(b_1{}^2 - \sum_{i=2}^{n} b_i{}^2\right) \geq 0.$$

$\square$

Epsilon 32. *If A, B, C, X, Y, Z denote the magnitudes of the corresponding angles of triangles ABC, and XYZ, respectively, then*

$$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y \geq 2.$$

Proof. By the Cosine Law in triangle ABC, we have $\cos A = (b^2 + c^2 - a^2)/2bc$. On the other hand, since $\sin A = 2S/bc$, we deduce that

$$\cot A = \cos A : \sin A = \frac{b^2 + c^2 - a^2}{2bc} : \frac{2S}{bc} = \frac{b^2 + c^2 - a^2}{4S}.$$

Analogously, we have that $\cot B = (c^2 + a^2 - b^2)/4S$, and so,

$$\cot A + \cot B = \frac{b^2 + c^2 - a^2}{4S} + \frac{c^2 + a^2 - b^2}{4S} = \frac{2c^2}{4S} = \frac{c^2}{2S}.$$

Now since $\cot Z = (x^2 + y^2 - z^2)/4T$, it follows that

$$\begin{aligned} & \cot A \cot Z + \cot B \cot Z = (\cot A + \cot B) \cdot \cot Z \\ = & \frac{c^2}{2S} \cdot \frac{x^2 + y^2 - z^2}{4T} \\ = & \frac{c^2 \left(x^2 + y^2 - z^2\right)}{8ST}. \end{aligned}$$

Similarly, we obtain

$$\cot B \cot X + \cot C \cot X = \frac{a^2 \left(y^2 + z^2 - x^2\right)}{8ST},$$

and

$$\cot C \cot Y + \cot A \cot Y = \frac{b^2 \left(z^2 + x^2 - y^2\right)}{8ST}.$$

Hence, we conclude that

$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y$

$$\begin{aligned} &= (\cot B \cot X + \cot C \cot X) + (\cot C \cot Y + \cot \\ &= \frac{a^2 \left(y^2 + z^2 \quad x^2\right)}{8ST} + \frac{b^2 \left(z^2 + x^2 \quad y^2\right)}{8ST} + \frac{c^2 \left(x \right.}{} \\ &= \frac{a^2 \left(y^2 + z^2 - x^2\right) + b^2 \left(z^2 + x^2 - y^2\right) + c^2 \left(x^2 \right.}{8ST} \end{aligned}$$

From the Neuberg-Pedoe Inequality, we have

$$a^2 \left(y^2 + z^2 - x^2\right) + b^2 \left(z^2 + x^2 - y^2\right) + c^2 \left(x^2 + y^2 - z^2\right) \geq 16ST,$$

and so

$$\cot A \cot Y + \cot A \cot Z + \cot B \cot Z + \cot B \cot X + \cot C \cot X + \cot C \cot Y \geq 2,$$

with equality if and only if the triangles ABC and XYZ are similar. $\square$

Epsilon 33. (Vasile Cârtoaje) Let a, b, c, x, y, z be nonnegative reals. Prove the inequality

$$(ay+az+bz+bx+cx+cy)^2 \geq 4(bc+ca+ab)(yz+zx+xy),$$

with equality if and only if $a:x=b:y=c:z$.

Proof. According to the Conway substitution theorem, since a, b, c are nonnegative reals, there exists a triangle ABC with area $S=\frac{1}{2}\sqrt{bc+ca+ab}$ and with its angles A, B, C satisfying $\cot A=\frac{a}{2S}$, $\cot B=\frac{b}{2S}$, $\cot C=\frac{c}{2S}$ (note that we *cannot* denote the sidelengths of triangle ABC by a, b, c here, since a, b, c already stand for something different). Similarly, since x, y, z are nonnegative reals, there exists a triangle XYZ with area $T=\frac{1}{2}\sqrt{yz+zx+xy}$ and with its angles X, Y, Z satisfying $\cot X=\frac{x}{2T}$, $\cot Y=\frac{y}{2T}$, $\cot Z=\frac{z}{2T}$. Now, by Epsilon 32, we have

$$\cot A\cot Y+\cot A\cot Z+\cot B\cot Z+\cot B\cot X+\cot C\cot X+\cot C\cot Y\geq 2,$$

which rewrites as

$$\frac{a}{2S}\cdot\frac{y}{2T}+\frac{a}{2S}\cdot\frac{z}{2T}+\frac{b}{2S}\cdot\frac{z}{2T}+\frac{b}{2S}\cdot\frac{x}{2T}+\frac{c}{2S}\cdot\frac{x}{2T}+\frac{c}{2T}\cdot\frac{y}{2T}\geq 2,$$

and thus,

$$\begin{aligned} & ay+az+bz+bx+cx+cy \\ \geq\ & 2\cdot 2S\cdot 2T \\ =\ & 2\cdot 2\cdot\frac{1}{2}\sqrt{bc+ca+ab}\cdot 2\cdot\frac{1}{2}\sqrt{yz+zx+xy} \\ =\ & 2\sqrt{(bc+ca+ab)(yz+zx+xy)}. \end{aligned}$$

Upon squaring, this becomes

$$(ay+az+bz+bx+cx+cy)^2\geq 4(bc+ca+ab)(yz+zx+xy).$$

$\square$

Epsilon 34. (Walter Janous, Crux Mathematicorum) If u, v, w, x, y, z are six reals such that the terms $y+z$, $z+x$, $x+y$, $v+w$, $w+u$, $u+v$, and $vw+wu+uv$ are all nonnegative, then

$$\frac{x}{y+z} \cdot (v+w) + \frac{y}{z+x} \cdot (w+u) + \frac{z}{x+y} \cdot (u+v) \geq \sqrt{3\,(vw+wu+uv)}.$$

Proof. According to the Conway substitution theorem, since the reals $v+w$, $w+u$, $u+v$ and $vw+wu+uv$ are all nonnegative, there exists a triangle ABC with sidelengths $a = \sqrt{v+w}$, $b = \sqrt{w+u}$, $c = \sqrt{u+v}$ and area $S = \frac{1}{2}\sqrt{vw+wu+uv}$. Applying the Extended Tsintsifas Inequality to this triangle ABC and to the reals x, y, z satisfying the condition that the reals $y+z$, $z+x$, $x+y$ are all positive, we obtain

$$\frac{x}{y+z} \cdot a^2 + \frac{y}{z+x} \cdot b^2 + \frac{z}{x+y} \cdot c^2 \geq 2\sqrt{3}S,$$

which rewrites as

$$\frac{x}{y+z} \cdot \left(\sqrt{v+w}\right)^2 + \frac{y}{z+x} \cdot \left(\sqrt{w+u}\right)^2 + \frac{z}{x+y} \cdot \left(\sqrt{u+v}\right)^2 \geq 2\sqrt{3} \cdot \frac{1}{2}\sqrt{vw+wu+uv},$$

and thus,

$$\frac{x}{y+z} \cdot (v+w) + \frac{y}{z+x} \cdot (w+u) + \frac{z}{x+y} \cdot (u+v) \geq \sqrt{3\,(vw+wu+uv)}.$$

□

Epsilon 35. (Tran Quang Hung) In any triangle ABC with sidelengths a, b, c, circumradius R, inradius r, and area S, we have that

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum\cos^2\frac{A}{2}-\sum\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

Proof. We know that Hadwiger-Finsler's Inequality states that

$$x^2+y^2+z^2-4T\sqrt{3} \geq (x-y)^2+(y-z)^2+(z-x)^2,$$

for any triangle XYZ with sidelengths x, y, z, and area T. Let us apply this for the triangle $XYZ = I_aI_bI_c$, where $X = I_a$, $Y = I_b$, $Z = I_c$ are the excenters of ABC. In this case, it is well-known that

$$x = 4R\cos\frac{A}{2},\ y = 4R\cos\frac{B}{2},\ z = 4R\cos\frac{C}{2},\ T = 2sR,$$

where s is the semiperimeter of triangle ABC. Therefore,

$$\begin{aligned}
& 16R^2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)-8s\sqrt{3} \\
\geq\ & 16R^2\left[\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)^2+\left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)^2+\left(\cos^2\frac{C}{2}-\cos^2\frac{A}{2}\right)^2\right],
\end{aligned}$$

which according to the well-known formulas

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\quad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\quad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}},$$

easily reduces to

$$a^2+b^2+c^2 \geq 4S\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2+16Rr\left(\sum\cos^2\frac{A}{2}-\sum\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

□

Epsilon 36. *For all $\theta \in \mathbb{R}$, we have*

$$\sin(3\theta) = 4\sin\theta \sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right).$$

Proof. It follows that

$$\begin{aligned}
\sin(3\theta) &= 3\sin\theta - 4\sin^3\theta \\
&= \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) \\
&= 4\sin\theta\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) \\
&= 4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right).
\end{aligned}$$

$\square$

Epsilon 37. *For all $A, B, C \in \mathbb{R}$ with $A + B + C = 2\pi$, we have*

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

Proof. Our job is to show that the quadratic eqaution

$$t^2 + (2\cos B \cos C)\, t + \cos^2 B + \cos^2 C - 1 = 0$$

has a root $t = \cos A$. We find that it admits roots

$$\begin{aligned} t &= \frac{-2\cos B \cos C \pm \sqrt{4\cos^2 B \cos^2 C - 4\left(\cos^2 B + \cos^2 C - 1\right)}}{2} \\ &= -\cos B \cos C \pm \sqrt{\left(1 - \cos^2 B\right)\left(1 - \cos^2 C\right)} \\ &= -\cos B \cos C \pm |\sin B \sin C|. \end{aligned}$$

Since we have

$$-\cos B \cos C + \sin B \sin C = -\cos(B + C) = -\cos(\pi - A) = \cos A,$$

we find that $t = \cos A$ satisfies the quadratic equation, as desired. $\square$

Epsilon 38. [SL 2005 KOR] *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \ge p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Solution. Let's euler this problem. Let ρ be the circumradius of the triangle ABC. It's easy to show that $BC = 2\rho \sin A$ and $EF = 2\rho \sin A \cos A$. Since $DQ = 2\rho \sin C \cos B \cos A$, $DR = 2\rho \sin B \cos C \cos A$, and $\angle FDE = \pi - 2A$, the Cosine Law gives us

$$\begin{aligned} QR^2 &= DQ^2 + DR^2 - 2DQ \cdot DR \cos(\pi - 2A) \\ &= 4\rho^2 \cos^2 A \left[(\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A)\right] \end{aligned}$$

or

$$QR = 2\rho \cos A \sqrt{f(A, B, C)},$$

where

$$f(A, B, C) = (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A).$$

So, what we need to attack is the following inequality:

$$\left(\sum_{\text{cyclic}} 2\rho \sin A\right)\left(\sum_{\text{cyclic}} 2\rho \cos A \sqrt{f(A, B, C)}\right) \ge \left(\sum_{\text{cyclic}} 2\rho \sin A \cos A\right)^2$$

or

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \cos A \sqrt{f(A, B, C)}\right) \ge \left(\sum_{\text{cyclic}} \sin A \cos A\right)^2.$$

Our job is now to find a reasonable lower bound of $\sqrt{f(A, B, C)}$. Once again, we express $f(A, B, C)$ as the sum of two squares. We observe that

$$\begin{aligned} f(A, B, C) &= (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A) \\ &= (\sin C \cos B + \sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C\,[-1 + \cos(2A)] \\ &= \sin^2(C + B) - 2 \sin C \cos B \sin B \cos C \cdot 2 \sin^2 A \\ &= \sin^2 A\,[1 - 4 \sin B \sin C \cos B \cos C]\,. \end{aligned}$$

So, we shall express $1 - 4 \sin B \sin C \cos B \cos C$ as the sum of two squares. The trick is to replace 1 with $\left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right)$. Indeed, we get

$$\begin{aligned} 1 - 4 \sin B \sin C \cos B \cos C &= \left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right) - 4 \sin B \sin C \cos B \cos C \\ &= (\sin B \cos C - \sin C \cos B)^2 + (\cos B \cos C - \sin B \sin C)^2 \\ &= \sin^2(B - C) + \cos^2(B + C) \\ &= \sin^2(B - C) + \cos^2 A. \end{aligned}$$

It therefore follows that

$$f(A, B, C) = \sin^2 A \left[\sin^2(B - C) + \cos^2 A\right] \ge \sin^2 A \cos^2 A$$

so that

$$\sum_{\text{cyclic}} \cos A \sqrt{f(A, B, C)} \ge \sum_{\text{cyclic}} \sin A \cos^2 A.$$

So, we can complete the proof if we establish that

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \sin A \cos^2 A\right) \ge \left(\sum_{\text{cyclic}} \sin A \cos A\right)^2.$$

Indeed, one sees that it's a direct consequence of The Cauchy-Schwarz Inequality

$$(p+q+r)(x+y+z) \geq (\sqrt{px}+\sqrt{qy}+\sqrt{rz})^2,$$

where p, q, r, x, y and z are positive real numbers.

$\square$

Remark 8.1. *Alternatively, one may obtain another lower bound of $f(A,B,C)$:*

$$\begin{aligned} f(A,B,C) &= (\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A) \\ &= (\sin C\cos B - \sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\,[1+\cos(2A)] \\ &= \sin^2(B-C) + 2\frac{\sin(2B)}{2}\cdot\frac{\sin(2C)}{2}\cdot 2\cos^2 A \\ &\geq \cos^2 A\sin(2B)\sin(2C). \end{aligned}$$

Then, we can use this to offer a lower bound of the perimeter of triangle PQR:

$$p(PQR) = \sum_{\text{cyclic}} 2\rho\cos A\sqrt{f(A,B,C)} \geq \sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}$$

So, one may consider the following inequality:

$$p(ABC)\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C} \geq p(DEF)^2$$

or

$$\left(2\rho\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(2\rho\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

or

$$\left(\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}}\cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

However, it turned out that this doesn't hold. Disprove this!

Epsilon 39. [IMO 2001/1 KOR] *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.*

Solution. The angle inequality $\angle CAB + \angle COP < 90^\circ$ can be written as $\angle COP < \angle PCO$. This can be shown if we establish the length inequality $OP > PC$. Since the power of P with respect to the circumcircle of ABC is $OP^2 = R^2 - BP \cdot PC$, where R is the circumradius of the triangle ABC, it becomes $R^2 - BP \cdot PC > PC^2$ or $R^2 > BC \cdot PC$. We euler this. It's an easy job to get $BC = 2R \sin A$ and $PC = 2R \sin B \cos C$. Hence, we show the inequality $R^2 > 2R \sin A \cdot 2R \sin B \cos C$ or $\sin A \sin B \cos C < \frac{1}{4}$. Since $\sin A < 1$, it suffices to show that $\sin A \sin B \cos C < \frac{1}{4}$. Finally, we use the angle condition $\angle C \geq \angle B + 30^\circ$ to obtain the trigonometric inequality

$$\sin B \cos C = \frac{\sin(B+C) - \sin(C-B)}{2} \leq \frac{1 - \sin(C-B)}{2} \leq \frac{1 - \sin 30^\circ}{2} = \frac{1}{4}.$$

$\square$

Epsilon 40. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2+b^2+c^2 \geq 4\sqrt{3}S.$$

Second Proof. [AE, p.171] Let ABC be a triangle with sides $BC=a$, $CA=b$ and $AB=c$. After taking the point P on the same side of BC as the vertex A so that $\triangle PBC$ is equilateral, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned} AP^2 &= b^2+c^2-2bc\cos\left|C-\frac{\pi}{6}\right| \\ &= b^2+c^2-2bc\cos\left(C-\frac{\pi}{6}\right) \\ &= b^2+c^2-bc\cos C-\sqrt{3}bc\sin C \\ &= b^2+c^2-\frac{b^2+c^2-a^2}{2}-2\sqrt{3}K \end{aligned}$$

which implies the geometric inequality

$$b^2+c^2-\frac{b^2+c^2-a^2}{2} \geq 2\sqrt{3}K$$

or equivalently

$$a^2+b^2+c^2 \geq 4\sqrt{3}S.$$

□

Epsilon 41. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Third Proof. [DP2] We take the point P on the same side of B_1C_1 as the vertex A_1 so that $\triangle PB_1C_1 \sim \triangle A_2B_2C_2$. Now, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned}
& a_2{}^2 A_1 P^2 \\
= \;& a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - 2a_1a_2b_1b_2 \cos|C_1 - C_2| \\
= \;& a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - 2a_1a_2b_1b_2 \cos(C_1 - C_2) \\
= \;& a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}(2a_1b_1\cos C_1)(2a_2b_2\cos C_2) - 8\left(\frac{1}{2}a_1b_1\sin C_1\right)\left(\frac{1}{2}a_2b_2\sin C_2\right) \\
= \;& a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) - 8F_1F_2,
\end{aligned}$$

which implies the geometric inequality

$$a_2{}^2 b_1{}^2 + b_2{}^2 a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) \geq 8F_1F_2$$

or equivalently

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

□

Epsilon 42. (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA + PB + PC \geq 2(PU + PV + PW).$$

Proof. ([MB] and [AK]) Let $d_1 = PA$, $d_2 = PB$, $d_3 = PC$, $l_1 = PU$, $l_2 = PV$, $l_3 = PW$, $2\theta_1 = \angle BPC$, $2\theta_2 = \angle CPA$, and $2\theta_3 = \angle APB$. We need to show that $d_1 + d_2 + d_3 \geq 2(l_1 + l_2 + l_3)$. It's easy to deduce the following identities

$$l_1 = \frac{2d_2d_3}{d_2 + d_3}\cos\theta_1,\ l_2 = \frac{2d_3d_1}{d_3 + d_1}\cos\theta_2,\ \text{ and }\ l_3 = \frac{2d_1d_2}{d_1 + d_2}\cos\theta_3,$$

It now follows that

$$l_1 + l_2 + l_3 \leq \sqrt{d_2d_3}\cos\theta_1 + \sqrt{d_3d_1}\cos\theta_2 + \sqrt{d_1d_2}\cos\theta_3 \leq \frac{1}{2}\left(d_1 + d_2 + d_3\right).$$

$\square$

Epsilon 43. ([AK], Abi-Khuzam) Let $x_1, \cdots, x_4$ be positive real numbers. Let $\theta_1, \cdots, \theta_4$ be real numbers such that $\theta_1 + \cdots + \theta_4 = \pi$. Then, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \le \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

Proof. Let $p = \frac{{x_1}^2+{x_2}^2}{2x_1x_2} + \frac{{x_3}^2+{x_4}^2}{2x_3x_4}$ $q = \frac{x_1x_2+x_3x_4}{2}$ and $\lambda = \sqrt{\frac{p}{q}}$. In the view of $\theta_1 + \theta_2 + (\theta_3 + \theta_4) = \pi$ and $\theta_3 + \theta_4 + (\theta_1 + \theta_2) = \pi$, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + \lambda \cos(\theta_3 + \theta_4) \le p\lambda = \sqrt{pq},$$

and

$$x_3 \cos\theta_3 + x_4 \cos\theta_4 + \lambda \cos(\theta_1 + \theta_2) \le \frac{q}{\lambda} = \sqrt{pq}.$$

Since $\cos(\theta_3 + \theta_4) + \cos(\theta_1 + \theta_2) = 0$, adding these two above inequalities yields

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \le 2\sqrt{pq} = \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

□

Epsilon 44. [IMO 1991/5 FRA] *Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.*

First Proof. Set $A_1 = A$, $A_2 = B$, $A_3 = C$, $A_4 = A$ and write $\angle PA_iA_{i+1} = \theta_i$. Let H_1, H_2, H_3 denote the feet of perpendiculars from P to the sides BC, CA, AB, respectively. Now, we assume to the contrary that $\theta_1, \theta_2, \theta_3 > \frac{\pi}{6}$. Since the angle sum of a triangle is 180°, it is immediate that $\theta_1, \theta_2, \theta_3 < \frac{5\pi}{6}$. Hence,

$$\frac{PH_i}{PA_{i+1}} = \sin\theta_i > \frac{1}{2},$$

for all $i = 1, 2, 3$. We now find that

$$2\left(PH_1 + PH_2 + PH_3\right) > PA_2 + PA_3 + PA_1,$$

which contradicts for The Erdős-Mordell Theorem. □

Epsilon 45. *Any triangle has the same Brocard angles.*

Proof. More strongly, we show that the isogonal conjugate of the first Brocard point is the second Brocard point. Let Ω_1, Ω_2 denote the Brocard points of a triangle ABC, respectively. Let ω_1, ω_2 be the corresponding Brocard angles. Take the isogonal conjugate point Ω of Ω_1. Then, by the definition of isogonal conjugate point, we find that

$$\angle \Omega BA = \angle \Omega_2 CB = \angle \Omega_2 AC = \omega_1.$$

Hence, we see that the interior point Ω is the the second Brocard point of ABC. By the *uniqueness* of the second Brocard point of ABC, we see that $\Omega = \Omega_2$ and that $\omega_1 = \omega_2$. $\square$

Epsilon 46. *The Brocard angle ω of the triangle ABC satisfies*

$$\cot\omega = \cot A + \cot B + \cot C.$$

Proof. Let Ω denote the first Brocard point of ABC. We only prove it in the case when ABC is acute. Let AH, PQ denote the altitude from A, Q, respectively. Both angles $\angle B$ and $\angle C$ are acute, the point H lies on the interior side of BC. Let $P \neq \Omega$ be the intersubsection point of the circumcircle of triangle ΩCA with ray $B\Omega$. Since $\angle APB = \angle AP\Omega = \angle AC\Omega = \omega = \angle\Omega BC = \angle PBC$, we find that AP is parallel to BC so that $AH = PQ$. Since $\angle A$ is acute or since $\angle PCB = \angle PBA + \angle C = \angle B + \angle C = 180^\circ - \angle A$ is obtuse, we see that the point H lies on the outside of side BC. Since the four points B, H, C, Q are collinear in this order, we have $BQ = BH + HC + CQ$. It thus follows that

$$\cot\omega = \frac{BQ}{PQ} = \frac{BH}{AH} + \frac{HC}{AH} + \frac{CQ}{PQ} = \cot A + \cot B + \cot C.$$

$\square$

Epsilon 47. (The Trigonometric Version of Ceva's Theorem) For an interior point P of a triangle $A_1A_2A_3$, we write

$$\begin{aligned}
&\angle A_3A_1A_2 = \alpha_1,\ \angle PA_1A_2 = \vartheta_1,\ \angle PA_1A_3 = \theta_1,\\
&\angle A_1A_2A_3 = \alpha_2,\ \angle PA_2A_3 = \vartheta_2,\ \angle PA_2A_1 = \theta_2,\\
&\angle A_2A_3A_1 = \alpha_3,\ \angle PA_3A_1 = \vartheta_3,\ \angle PA_3A_2 = \theta_3.
\end{aligned}$$

Then, we find a *hidden* symmetry:

$$\frac{\sin\vartheta_1}{\sin\theta_1}\cdot\frac{\sin\vartheta_2}{\sin\theta_2}\cdot\frac{\sin\vartheta_3}{\sin\theta_3} = 1$$

or equivalently

$$\frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3} = [\cot\vartheta_1 - \cot\alpha_1]\,[\cot\vartheta_2 - \cot\alpha_2]\,[\cot\vartheta_3 - \cot\alpha_3]\,.$$

Proof. Applying The Sine Law, we have

$$\frac{\sin\vartheta_1}{\sin\theta_1} = \frac{PA_2}{PA_1},\ \frac{\sin\vartheta_2}{\sin\theta_2} = \frac{PA_3}{PA_2},\ \frac{\sin\vartheta_3}{\sin\theta_3} = \frac{PA_1}{PA_3}.$$

It follows that

$$\frac{\sin\vartheta_1}{\sin\theta_1}\cdot\frac{\sin\vartheta_2}{\sin\theta_2}\cdot\frac{\sin\vartheta_3}{\sin\theta_3} = \frac{PA_2}{PA_1}\cdot\frac{PA_3}{PA_2}\cdot\frac{PA_1}{PA_3} = 1.$$

We now observe that, for $i = 1, 2, 3$,

$$\cot\vartheta_i - \cot\alpha_i = \frac{\cos\vartheta_i}{\sin\vartheta_i} - \frac{\cos\alpha_i}{\sin\alpha_i} = \frac{\sin(\alpha_i - \vartheta_i)}{\sin\alpha_i\sin\vartheta_i} = \frac{\sin\theta_i}{\sin\alpha_i\sin\vartheta_i}.$$

It therefore follows that

$$\begin{aligned}
& [\cot\vartheta_1 - \cot\alpha_1]\,[\cot\vartheta_2 - \cot\alpha_2]\,[\cot\vartheta_3 - \cot\alpha_3]\\
= \ & \frac{\sin\theta_1}{\sin\alpha_1\sin\vartheta_1}\cdot\frac{\sin\theta_2}{\sin\alpha_2\sin\vartheta_2}\cdot\frac{\sin\theta_3}{\sin\alpha_3\sin\vartheta_3}\\
= \ & \frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3}\cdot\frac{\sin\theta_1}{\sin\vartheta_1}\cdot\frac{\sin\theta_2}{\sin\vartheta_2}\cdot\frac{\sin\theta_3}{\sin\vartheta_3}\\
= \ & \frac{1}{\sin\alpha_1\sin\alpha_2\sin\alpha_3}.
\end{aligned}$$

□

Epsilon 48. *Let P be an interior point of a triangle ABC. Show that*

$$\cot(\angle PAB) + \cot(\angle PBC) + \cot(\angle PCA) \geq 3\sqrt{3}.$$

Proof. Set $A_1 = A$, $A_2 = B$, $A_3 = C$, $A_4 = A$ and write $\angle A_i = \alpha_i$ and $\angle PA_iA_{i+1} = \vartheta_i$ for $i = 1, 2, 3$. Our job is to establish the inequality

$$\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3 \geq 3\sqrt{3}.$$

We begin with The Trigonometric Version of Ceva's Theorem

$$\frac{1}{\sin\alpha_1 \sin\alpha_2 \sin\alpha_3} = [\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3].$$

We first apply The AM-GM Inequality and Jensen's Inequality to deduce

$$\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \leq \left(\frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{3}\right)^3 \leq \sin^3\left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^3$$

or

$$\left(\frac{2}{\sqrt{3}}\right)^3 \leq [\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3].$$

Since $\vartheta_i \in (0, \alpha_i)$, the monotonicity of the cotangent function shows that $\cot\alpha_i - \cot\vartheta_i$ is positive. Hence, by The AM-GM Inequality, the above inequality guarantees that

$$\begin{aligned}
\frac{2}{\sqrt{3}} &\leq \sqrt[3]{[\cot\vartheta_1 - \cot\alpha_1][\cot\vartheta_2 - \cot\alpha_2][\cot\vartheta_3 - \cot\alpha_3]} \\
&\leq \frac{[\cot\vartheta_1 - \cot\alpha_1] + [\cot\vartheta_2 - \cot\alpha_2] + [\cot\vartheta_3 - \cot\alpha_3]}{3} \\
&= \frac{[\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3] - [\cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3]}{3}
\end{aligned}$$

or

$$\cot\vartheta_1 + \cot\vartheta_2 + \cot\vartheta_3 \geq \cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3 + 2\sqrt{3}.$$

Since we know $\cot\alpha_1 + \cot\alpha_2 + \cot\alpha_3 \geq \sqrt{3}$, we get the desired inequality. $\square$

Epsilon 49. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Fourth Proof. ([RW], R. Weitzenböck) Let ABC be a triangle with sides a, b, and c. To euler it, we toss the picture on the real plane $\mathbb{R}^2$ with the coordinates $A(\alpha, \beta)$, $B\left(-\frac{a}{2}, 0\right)$ and $C\left(\frac{a}{2}, 0\right)$. Now, we obtain

$$\left(a^2 + b^2 + c^2\right)^2 - \left(4\sqrt{3}S\right)^2 = \left(\frac{3}{2}a^2 + \left(\alpha^2 - \beta^2\right)\right)^2 + 16\alpha^2\beta^2 \geq 0.$$

$\square$

Epsilon 50. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Fourth Proof. (By a participant from KMO[16] summer program.) We toss $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ onto the real plane $\mathbb{R}^2$:

$$A_1(0, p_1),\ B_1(p_2, 0),\ C_1(p_3, 0),\ A_2(0, q_1),\ B_2(q_2, 0),\ \text{and } C_2(q_3, 0).$$

It therefore follows from the inequality $x^2 + y^2 \geq 2|xy|$ that

$$\begin{aligned}
& a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \\
= \quad & (p_3 - p_2)^2(2q_1{}^2 + 2q_1q_2) + (p_1{}^2 + p_3{}^2)(2q_2{}^2 - 2q_2q_3) + (p_1{}^2 + p_2{}^2)(2q_3{}^2 - 2q_2q_3) \\
= \quad & 2(p_3 - p_2)^2 q_1{}^2 + 2(q_3 - q_2)^2 p_1{}^2 + 2(p_3q_2 - p_2q_3)^2 \\
\geq \quad & 2((p_3 - p_2)q_1)^2 + 2((q_3 - q_2)p_1)^2 \\
\geq \quad & 4|(p_3 - p_2)q_1| \cdot |(q_3 - q_2)p_1| \\
= \quad & 16F_1F_2\ .
\end{aligned}$$

□

[16] Korean Mathematical Olympiads

Epsilon 51. (USA 2003) Let ABC be a triangle. A circle passing through A and B intersects the segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Solution. (Darij Grinberg) By Ceva's theorem, applied to the triangle BCF and the concurrent cevians BM, CA and FE (in fact, these cevians concur at the point D), we have

$$\frac{MF}{MC} \cdot \frac{EC}{EB} \cdot \frac{AB}{AF} = 1.$$

Hence, $\frac{MF}{MC} = \frac{AF}{AB} \cdot \frac{EB}{EC} = \frac{AF}{AB} : \frac{EC}{EB}$. Thus, $MF = MC$ holds if and only if $\frac{AF}{AB} = \frac{EC}{EB}$. But by Thales' theorem, $\frac{AF}{AB} = \frac{EC}{EB}$ is equivalent to $AE|FC$, and obviously we have $AE|FC$ if and only if $\angle EAC = \angle ACF$. Now, since the points A, B, D and E lie on one circle, we have that $\angle EAD = \angle EBD$, what rewrites as $\angle EAC = \angle CBM$. On other hand, we trivially have that $\angle ACF = \angle DCM$. Thus, $\angle EAC = \angle ACF$ if and only if $\angle CBM = \angle DCM$. Now, as it is clear that $\angle CMB = \angle DMC$, we have $\angle CBM = \angle DCM$ if and only if the triangles CMB and DMC are similar. But, the triangles CMB and DMC are similar if and only if $\frac{MB}{MC} = \frac{MC}{MD}$. This is finally equivalent to $MB \cdot MD = MC^2$, and so, by combining all these equivalences, the conclusion follows. $\square$

Epsilon 52. [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $\overline{BC}\cdot\overline{PB}\cdot\overline{PC}+\overline{AB}\cdot\overline{PA}\cdot\overline{PB}+\overline{CA}\cdot\overline{PC}\cdot\overline{PA}\geq\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$ and

(2) $\overline{PA}^3\cdot\overline{BC}+\overline{PB}^3\cdot\overline{CA}+\overline{PC}^3\cdot\overline{AB}\geq 3\overline{PG}\cdot\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$.

Solution. We only check the first inequality. We regard A, B, C, P as complex numbers and assume that P corresponds to 0. We're required to prove that

$$|(B-C)BC|+|(A-B)AB|+|(C-A)CA|\geq|(B-C)(C-A)(A-B)|.$$

It remains to apply The Triangle Inequality to the algebraic identity

$$(B-C)BC+(A-B)AB+(C-A)CA=-(B-C)(C-A)(A-B).$$

□

Epsilon 53. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Fifth Proof. ([GC], G. Chang) We regard A, B, C, A', B', C' as complex numbers and assume that C corresponds to 0. Rewriting the both sides in the inequality in terms of complex numbers, we get

$$\begin{aligned} & a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \\ = \quad & 2\left(|A'|^2|B|^2 + |A|^2|B'|^2\right) - \left(A\overline{B} + \overline{A}B\right)\left(A'\overline{B'} + \overline{A'}B\right) \end{aligned}$$

and

$$16F_1F_2 = \pm\left(\overline{A}B - A\overline{B}\right)\left(A'\overline{B'} + \overline{A'}B'\right),$$

where the sign begin chose to make the right hand positive. According to whether the triangle ABC and the triangle $A'B'C'$ have the same orientation or not, we obtain either

$$a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)-16F_1F_2 = 2\left|AB' - A'B\right|^2$$

or

$$a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)-16F_1F_2 = 2\left|A\overline{B'} - \overline{A'}B\right|^2.$$

This completes the proof. $\square$

Epsilon 54. [SL 2002 KOR] *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $\overline{AB} + \overline{AC} \geq 4\overline{DE}$.*

Solution. Let $\overline{AF} = x, \overline{BF} = y, \overline{CF} = z$ and let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. We can toss the pictures on $\mathbb{C}$ so that the points F, A, B, C, D, and E are represented by the complex numbers 0, x, $y\omega$, $z\omega^2$, d, and e. It's an easy exercise to establish that $\overline{DF} = \frac{xz}{x+z}$ and $\overline{EF} = \frac{xy}{x+y}$. This means that $d = -\frac{xz}{x+z}\omega$ and $e = -\frac{xy}{x+y}\omega$. We're now required to prove that

$$|x - y\omega| + |z\omega^2 - x| \geq 4\left|\frac{-zx}{z+x}\omega + \frac{xy}{x+y}\omega^2\right|.$$

Since $|\omega| = 1$ and $\omega^3 = 1$, we have $|z\omega^2 - x| = |\omega(z\omega^2 - x)| = |z - x\omega|$. Therefore, we need to prove

$$|x - y\omega| + |z - x\omega| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|.$$

More strongly, we establish that $|(x - y\omega) + (z - x\omega)| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|$ or $|p - q\omega| \geq |r - s\omega|$, where $p = z + x$, $q = y + x$, $r = \frac{4zx}{z+x}$ and $s = \frac{4xy}{x+y}$. It's clear that $p \geq r > 0$ and $q \geq s > 0$. It follows that

$$|p - q\omega|^2 - |r - s\omega|^2 = (p - q\omega)\overline{(p - q\omega)} - (r - s\omega)\overline{(r - s\omega)} = (p^2 - r^2) + (pq - rs) + (q^2 - s^2) \geq 0.$$

It's easy to check that the equality holds if and only if $\triangle ABC$ is equilateral. $\square$

Epsilon 55. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

First Solution. Choose $A, B, C \in \left(0, \frac{\pi}{2}\right)$ with $a = \sqrt{2}\tan A$, $b = \sqrt{2}\tan B$, and $c = \sqrt{2}\tan C$. Using the trigonometric identity $1+\tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C.$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\cos B\cos C - \cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying The AM-GM Inequality and Jesen's Inequality, we have

$$\cos A\cos B\cos C \leq \left(\frac{\cos A+\cos B+\cos C}{3}\right)^3 \leq \cos^3\theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3\theta(\cos^3\theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{or} \quad \cos^3\theta - \cos 3\theta = 3\cos\theta - 3\cos^3\theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4\theta\left(1-\cos^2\theta\right),$$

which follows from The AM-GM Inequality

$$\left(\frac{\cos^2\theta}{2}\cdot\frac{\cos^2\theta}{2}\cdot\left(1-\cos^2\theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2\theta}{2}+\frac{\cos^2\theta}{2}+\left(1-\cos^2\theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. $\square$

Epsilon 56. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

First Solution. We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity :

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying The AM-GM Inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3 \left(\cos B \cos C \cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3 \left(\cos C \cos D \cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3 \left(\cos D \cos A \cos B\right)^{\frac{2}{3}}, \text{ and } \sin^2 D \geq 3 \left(\cos A \cos B \cos C\right)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! □

Epsilon 57. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\le\frac{3}{2}.$$

First Solution. We give a convexity proof. We can write $x=\tan A$, $y=\tan B$, $z=\tan C$, where $A,B,C\in\left(0,\frac{\pi}{2}\right)$. Using the fact that $1+\tan^2\theta=\left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of A, B, C :

$$\cos A+\cos B+\cos C\le\frac{3}{2}.$$

It follows from $\tan(\pi-C)=-z=\frac{x+y}{1-xy}=\tan(A+B)$ and from $\pi-C, A+B\in(0,\pi)$ that $\pi-C=A+B$ or $A+B+C=\pi$. Hence, it suffices to show the following. $\square$

Epsilon 58. (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.

Solution. Notice that $a, b, c > 1$ implies that $a^2 + b^2 + c^2 + abc > 4$. If $a \leq 1$, then we have $ab + bc + ca - abc \geq (1 - a)bc \geq 0$. We now prove that $ab + bc + ca - abc \leq 2$. Letting $a = 2p$, $b = 2q$, $c = 2r$, we get $p^2 + q^2 + r^2 + 2pqr = 1$. By the above exercise, we can write

$$a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C \text{ for some } A, B, C \in \left[0, \frac{\pi}{2}\right] \text{ with } A + B + C = \pi.$$

We are required to prove

$$\cos A \cos B + \cos B \cos C + \cos C \cos A - 2\cos A \cos B \cos C \leq \frac{1}{2}.$$

One may assume that $A \geq \frac{\pi}{3}$ or $1 - 2\cos A \geq 0$. Note that

$$\cos A \cos B + \cos B \cos C + \cos C \cos A - 2\cos A \cos B \cos C = \cos A(\cos B + \cos C) + \cos B \cos C(1 - 2\cos A).$$

We apply Jensen's Inequality to deduce $\cos B + \cos C \leq \frac{3}{2} - \cos A$. Note that $2\cos B \cos C = \cos(B - C) + \cos(B + C) \leq 1 - \cos A$. These imply that

$$\cos A(\cos B + \cos C) + \cos B \cos C(1 - 2\cos A) \leq \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1 - \cos A}{2}\right)(1 - 2\cos A).$$

However, it's easy to verify that $\cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1 - \cos A}{2}\right)(1 - 2\cos A) = \frac{1}{2}$. □

Epsilon 59. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

First Solution. To remove the square roots, we make the following substitution :

$$x=\frac{a}{\sqrt{a^2+8bc}},\ \ y=\frac{b}{\sqrt{b^2+8ca}},\ \ z=\frac{c}{\sqrt{c^2+8ab}}.$$

Clearly, $x,y,z\in(0,1)$. Our aim is to show that $x+y+z\geq 1$. We notice that

$$\frac{a^2}{8bc}=\frac{x^2}{1-x^2},\ \ \frac{b^2}{8ac}=\frac{y^2}{1-y^2},\ \ \frac{c^2}{8ab}=\frac{z^2}{1-z^2}\ \ \Longrightarrow\ \ \frac{1}{512}=\left(\frac{x^2}{1-x^2}\right)\left(\frac{y^2}{1-y^2}\right)\left(\frac{z^2}{1-z^2}\right).$$

Hence, we need to show that

$$x+y+z\geq 1,\ \text{where } 0<x,y,z<1 \text{ and } (1-x^2)(1-y^2)(1-z^2)=512(xyz)^2.$$

However, $1>x+y+z$ implies that, by The AM-GM Inequality,

$(1-x^2)(1-y^2)(1-z^2)>((x+y+z)^2-x^2)((x+y+z)^2-y^2)((x+y+z)^2-z^2)=(x+x+y+z)(y+z)$
$(x+y+y+z)(z+x)(x+y+z+z)(x+y)\geq 4(x^2yz)^{\frac{1}{4}}\cdot 2(yz)^{\frac{1}{2}}\cdot 4(y^2zx)^{\frac{1}{4}}\cdot 2(zx)^{\frac{1}{2}}\cdot 4(z^2xy)^{\frac{1}{4}}\cdot 2(xy)^{\frac{1}{2}}$
$=512(xyz)^2$. This is a contradiction ! $\square$

Epsilon 60. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Second Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

It follows from The Cauchy-Schwarz Inequality that

$$[(y+z) + (z+x) + (x+y)]\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq (x+y+z)^2$$

so that, by The AM-GM Inequality,

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$

□

Epsilon 61. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

Second Solution. The starting point is letting $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. We find that $a+b+c=abc$ is equivalent to $1=xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\frac{x}{\sqrt{(x+y)(x+z)}}=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\leq\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right).$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\leq\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)\quad\text{and}\quad\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right).$$

Adding these three yields the required result. $\square$

Epsilon 62. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1.$$

Second Solution. ([IV], Ilan Vardi) Since $abc = 1$, we may assume that $a \ge 1 \ge b$. [17] It follows that

$$1-\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = \left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right)+\frac{(a-1)(1-b)}{a}.$$ [18]

□

Third Solution. As in the first solution, after the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for x, y, $z > 0$, we can rewrite it as $xyz \ge (y+z-x)(z+x-y)(x+y-z)$. Without loss of generality, we can assume that $z \ge y \ge x$. Set $y - x = p$ and $z - x = q$ with $p, q \ge 0$. It's straightforward to verify that

$$xyz-(y+z-x)(z+x-y)(x+y-z) = (p^2-pq+q^2)x+(p^3+q^3-p^2q-pq^2).$$

Since $p^2 - pq + q^2 \ge (p-q)^2 \ge 0$ and $p^3+q^3-p^2q-pq^2 = (p-q)^2(p+q) \ge 0$, we get the result. □

Fourth Solution. (From the IMO 2000 Short List) Using the condition $abc = 1$, it's straightforward to verify the equalities

$$2 = \frac{1}{a}\left(a-1+\frac{1}{b}\right)+c\left(b-1+\frac{1}{c}\right),$$
$$2 = \frac{1}{b}\left(b-1+\frac{1}{c}\right)+a\left(c-1+\frac{1}{a}\right),$$
$$2 = \frac{1}{c}\left(c-1+\frac{1}{a}\right)+b\left(a-1+\frac{1}{c}\right).$$

In particular, they show that at most one of the numbers $u = a-1+\frac{1}{b}$, $v = b-1+\frac{1}{c}$, $w = c-1+\frac{1}{a}$ is negative. If there is such a number, we have

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = uvw < 0 < 1.$$

And if $u, v, w \ge 0$, The AM-GM Inequality yields

$$2 = \frac{1}{a}u+cv \ge 2\sqrt{\frac{c}{a}uv},\;\; 2 = \frac{1}{b}v+aw \ge 2\sqrt{\frac{a}{b}vw},\;\; 2 = \frac{1}{c}w+aw \ge 2\sqrt{\frac{b}{c}wu}.$$

Thus, $uv \le \frac{a}{c}$, $vw \le \frac{b}{a}$, $wu \le \frac{c}{b}$, so $(uvw)^2 \le \frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b} = 1$. Since $u, v, w \ge 0$, this completes the proof. □

[17] Why? Note that the inequality is not symmetric in the three variables. Check it!

[18] For a verification of the identity, see [IV].

Epsilon 63. *Let a, b, c be positive real numbers satisfying $a+b+c=1$. Show that*

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab}\le 1+\frac{3\sqrt{3}}{4}.$$

Solution. We want to establish that

$$\frac{1}{1+\frac{bc}{a}}+\frac{1}{1+\frac{ca}{b}}+\frac{\sqrt{\frac{ab}{c}}}{1+\frac{ab}{c}}\le 1+\frac{3\sqrt{3}}{4}.$$

Set $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ca}{b}}$, $z=\sqrt{\frac{ab}{c}}$. We need to prove that

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{z}{1+z^2}\le 1+\frac{3\sqrt{3}}{4},$$

where $x,y,z>0$ and $xy+yz+zx=1$. It's not hard to show that there exists $A,B,C\in(0,\pi)$ with

$$x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}, \text{ and } A+B+C=\pi.$$

The inequality becomes

$$\frac{1}{1+\left(\tan\frac{A}{2}\right)^2}+\frac{1}{1+\left(\tan\frac{B}{2}\right)^2}+\frac{\tan\frac{C}{2}}{1+\left(\tan\frac{C}{2}\right)^2}\le 1+\frac{3\sqrt{3}}{4}$$

or

$$1+\frac{1}{2}\left(\cos A+\cos B+\sin C\right)\le 1+\frac{3\sqrt{3}}{4}$$

or

$$\cos A+\cos B+\sin C\le\frac{3\sqrt{3}}{2}.$$

□

Note that $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$. Since $\left|\frac{A-B}{2}\right|<\frac{\pi}{2}$, this means that

$$\cos A+\cos B\le 2\cos\left(\frac{A+B}{2}\right)=2\cos\left(\frac{\pi-C}{2}\right).$$

It will be enough to show that

$$2\cos\left(\frac{\pi-C}{2}\right)+\sin C\le\frac{3\sqrt{3}}{2},$$

where $C\in(0,\pi)$. This is a one-variable inequality.[19] It's left as an exercise for the reader.

[19] *Differentiate!* Shiing-Shen Chern

Epsilon 64. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

Second Solution. (given by Jeong Soo Sim at the KMO Weekend Program 2007) We need to prove the inequality $a^4b^4c^4d^4 \geq 81$. After making the substitution

$$A=\frac{1}{1+a^4},\ B=\frac{1}{1+b^4},\ C=\frac{1}{1+c^4},\ D=\frac{1}{1+d^4},$$

we obtain

$$a^4=\frac{1-A}{A},\ b^4=\frac{1-B}{B},\ c^4=\frac{1-C}{C},\ d^4=\frac{1-D}{D}.$$

The constraint becomes $A+B+C+D=1$ and the inequality can be written as

$$\frac{1-A}{A}\cdot\frac{1-B}{B}\cdot\frac{1-C}{C}\cdot\frac{1-D}{D}\geq 81.$$

or

$$\frac{B+C+D}{A}\cdot\frac{C+D+A}{B}\cdot\frac{D+A+B}{C}\cdot\frac{A+B+C}{D}\geq 81.$$

or

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 81ABCD.$$

However, this is an immediate consequence of The AM-GM Inequality:

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 3\,(BCD)^{\frac{1}{3}}\cdot 3\,(CDA)^{\frac{1}{3}}\cdot 3\,(DAB)^{\frac{1}{3}}\cdot 3\,(ABC)^{\frac{1}{3}}.$$

$\square$

Epsilon 65. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

First Solution. We begin with the algebraic substitution $a=\sqrt{x-1}$, $b=\sqrt{y-1}$, $c=\sqrt{z-1}$. Then, the condition becomes

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}=2 \quad \Leftrightarrow \quad a^2b^2+b^2c^2+c^2a^2+2a^2b^2c^2=1$$

and the inequality is equivalent to

$$\sqrt{a^2+b^2+c^2+3} \geq a+b+c \quad \Leftrightarrow \quad ab+bc+ca \leq \frac{3}{2}.$$

Let $p=bc$, $q=ca$, $r=ab$. Our job is to prove that $p+q+r \leq \frac{3}{2}$ where $p^2+q^2+r^2+2pqr=1$. Now, we can make the trigonometric substitution

$$p=\cos A,\ q=\cos B,\ r=\cos C \ \text{ for some } A, B, C \in \left(0, \frac{\pi}{2}\right) \text{ with } A+B+C=\pi.$$

What we need to show is now that $\cos A+\cos B+\cos C \leq \frac{3}{2}$. It follows from Jensen's Inequality. $\square$

Epsilon 66. (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + 1.$$

Solution. After writing $x = \frac{a}{b}$ and $y = \frac{c}{b}$, we get

$$\frac{c}{a} = \frac{y}{x}, \quad \frac{a+b}{b+c} = \frac{x+1}{1+y}, \quad \frac{b+c}{c+a} = \frac{1+y}{y+x}.$$

One may rewrite the inequality as

$$x^3y^2 + x^2 + x + y^3 + y^2 \geq x^2y + 2xy + 2xy^2.$$

Apply The AM-GM Inequality to obtain

$$\frac{x^3y^2 + x}{2} \geq x^2y, \quad \frac{x^3y^2 + x + y^3 + y^3}{2} \geq 2xy^2, \quad x^2 + y^2 \geq 2xy.$$

Adding these three inequalities, we get the result. The equality holds if and only if $x = y = 1$ or $a = b = c$. $\square$

Epsilon 67. [SL 2001] *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1+x_1{}^2} + \frac{x_2}{1+x_1{}^2+x_2{}^2} + \cdots + \frac{x_n}{1+x_1{}^2+\cdots+x_n{}^2} < \sqrt{n}.$$

First Solution. We only consider the case when $x_1, \cdots, x_n$ are all nonnegative real numbers.(Why?)[20] Let $x_0 = 1$. After the substitution $y_i = x_0{}^2 + \cdots + x_i{}^2$ for all $i = 0, \cdots, n$, we obtain $x_i = \sqrt{y_i - y_{i-1}}$. We need to prove the following inequality

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} < \sqrt{n}.$$

Since $y_i \geq y_{i-1}$ for all $i = 1, \cdots, n$, we have an upper bound of the left hand side:

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} \leq \sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{\sqrt{y_i y_{i-1}}} = \sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}}$$

We now apply the Cauchy-Schwarz inequality to give an upper bound of the last term:

$$\sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}} \leq \sqrt{n \sum_{i=0}^{n} \left(\frac{1}{y_{i-1}} - \frac{1}{y_i}\right)} = \sqrt{n \left(\frac{1}{y_0} - \frac{1}{y_n}\right)}.$$

Since $y_0 = 1$ and $y_n > 0$, this yields the desired upper bound $\sqrt{n}$.

□

Second Solution. We may assume that $x_1, \cdots, x_n$ are all nonnegative real numbers. Let $x_0 = 0$. We make the following *algebraic* substitution

$$t_i = \frac{x_i}{\sqrt{x_0{}^2 + \cdots + x_i{}^2}},\ c_i = \frac{1}{\sqrt{1+t_i{}^2}} \text{ and } s_i = \frac{t_i}{\sqrt{1+t_i{}^2}}$$

for all $i = 0, \cdots, n$. It's an easy exercise to show that $\frac{x_i}{x_0{}^2+\cdots+x_i{}^2} = c_0 \cdots c_i s_i$. Since $s_i = \sqrt{1 - c_i{}^2}$, the desired inequality becomes

$$c_0 c_1 \sqrt{1 - c_1{}^2} + c_0 c_1 c_2 \sqrt{1 - c_2{}^2} + \cdots + c_0 c_1 \cdots c_n \sqrt{1 - c_n{}^2} < \sqrt{n}.$$

Since $0 < c_i \leq 1$ for all $i = 1, \cdots, n$, we have

$$\sum_{i=1}^{n} c_0 \cdots c_i \sqrt{1 - c_i{}^2} \leq \sum_{i=1}^{n} c_0 \cdots c_{i-1} \sqrt{1 - c_i{}^2} = \sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2}.$$

Since $c_0 = 1$, by The Cauchy-Schwarz Inequality, we obtain

$$\sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2} \leq \sqrt{n \sum_{i=1}^{n} [(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2]} = \sqrt{n\,[1 - (c_0 \cdots c_n)^2]}.$$

□

[20] $\frac{x_1}{1+x_1{}^2} + \frac{x_2}{1+x_1{}^2+x_2{}^2} + \cdots + \frac{x_n}{1+x_1{}^2+\cdots+x_n{}^2} \leq \frac{|x_1|}{1+x_1{}^2} + \frac{|x_2|}{1+x_1{}^2+x_2{}^2} + \cdots + \frac{|x_n|}{1+x_1{}^2+\cdots+x_n{}^2}.$

Epsilon 68. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Solution. We don't employ The Ravi Substitution! It follows from the triangle inequality that

$$\sum_{\text{cyclic}} \frac{a}{b+c} < \sum_{\text{cyclic}} \frac{a}{\frac{1}{2}(a+b+c)} = 2.$$

□

Epsilon 69. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Second Solution. Let's try to find a new lower bound of $(x+y+z)^2$ where $x,y,z>0$. There are well-known lower bounds such as $3(xy+yz+zx)$ and $9(xyz)^{\frac{2}{3}}$. Here, we break the symmetry. We notice that

$$(x+y+z)^2 = x^2+y^2+z^2+xy+xy+yz+yz+zx+zx.$$

We apply The AM-GM Inequality to the right hand side except the term x^2:

$$y^2+z^2+xy+xy+yz+yz+zx+zx \geq 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}\ .$$

It follows that

$$(x+y+z)^2 \geq x^2+8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}} = x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right).$$

We proved the estimation, for $x,y,z>0$,

$$x+y+z \geq \sqrt{x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{x^{\frac{3}{4}}}{\sqrt{x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}}} \geq \sum_{\text{cyclic}}\frac{x}{x+y+z}=1.$$

After the substitution $x=a^{\frac{4}{3}}$, $y=b^{\frac{4}{3}}$, and $z=c^{\frac{4}{3}}$, it now becomes the inequality

$$\sum_{\text{cyclic}}\frac{a}{\sqrt{a^2+8bc}}\geq 1.$$

□

Epsilon 70. [IMO 2005/3 KOR] *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

First Solution. It's equivalent to the following inequality

$$\left(\frac{x^2-x^5}{x^5+y^2+z^2}+1\right)+\left(\frac{y^2-y^5}{y^5+z^2+x^2}+1\right)+\left(\frac{z^2-z^5}{z^5+x^2+y^2}+1\right) \leq 3$$

or

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \leq 3.$$

With The Cauchy-Schwarz Inequality and the fact that $xyz \geq 1$, we have

$$(x^5+y^2+z^2)(yz+y^2+z^2) \geq (x^2+y^2+z^2)^2$$

or

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

Taking the cyclic sum, we reach

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \leq 2+\frac{xy+yz+zx}{x^2+y^2+z^2} \leq 3.$$

$\square$

Second Solution. The main idea is to think of 1 as follows :

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{y^5+z^2+x^2}+\frac{z^5}{z^5+x^2+y^2} \geq 1 \geq \frac{x^2}{x^5+y^2+z^2}+\frac{y^2}{y^5+z^2+x^2}+\frac{z^2}{z^5+x^2+y^2}.$$

We first show the left-hand. It follows from $y^4+z^4 \geq y^3z+yz^3 = yz(y^2+z^2)$ that

$$x(y^4+z^4) \geq xyz(y^2+z^2) \geq y^2+z^2 \quad \text{or} \quad \frac{x^5}{x^5+y^2+z^2} \geq \frac{x^5}{x^5+xy^4+xz^4} = \frac{x^4}{x^4+y^4+z^4}.$$

Taking the cyclic sum, we have the required inequality. It remains to show the right-hand. As in the first solution, The Cauchy-Schwarz Inequality and $xyz \geq 1$ imply that

$$(x^5+y^2+z^2)(yz+y^2+z^2) \geq (x^2+y^2+z^2)^2 \quad \text{or} \quad \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

Taking the cyclic sum, we have

$$\sum_{\text{cyclic}} \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5+y^2+z^2}.$$

Our job is now to establish the following homogeneous inequality

$$1 \geq \sum_{\text{cyclic}} \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2} \Leftrightarrow (x^2+y^2+z^2)^2 \geq 2\sum_{\text{cyclic}} x^2y^2+\sum_{\text{cyclic}} x^2yz \Leftrightarrow \sum_{\text{cyclic}} x^4 \geq \sum_{\text{cyclic}} x^2yz.$$

However, by The AM-GM Inequality, we obtain

$$\sum_{\text{cyclic}} x^4 = \sum_{\text{cyclic}} \frac{x^4+y^4}{2} \geq \sum_{\text{cyclic}} x^2y^2 = \sum_{\text{cyclic}} x^2\left(\frac{y^2+z^2}{2}\right) \geq \sum_{\text{cyclic}} x^2yz.$$

$\square$

Remark 8.2. *Here is an alternative way to reach the right hand side inequality. We claim that*

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

We do this by proving

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2yz}{x^4+y^3z+yz^3}$$

because $xyz \geq 1$ *implies that*

$$\frac{x^2yz}{x^4+y^3z+yz^3} = \frac{x^2}{\frac{x^5}{xyz}+y^2+z^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

Hence, we need to show the homogeneous inequality

$$(2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) \geq 4x^2yz(x^2+y^2+z^2)^2.$$

However, this is a straightforward consequence of The AM-GM Inequality.

$$\begin{aligned}
& (2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) - 4x^2yz(x^2+y^2+z^2)^2 \\
= \ & (x^8+x^4y^4+x^6y^2+x^6y^2+y^7z+y^3z^5)+(x^8+x^4z^4+x^6z^2+x^6z^2+yz^7+y^5z^3) \\
& +2(x^6y^2+x^6z^2)-6x^4y^3z-6x^4yz^3-2x^6yz \\
\geq \ & 6\sqrt[6]{x^8\cdot x^4y^4\cdot x^6y^2\cdot x^6y^2\cdot y^7z\cdot y^3z^5}+6\sqrt[6]{x^8\cdot x^4z^4\cdot x^6z^2\cdot x^6z^2\cdot yz^7\cdot y^5z^3} \\
& +2\sqrt{x^6y^2\cdot x^6z^2}-6x^4y^3z-6x^4yz^3-2x^6yz \\
= \ & 0.
\end{aligned}$$

Taking the cyclic sum, we obtain

$$1 = \sum_{\text{cyclic}} \frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5+y^2+z^2}.$$

Third Solution. (by an IMO 2005 contestant Iurie Boreico[21] from Moldova) We establish that

$$\frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{x^5-x^2}{x^3(x^2+y^2+z^2)}.$$

It follows immediately from the identity

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyclic}} \frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-\frac{1}{x}\right) \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-yz\right) \geq 0.$$

$\square$

[21]He received the special prize for this solution.

Epsilon 71. (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Solution. (by Sanghoon at the KMO Weekend Program 2007) We need the following lemma:

Lemma 8.2. *For all $p, q, \omega_1, \omega_2 > 0$, we have*

$$\frac{pq}{p+q} \leq \frac{{\omega_1}^2 p + {\omega_2}^2 q}{(\omega_1 + \omega_2)^2}.$$

Proof. After expanding, it becomes

$$(p+q)\left({\omega_1}^2 p + {\omega_2}^2 q\right) - (\omega_1 + \omega_2)^2 pq \geq 0.$$

However, it can be written as

$$(\omega_1 p - \omega_2 q)^2 \geq 0.$$

□

Now, taking $(p, q, \omega_1, \omega_2) = (a, x, x+y+z, a+b+c)$ in the lemma, we get

$$\frac{ax}{a+x} \leq \frac{(x+y+z)^2 a + (a+b+c)^2 x}{(x+y+z+a+b+c)^2}.$$

Similarly, we obtain

$$\frac{by}{b+y} \leq \frac{(x+y+z)^2 b + (a+b+c)^2 y}{(x+y+z+a+b+c)^2}$$

and

$$\frac{cz}{c+z} \leq \frac{(x+y+z)^2 c + (a+b+c)^2 z}{(x+y+z+a+b+c)^2}.$$

Adding the above three inequalities, we get

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(x+y+z)^2(a+b+c) + (a+b+c)^2(x+y+z)}{(x+y+z+a+b+c)^2}.$$

or

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z},$$

as desired.

□

Epsilon 72. (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

Proof. Establish the inequality

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq 3\left(\frac{a}{a+b+c}\right).$$

□

Epsilon 73. (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Proof. Use the auxiliary inequality

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}.$$

$\square$

Epsilon 74. (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Solution. Employ the following inequality

$$\frac{1}{1+t^2} \leq -\frac{27}{50}\left(t-2\right),$$

where $t \leq 1$. $\square$

Epsilon 75. (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Solution. Because of the homogeneity of the inequality, we may normalize to $a+b+c=1$. It takes the form

$$\frac{(1-2a)^2}{(1-a)^2+a^2}+\frac{(1-2b)^2}{(1-b)^2+b^2}+\frac{(1-2c)^2}{(1-c)^2+c^2}\geq\frac{3}{5}$$

or

$$\frac{1}{2a^2-2a+1}+\frac{1}{2b^2-2b+1}+\frac{1}{2c^2-2c+1}\leq\frac{27}{5}.$$

We find that the equation of the tangent line of $f(x)=\frac{1}{2x^2-2x+1}$ at $x=\frac{1}{3}$ is given by

$$y=\frac{54}{25}x+\frac{27}{25}$$

and that

$$f(x)-\left(\frac{54}{25}x+\frac{27}{25}\right)=-\frac{2(3x-1)^2(6x+1)}{25(2x^2-2x+1)}\leq 0.$$

for all $x>0$. It follows that

$$\sum_{\text{cyclic}}f(a)\leq\sum_{\text{cyclic}}\frac{54}{25}a+\frac{27}{25}=\frac{27}{5}.$$

□

Epsilon 76. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x+y+z = 1$. Prove that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.*

First Solution. Using the constraint $x+y+z = 1$, we reduce the inequality to homogeneous one:

$$0 \leq (xy + yz + zx)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \leq xyz + \sum_{\text{sym}} x^2y.$$

The right hand side inequality simplifies to

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2y \geq 0.$$

In the view of

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2y = \left(2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) + 5 \left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right),$$

it's enough to show that

$$2 \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y$$

and

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y.$$

The first inequality follows from

$$2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y = \sum_{\text{cyclic}} (x^3 + y^3) - \sum_{\text{cyclic}} (x^2y + xy^2) = \sum_{\text{cyclic}} (x^3 + y^3 - x^2y - xy^2) \geq 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyclic}} x(x - y)(x - z) \geq 0,$$

which is a particular case of Schur's Theorem. $\square$

Epsilon 77. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Second Solution. After the algebraic substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we are required to prove that

$$\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}},$$

where $a, b, c \in (0,1)$ and $a+b+c=2$. Using the constraint $a+b+c=2$, we obtain a homogeneous inequality

$$\sqrt{\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{\frac{a+b+c}{2}-a}{a}}+\sqrt{\frac{\frac{a+b+c}{2}-b}{b}}+\sqrt{\frac{\frac{a+b+c}{2}-c}{c}}$$

or

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}},$$

which immediately follows from The Cauchy-Schwarz Inequality

$$\sqrt{[(b+c-a)+(c+a-b)+(a+b-c)]\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}.$$

□

Epsilon 78. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

First Solution. By Schur's Inequality and The AM-GM Inequality, we have

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{cyclic}} x^2y + xy^2 \geq \sum_{\text{cyclic}} 2(xy)^{\frac{3}{2}}.$$

$\square$

Epsilon 79. *Let $t \in (0, 3]$. For all $a, b, c \geq 0$, we have*

$$(3-t) + t(abc)^{\frac{2}{t}} + \sum_{\text{cyclic}} a^2 \geq 2 \sum_{\text{cyclic}} ab.$$

Proof. After setting $x = a^{\frac{2}{3}}$, $y = b^{\frac{2}{3}}$, $z = c^{\frac{2}{3}}$, it becomes

$$3 - t + t(xyz)^{\frac{3}{t}} + \sum_{\text{cyclic}} x^3 \geq 2 \sum_{\text{cyclic}} (xy)^{\frac{3}{2}}.$$

By the previous epsilon, it will be enough to show that

$$3 - t + t(xyz)^{\frac{3}{t}} \geq 3xyz,$$

which is a straightforward consequence of the weighted AM-GM inequality :

$$\frac{3-t}{3} \cdot 1 + \frac{t}{3}(xyz)^{\frac{3}{t}} \geq 1^{\frac{3-t}{3}} \left((xyz)^{\frac{3}{t}}\right)^{\frac{t}{3}} = 3xyz.$$

One may check that the equality holds if and only if $a = b = c = 1$. □

Remark 8.3. *In particular, we obtain non-homogeneous inequalities*

$$\frac{5}{2} + \frac{1}{2}(abc)^4 + a^2 + b^2 + c^2 \geq 2(ab + bc + ca),$$
$$2 + (abc)^2 + a^2 + b^2 + c^2 \geq 2(ab + bc + ca),$$
$$1 + 2abc + a^2 + b^2 + c^2 \geq 2(ab + bc + ca).$$

Epsilon 80. (APMO 2004/5) *Prove that, for all positive real numbers a, b, c,*

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Second Solution. After expanding, it becomes

$$8+(abc)^2+2\sum_{\text{cyclic}} a^2b^2+4\sum_{\text{cyclic}} a^2 \geq 9\sum_{\text{cyclic}} ab.$$

From the inequality $(ab-1)^2+(bc-1)^2+(ca-1)^2 \geq 0$, we obtain

$$6+2\sum_{\text{cyclic}} a^2b^2 \geq 4\sum_{\text{cyclic}} ab.$$

Hence, it will be enough to show that

$$2+(abc)^2+4\sum_{\text{cyclic}} a^2 \geq 5\sum_{\text{cyclic}} ab.$$

Since $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$, it will be enough to show that

$$2+(abc)^2+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab,$$

which is a particular case of the previous epsilon. $\square$

Epsilon 81. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*
$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

Second Solution. It is equivalent to the following homogeneous inequality:
$$\left(a-(abc)^{1/3}+\frac{(abc)^{2/3}}{b}\right)\left(b-(abc)^{1/3}+\frac{(abc)^{2/3}}{c}\right)\left(c-(abc)^{1/3}+\frac{(abc)^{2/3}}{a}\right) \leq abc.$$
After the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it becomes
$$\left(x^3-xyz+\frac{(xyz)^2}{y^3}\right)\left(y^3-xyz+\frac{(xyz)^2}{z^3}\right)\left(z^3-xyz+\frac{(xyz)^2}{x^3}\right) \leq x^3y^3z^3,$$
which simplifies to
$$\left(x^2y-y^2z+z^2x\right)\left(y^2z-z^2x+x^2y\right)\left(z^2x-x^2y+y^2z\right) \leq x^3y^3z^3$$
or
$$3x^3y^3z^3+\sum_{\text{cyclic}} x^6y^3 \geq \sum_{\text{cyclic}} x^4y^4z+\sum_{\text{cyclic}} x^5y^2z^2$$
or
$$3(x^2y)(y^2z)(z^2x)+\sum_{\text{cyclic}} (x^2y)^3 \geq \sum_{\text{sym}} (x^2y)^2(y^2z)$$
which is a special case of Schur's Inequality. □

Epsilon 82. (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \leq 1.$$

Solution. We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}}+\frac{1}{b+c+(abc)^{1/3}}+\frac{1}{c+a+(abc)^{1/3}} \leq \frac{1}{(abc)^{1/3}}.$$

We make the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\frac{1}{x^3+y^3+xyz}+\frac{1}{y^3+z^3+xyz}+\frac{1}{z^3+x^3+xyz} \leq \frac{1}{xyz}$$

which is equivalent to

$$xyz\sum_{\text{cyclic}}(x^3+y^3+xyz)(y^3+z^3+xyz) \leq (x^3+y^3+xyz)(y^3+z^3+xyz)(z^3+x^3+xyz)$$

or

$$\sum_{\text{sym}} x^6y^3 \geq \sum_{\text{sym}} x^5y^2z^2 \quad !$$

We now obtain

$$\begin{aligned}
\sum_{\text{sym}} x^6y^3 &= \sum_{\text{cyclic}} x^6y^3 + y^6x^3 \\
&\geq \sum_{\text{cyclic}} x^5y^4 + y^5x^4 \\
&= \sum_{\text{cyclic}} x^5(y^4 + z^4) \\
&\geq \sum_{\text{cyclic}} x^5(y^2z^2 + y^2z^2) \\
&= \sum_{\text{sym}} x^5y^2z^2.
\end{aligned}$$

□

Epsilon 83. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that $a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Let x, y, z be positive real numbers. Then, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Solution. We distinguish two cases.

Case 1. $b_1 \geq a_2$: It follows from $a_1 \geq a_1+a_2-b_1$ and from $a_1 \geq b_1$ that $a_1 \geq max(a_1+a_2-b_1, b_1)$ so that $max(a_1, a_2) = a_1 \geq max(a_1+a_2-b_1, b_1)$. From $a_1+a_2-b_1 \geq b_1+a_3-b_1 = a_3$ and $a_1 + a_2 - b_1 \geq b_2 \geq b_3$, we have $max(a_1 + a_2 - b_1, a_3) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} z^{a_3}(x^{a_1} y^{a_2} + x^{a_2} y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3}(x^{a_1+a_2-b_1} y^{b_1} + x^{b_1} y^{a_1+a_2-b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1}(y^{a_1+a_2-b_1} z^{a_3} + y^{a_3} z^{a_1+a_2-b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1}(y^{b_2} z^{b_3} + y^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

Case 2. $b_1 \leq a_2$: It follows from $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$ that $b_1 \geq a_2 + a_3 - b_1$ and that $a_1 \geq a_2 \geq b_1 \geq a_2 + a_3 - b_1$. Therefore, we have $max(a_2, a_3) \geq max(b_1, a_2 + a_3 - b_1)$ and $max(a_1, a_2 + a_3 - b_1) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} x^{a_1}(y^{a_2} z^{a_3} + y^{a_3} z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1}(y^{b_1} z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1} z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1}(x^{a_1} z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1} z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1}(x^{b_2} z^{b_3} + x^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

□

Epsilon 84. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Third Solution. It's equivalent to

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2(abc)^{4/3}}.$$

Set $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\sum_{\text{cyclic}} \frac{1}{x^9(y^3+z^3)} \geq \frac{3}{2x^4y^4z^4}.$$

Clearing denominators, this can be rewritten as

$$\sum_{\text{sym}} x^{12}y^{12} + 2\sum_{\text{sym}} x^{12}y^9z^3 + \sum_{\text{sym}} x^9y^9z^6 \geq 3\sum_{\text{sym}} x^{11}y^8z^5 + 6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}} x^{12}y^{12} - \sum_{\text{sym}} x^{11}y^8z^5\right) + 2\left(\sum_{\text{sym}} x^{12}y^9z^3 - \sum_{\text{sym}} x^{11}y^8z^5\right) + \left(\sum_{\text{sym}} x^9y^9z^6 - \sum_{\text{sym}} x^8y^8z^8\right) \geq 0,$$

By Muirhead's Theorem, every term on the left hand side is nonnegative. $\square$

Epsilon 85. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}.$$

Second Solution. It's equivalent to

$$4\sum_{\text{sym}}x^5y+2\sum_{\text{cyclic}}x^4yz+6x^2y^2z^2-\sum_{\text{sym}}x^4y^2-6\sum_{\text{cyclic}}x^3y^3-2\sum_{\text{sym}}x^3y^2z\geq 0.$$

We rewrite this as following

$$\left(\sum_{\text{sym}}x^5y-\sum_{\text{sym}}x^4y^2\right)+3\left(\sum_{\text{sym}}x^5y-\sum_{\text{sym}}x^3y^3\right)+2xyz\left(3xyz+\sum_{\text{cyclic}}x^3-\sum_{\text{sym}}x^2y\right)\geq 0.$$

By Muirhead's Theorem and Schur's Inequality, it's a sum of three nonnegative terms. $\square$

Epsilon 86. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Solution. Using $xy + yz + zx = 1$, we homogenize the given inequality as following :

$$(xy + yz + zx)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)^2 \geq \left(\frac{5}{2}\right)^2$$

or

$$4\sum_{\text{sym}} x^5y + \sum_{\text{sym}} x^4yz + 14\sum_{\text{sym}} x^3y^2z + 38x^2y^2z^2 \geq \sum_{\text{sym}} x^4y^2 + 3\sum_{\text{sym}} x^3y^3$$

or

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + xyz\left(\sum_{\text{sym}} x^3 + 14\sum_{\text{sym}} x^2y + 38xyz\right) \geq 0.$$

By Muirhead's Theorem, we get the result. In the above inequality, without the condition $xy + yz + zx = 1$, the equality holds if and only if $x = y, z = 0$ *or* $y = z, x = 0$ *or* $z = x, y = 0$. Since $xy + yz + zx = 1$, the equality occurs when $(x, y, z) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$. ☐

Epsilon 87. [SC] If m_a,m_b,m_c are medians and r_a,r_b,r_c the exradii of a triangle, prove that

$$\frac{r_ar_b}{m_am_b}+\frac{r_br_c}{m_bm_c}+\frac{r_cr_a}{m_cm_a}\geq 3.$$

Solution. Set $2s=a+b+c$. Using the well-known identities

$$r_a=\sqrt{\frac{s(s-b)(s-c)}{s-a}},\ m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2},\ etc.$$

we obtain

$$\sum_{\text{cyclic}}\frac{r_br_c}{m_bm_c}=\sum_{\text{cyclic}}\frac{4s(s-a)}{\sqrt{(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}}.$$

Applying the AM-GM inequality, we obtain

$$\sum_{\text{cyclic}}\frac{r_br_c}{m_bm_c}\geq\sum_{\text{cyclic}}\frac{8s(s-a)}{(2c^2+2a^2-b^2)+(2a^2+2b^2-c^2)}=\sum_{\text{cyclic}}\frac{2(a+b+c)(b+c-a)}{4a^2+b^2+c^2}.$$

Thus, it will be enough to show that

$$\sum_{\text{cyclic}}\frac{2(a+b+c)(b+c-a)}{4a^2+b^2+c^2}\geq 3.$$

After expanding the above inequality, we see that it becomes

$$2\sum_{\text{cyclic}}a^6+4\sum_{\text{cyclic}}a^4bc+20\sum_{\text{sym}}a^3b^2c+68\sum_{\text{cyclic}}a^3b^3+16\sum_{\text{cyclic}}a^5b\geq 276a^2b^2c^2+27\sum_{\text{cyclic}}a^4b^2.$$

We note that this cannot be proven by just applying Muirhead's Theorem. Since a, b, c are the sides of a triangle, we can make The Ravi Substitution $a=y+z$, $b=z+x$, $c=x+y$, where $x,y,z>0$. After some brute-force algebra, we can rewrite the above inequality as

$$25\sum_{\text{sym}}x^6+230\sum_{\text{sym}}x^5y+115\sum_{\text{sym}}x^4y^2+10\sum_{\text{sym}}x^3y^3+80\sum_{\text{sym}}x^4yz$$
$$\geq 336\sum_{\text{sym}}x^3y^2z+124\sum_{\text{sym}}x^2y^2z^2.$$

Now, by Muirhead's Theorem, we get the result ! ☐

Epsilon 88. *Let $\mathcal{P}(u,v,w) \in \mathbb{R}[u,v,w]$ be a homogeneous symmetric polynomial with degree 3. Then the following two statements are equivalent.*

(a) $\mathcal{P}(1,1,1),\ \mathcal{P}(1,1,0),\ \mathcal{P}(1,0,0) \geq 0$.
(b) $\mathcal{P}(x,y,z) \geq 0$ *for all* $x,y,z \geq 0$.

Proof. [SR1] We only prove that (a) implies (b). Let

$$P(u,v,w) = A\sum_{\text{cyclic}} u^3 + B\sum_{\text{sym}} u^2v + Cuvw.$$

Letting $p = P(1,1,1) = 3A+6B+C$, $q = P(1,1,0) = A+B$, and $r = P(1,0,0) = A$, we have $A = r$, $B = q-r$, $C = p-6q+3r$, and $p,q,r \geq 0$. For $x,y,z \geq 0$, we have

$$\begin{aligned} P(x,y,z) &= r\sum_{\text{cyclic}} x^3 + (q-r)\sum_{\text{sym}} x^2y + (p-6q+3r)xyz \\ &= r\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + q\left(\sum_{\text{sym}} x^2y - \sum_{\text{sym}} xyz\right) + pxyz \\ &\geq 0. \end{aligned}$$

□

Remark 8.4. *Here is an alternative way to prove the inequality* $P(x,y,z) \geq 0$.

Case 1. $q \geq r$ *: We compute*

$$P(x,y,z) = \frac{r}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (q-r)\left(\sum_{\text{sym}} x^2y - \sum_{\text{sym}} xyz\right) + pxyz.$$

Every term on the right hand side is nonnegative.

Case 2. $q \leq r$ *: We compute*

$$P(x,y,z) = \frac{q}{2}\left(\sum_{\text{sym}} x^3 - \sum_{\text{sym}} xyz\right) + (r-q)\left(\sum_{\text{cyclic}} x^3 + 3xyz - \sum_{\text{sym}} x^2y\right) + pxyz.$$

Every term on the right hand side is nonnegative.

Epsilon 89. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Third Solution. We offer a convexity proof. We make the substitution

$$x=\frac{a}{a+b+c},\ y=\frac{b}{a+b+c},\ z=\frac{c}{a+b+c}.$$

The inequality becomes

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq 1,$$

where $f(t)=\frac{1}{\sqrt{t}}$.[22] Since f is convex on $\mathbb{R}^+$ and $x+y+z=1$, we apply (the weighted) Jensen's Inequality to obtain

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq f(x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy)).$$

Note that $f(1)=1$. Since the function f is strictly decreasing, it suffices to show that

$$1\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

Using $x+y+z=1$, we homogenize it as

$$(x+y+z)^3\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

However, it is easily seen from

$$(x+y+z)^3-x(x^2+8yz)-y(y^2+8zx)-z(z^2+8xy)=3[x(y-z)^2+y(z-x)^2+z(x-y)^2]\geq 0.$$

□

Fourth Solution. We begin with the substitution

$$x=\frac{bc}{a^2},y=\frac{ca}{b^2},z=\frac{ab}{c^2}.$$

Then, we get $xyz=1$ and the inequality becomes

$$\frac{1}{\sqrt{1+8x}}+\frac{1}{\sqrt{1+8y}}+\frac{1}{\sqrt{1+8z}}\geq 1$$

which is equivalent to

$$\sum_{\text{cyclic}}\sqrt{(1+8x)(1+8y)}\geq\sqrt{(1+8x)(1+8y)(1+8z)}.$$

After squaring both sides, it's equivalent to

$$8(x+y+z)+2\sqrt{(1+8x)(1+8y)(1+8z)}\sum_{\text{cyclic}}\sqrt{1+8x}\geq 510.$$

Recall that $xyz=1$. The AM-GM Inequality gives us $x+y+z\geq 3$,

$$(1+8x)(1+8y)(1+8z)\geq 9x^{\frac{8}{9}}\cdot 9y^{\frac{8}{9}}\cdot 9z^{\frac{8}{9}}=729\ \text{ and }\ \sum_{\text{cyclic}}\sqrt{1+8x}\geq\sum_{\text{cyclic}}\sqrt{9x^{\frac{8}{9}}}\geq 9(xyz)^{\frac{4}{27}}=9.$$

Using these three inequalities, we get the result. □

[22] Dividing by $a+b+c$ gives the equivalent inequality $\sum_{\text{cyclic}}\frac{\frac{a}{a+b+c}}{\sqrt{\frac{a^2}{(a+b+c)^2}+\frac{8bc}{(a+b+c)^2}}}\geq 1$.

Epsilon 90. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Second Solution. We present a convexity proof. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

Since it's homogeneous, we can restrict our attention to the case $x+y+z=1$. Then, it becomes

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq 1,$$

where $f(t)=t^2$. Since f is convex on $\mathbb{R}$, we apply (the weighted) Jensen's Inequality to obtain

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq f\left(y\cdot\frac{x}{y}+z\cdot\frac{y}{z}+x\cdot\frac{z}{x}\right)=f(1)=1.$$

□

Epsilon 91. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)\,(ab^2+bc^2+ca^2)} \geq abc + \sqrt[3]{(a^3+abc)\,(b^3+abc)\,(c^3+abc)}$$

First Solution. Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)} \geq 1 + \sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, we obtain the constraint $xyz = 1$. It takes the form

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

From the constraint $xyz = 1$, we obtain the identity

$$\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right) = \left(\frac{x+z}{z}\right)\left(\frac{y+x}{x}\right)\left(\frac{z+y}{y}\right) = (z+x)(x+y)(y+z).$$

Hence, we are required to prove that

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \sqrt[3]{(x+y)(y+z)(z+x)}.$$

Now, we offer two ways to finish the proof.

First Method. Observe that

$$(x+y+z)\,(xy+yz+zx) = (x+y)(y+z)(z+x) + xyz = (x+y)(y+z)(z+x) + 1.$$

Letting $p = \sqrt[3]{(x+y)(y+z)(z+x)}$, the inequality we want to prove now becomes

$$\sqrt{p^3+1} \geq 1 + p.$$

Applying The AM-GM Inequality yields

$$p \geq \sqrt[3]{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}} = 2.$$

It follows that

$$(p^3+1) - (1+p)^2 = p(p+1)(p-2) \geq 0,$$

as desired.

Second Method. More strongly, we establish that, for all $x, y, z > 0$,

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \frac{1}{3}\left(\frac{y+z}{\sqrt{yz}} + \frac{z+x}{\sqrt{zx}} + \frac{x+y}{\sqrt{xy}}\right).$$

However, an application of The Cauchy-Schwarz Inequality yields

$$[x+(y+z)]\,[yz+x(y+z)] \geq \left(\sqrt{xyz}+\sqrt{x(y+z)^2}\right)^2 = \left(1+\frac{y+z}{\sqrt{yz}}\right)^2$$

or

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \frac{y+z}{\sqrt{yz}}.$$

Similarly, we also have

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \frac{z+x}{\sqrt{zx}}.$$

and

$$\sqrt{(x+y+z)\,(xy+yz+zx)} \geq 1 + \frac{x+y}{\sqrt{xy}}.$$

Adding these three, we get the desired inequality. $\square$

Epsilon 92. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$.*

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

First Solution. (Marcin E. Kuczma[23]) For $x_1 = \cdots = x_n = 0$, it holds for any $C \geq 0$. Hence, we consider the case when $x_1 + \cdots + x_n > 0$. Since the inequality is homogeneous, we may normalize to $x_1 + \cdots + x_n = 1$. From the assumption $x_1 + \cdots + x_n = 1$, we have

$$\begin{aligned}
\mathcal{F}(x_1, \cdots, x_n) &= \sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \\
&= \sum_{1\leq i<j\leq n} {x_i}^3 x_j + \sum_{1\leq i<j\leq n} x_i {x_j}^3 \\
&= \sum_{1\leq i\leq n} {x_i}^3 \sum_{j\neq i} x_i \\
&= \sum_{1\leq i\leq n} {x_i}^3 (1 - x_i) \\
&= \sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3).
\end{aligned}$$

We claim that $C = \frac{1}{8}$. It suffices to show that $\mathcal{F}(x_1, \cdots, x_n) \leq \frac{1}{8} = \mathcal{F}\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right)$.

Lemma 8.3. $0 \leq x \leq y \leq \frac{1}{2}$ *implies* $x^2 - x^3 \leq y^2 - y^3$.

Proof. Since $x + y \leq 1$, we get $x + y \geq (x+y)^2 \geq x^2 + xy + y^2$. Since $y - x \geq 0$, this implies that $y^2 - x^2 \geq y^3 - x^3$ or $y^2 - y^3 \geq x^2 - x^3$, as desired. □

Case 1. $\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n$*:*

$$\sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3) \leq \sum_{i=1}^{n} x_i \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3\right) = \frac{1}{8}\sum_{i=1}^{n} x_i = \frac{1}{8}.$$

Case 2. $x_1 \geq \frac{1}{2} \geq x_2 \geq \cdots \geq x_n$*: Let $x_1 = x$ and $y = 1 - x = x_2 + \cdots + x_n$. Since $y \geq x_2, \cdots, x_n$,*

$$\mathcal{F}(x_1, \cdots, x_n) = x^3 y + \sum_{i=2}^{n} x_i ({x_i}^2 - {x_i}^3) \leq x^3 y + \sum_{i=2}^{n} x_i (y^2 - y^3) = x^3 y + y(y^2 - y^3).$$

Since $x^3 y + y(y^2 - y^3) = x^3 y + y^3(1 - y) = xy(x^2 + y^2)$, it remains to show that

$$xy(x^2 + y^2) \leq \frac{1}{8}.$$

Using $x + y = 1$, we homogenize the above inequality as following.

$$xy(x^2 + y^2) \leq \frac{1}{8}(x+y)^4.$$

However, we immediately find that $(x+y)^4 - 8xy(x^2 + y^2) = (x-y)^4 \geq 0$.

□

[23] I slightly modified his solution in [AS].

Epsilon 93. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

Second Solution. By The Cauchy-Schwarz Inequality, we have

$$\left(\sum_{i=1}^{n} a_i + b_i\right)\left(\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i}\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2$$

or

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} \geq \frac{\left(\sum_{i=1}^{n}\right)^2}{\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i} = \frac{1}{2}\sum_{i=1}^{n} a_i$$

□

Epsilon 94. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Second Solution. The Cauchy-Schwarz Inequality shows that

$$\begin{aligned}
\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} &\leq \sqrt{3\left(a^2 + b + a + b^2 + 1 + ab\right)} \\
&= \sqrt{3\left(a^2 + ab + b^2 + a + b + 1\right)} \\
&\leq \sqrt{3\left((a + b)^2 + a + b + 1\right)} \\
&= 3.
\end{aligned}$$

$\square$

Epsilon 95. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Third Solution. We first note that

$$\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 1.$$

Apply The Cauchy-Schwarz Inequality to deduce

$$\sqrt{x+y+z} = \sqrt{(x+y+z)\left(\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z}\right)} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

□

Epsilon 96. (Gazeta Matematicã) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Solution. We obtain the chain of equalities and inequalities

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyclic}} \sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)} \\
&\geq \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) \qquad \text{(Cauchy} - \text{Schwarz)} \\
&= \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right) \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} \qquad \text{(AM} - \text{GM)} \\
&\geq \sqrt{2} \sum_{\text{cyclic}} \sqrt{a^4+\frac{a^2bc}{2}} \qquad \text{(Cauchy} - \text{Schwarz)} \\
&= \sum_{\text{cyclic}} \sqrt{2a^4+a^2bc}\,.
\end{aligned}$$

□

Epsilon 97. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}$$

Second Solution. (based on work by an winter program participant) We obtain

$$\begin{aligned}
& \sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \\
= \ & \frac{1}{2}\sqrt{[b(a^2 + bc) + c(b^2 + ca) + a(c^2 + ab)]\,[c(a^2 + bc) + a(b^2 + ca) + b(c^2 + ab)]} \\
\geq \ & \frac{1}{2}\left(\sqrt{bc}(a^2 + bc) + \sqrt{ca}(b^2 + ca) + \sqrt{ab}(c^2 + ab)\right) && \text{(Cauchy} - \text{Schwarz)} \\
\geq \ & \frac{3}{2}\sqrt[3]{\sqrt{bc}(a^2 + bc) \cdot \sqrt{ca}(b^2 + ca) \cdot \sqrt{ab}(c^2 + ab)} && \text{(AM} - \text{GM)} \\
= \ & \frac{1}{2}\sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)} + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)} \\
\geq \ & \frac{1}{2}\sqrt[3]{2\sqrt{a^3 \cdot abc} \cdot 2\sqrt{b^3 \cdot abc} \cdot 2\sqrt{c^3 \cdot abc}} + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)} && \text{(AM} - \text{GM)} \\
= \ & abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}.
\end{aligned}$$

□

Epsilon 98. (Andrei Ciupan, Romanian Junior Balkan MO 2007 Team Selection Tests) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c\geq ab+bc+ca$.

First Solution. By applying The Cauchy-Schwarz Inequality, we obtain

$$(a+b+1)(a+b+c^2)\geq (a+b+c)^2$$

or

$$\frac{1}{a+b+1}\leq \frac{c^2+a+b}{(a+b+c)^2}.$$

Now by summing cyclically, we obtain

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2}$$

But from the condition, we can see that

$$a^2+b^2+c^2+2(a+b+c)\geq (a+b+c)^2,$$

and therefore

$$a+b+c\geq ab+bc+ca.$$

We see that the equality occurs if and only if $a=b=c=1$. $\square$

Second Solution. (Cezar Lupu) We first observe that

$$2\geq \sum_{\text{cyclic}}\left(1-\frac{1}{a+b+1}\right)=\sum_{\text{cyclic}}\frac{a+b}{a+b+1}=\sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}.$$

Apply The Cauchy-Schwarz Inequality to get

$$\begin{aligned}
2 &\geq \sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}\\
&\geq \frac{\left(\sum_{\text{cyclic}}a+b\right)^2}{\sum_{\text{cyclic}}(a+b)^2+a+b}\\
&= \frac{4\sum_{\text{cyclic}}a^2+8\sum_{\text{cyclic}}ab}{2\sum_{\text{cyclic}}a^2+2\sum_{\text{cyclic}}ab+2\sum_{\text{cyclic}}a}
\end{aligned}$$

or

$$a+b+c\geq ab+bc+ca.$$

$\square$

Epsilon 99. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Proof. Because of the homogeneity of the inequality, we may rescale $x_{1j}, \cdots, x_{mj}$ so that $x_{1j} + \cdots + x_{mj} = 1$ for each $j \in \{1, \cdots, n\}$. Then, we need to show that

$$\prod_{j=1}^{n} 1^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

The Weighted AM-GM Inequality provides that

$$\sum_{j=1}^{n} \omega_j x_{ij} \geq \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad (i \in \{1, \cdots, m\}) \implies \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

However, we immediately have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j x_{ij} = \sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} \omega_j = 1.$$

□

Epsilon 100. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a* **continuous** *function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.
(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.
(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a,b]$.
(4) For all $k \in \{0,1,2,\cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a,b]$.
(5) We have $\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$.
(6) We have $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

Solution. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is obvious.

(2) $\Rightarrow$ (1) **:** Let $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. One may see that there exist positive rational sequences $\{r_k(1)\}_{k\in\mathbb{N}}, \cdots, \{r_k(n)\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} r_k(j) = w_j \ \ (1 \leq j \leq n) \ \text{ and } \ r_k(1) + \cdots + r_k(n) = 1 \ \text{ for all } \ k \in \mathbb{N}.$$

By the hypothesis in (2), we obtain $r_k(1)f(x_1) + \cdots + r_k(n)f(x_n) \geq f(r_k(1)\, x_1 + \cdots + r_k(n)\, x_n)$. Since f is continuous, taking $k \to \infty$ to both sides yields the inequality

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

(3) $\Rightarrow$ (2) **:** Let $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$. We can find a positive integer $N \in \mathbb{N}$ so that $Nr_1, \cdots, Nr_n \in \mathbb{N}$. For each $i \in \{1, \cdots, n\}$, we can write $r_i = \frac{p_i}{N}$, where $p_i \in \mathbb{N}$. It follows from $r_1 + \cdots + r_n = 1$ that $N = p_1 + \cdots + p_n$. Then, (3) implies that

$$\begin{aligned}
& r_1 f(x_1) + \cdots + r_n f(x_n) \\
= \ & \frac{\overbrace{f(x_1) + \cdots + f(x_1)}^{p_1 \text{ terms}} + \cdots + \overbrace{f(x_n) + \cdots + f(x_n)}^{p_n \text{ terms}}}{N} \\
\geq \ & f\left(\frac{\overbrace{x_1 + \cdots + x_1}^{p_1 \text{ terms}} + \cdots + \overbrace{x_n + \cdots + x_n}^{p_n \text{ terms}}}{N}\right) \\
= \ & f(r_1\, x_1 + \cdots + r_n\, x_n).
\end{aligned}$$

(4) $\Rightarrow$ (3) **:** Let $y_1, \cdots, y_N \in [a, b]$. Take a large $k \in \mathbb{N}$ so that $2^k > N$. Let $a = \frac{y_1+\cdots+y_N}{N}$. Then, (4) implies that

$$\begin{aligned}
& \frac{f(y_1) + \cdots + f(y_N) + (2^k - n)f(a)}{2^k} \\
= \quad & \frac{f(y_1) + \cdots + f(y_N) + \overbrace{f(a) + \cdots + f(a)}^{(2^k - N) \text{ terms}}}{2^k} \\
\geq \quad & f\left(\frac{y_1 + \cdots + y_N + \overbrace{a + \cdots + a}^{(2^k - N) \text{ terms}}}{2^k}\right) \\
= \quad & f(a)
\end{aligned}$$

so that

$$f(y_1) + \cdots + f(y_N) \geq Nf(a) = Nf\left(\frac{y_1 + \cdots + y_N}{N}\right).$$

(5) $\Rightarrow$ (4) **:** We use induction on k. In case $k = 0, 1, 2$, it clearly holds. Suppose that (4) holds for some $k \geq 2$. Let $y_1, \cdots, y_{2^{k+1}} \in [a, b]$. By the induction hypothesis, we obtain

$$\begin{aligned}
& f(y_1) + \cdots + f(y_{2^k}) + f(y_{2^k+1}) + \cdots + f(y_{2^{k+1}}) \\
\geq \quad & 2^k f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right) + 2^k f\left(\frac{y_{2^k+1} + \cdots + y_{2^{k+1}}}{2^k}\right) \\
= \quad & 2^{k+1} \frac{f\left(\frac{y_1+\cdots+y_{2^k}}{2^k}\right) + f\left(\frac{y_{2^k+1}+\cdots+y_{2^{k+1}}}{2^k}\right)}{2} \\
\geq \quad & 2^{k+1} f\left(\frac{\frac{y_1+\cdots+y_{2^k}}{2^k} + \frac{y_{2^k+1}+\cdots+y_{2^{k+1}}}{2^k}}{2}\right) \\
= \quad & 2^{k+1} f\left(\frac{y_1 + \cdots + y_{2^{k+1}}}{2^{k+1}}\right).
\end{aligned}$$

Hence, (4) holds for $k + 1$. This completes the induction.

So far, we've established that (1), (2), (3), (4), (5) are all equivalent. Since (1) $\Rightarrow$ (6) $\Rightarrow$ (5) is obvious, this completes the proof. $\square$

Epsilon 101. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Second Solution. After employing the substitution

$$x = e^{\frac{p}{3}},\ y = e^{\frac{q}{3}},\ z = e^{\frac{r}{3}},$$

the inequality becomes

$$3e^{\frac{p+q+r}{3}} + e^p + e^q + e^r \geq 2\left(e^{\frac{q+r}{2}} + e^{\frac{r+p}{2}} + e^{\frac{p+q}{2}}\right)$$

It is a straightforward consequence of Popoviciu's Inequality. $\square$

Epsilon 102. *Let ABC be an acute triangle. Show that*

$$\cos A + \cos B + \cos C \geq 1.$$

Proof. Observe that $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ majorize (A, B, C). Since $-cosx$ is convex on $\left(0, \frac{\pi}{2}\right)$, The Hardy-Littlewood-Pólya Inequality implies that

$$\cos A + \cos B + \cos C \geq \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + \cos 0 = 1.$$

$\square$

Epsilon 103. *Let ABC be a triangle. Show that*

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\le 1.$$

Proof. Observe that $(\pi, 0, 0)$ majorizes (A, B, C). The convexity of $\tan^2\left(\frac{x}{4}\right)$ on $[0, \pi]$ yields the estimation:

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\le \tan^2\left(\frac{\pi}{4}\right)+\tan^2 0+\tan^2 0=1.$$

$\square$

Epsilon 104. *Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.*

Proof. [NP, p.33] Since the inequality is symmetric, we may assume that $x \geq y \geq z$. We consider the two cases. In the case when $x \geq \frac{x+y+z}{3} \geq y \geq z$, the majorization

$$\left(x, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, y, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. In the case when $x \geq y \geq \frac{x+y+z}{3} \geq z$, the majorization

$$\left(x, y, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. $\square$

Epsilon 105. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$.*

Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

Second Solution. (Kin Y. Li[24]) According to the homogenity of the inequality, we may normalize to $x_1 + \cdots + x_n = 1$. Our job is to maximize

$$\begin{aligned}
\mathcal{F}(x_1, \cdots, x_n) &= \sum_{1\leq i<j\leq n} x_i x_j \left(x_i^2 + x_j^2\right) \\
&= \sum_{1\leq i<j\leq n} {x_i}^3 x_j + \sum_{1\leq i<j\leq n} x_i {x_j}^3 \\
&= \sum_{1\leq i\leq n} {x_i}^3 \sum_{j\neq i} x_i \\
&= \sum_{1\leq i\leq n} {x_i}^3 (1 - x_i) \\
&= \sum_{i=1}^{n} f(x_i),
\end{aligned}$$

where $f(t) = t^3 - t^4$ is a convex function on $\left[0, \frac{1}{2}\right]$. Since the inequality is symmetric, we can restrict our attention to the case $x_1 \geq x_2 \geq \cdots \geq x_n$. If $\frac{1}{2} \geq x_1$, then we see that $\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots 0\right)$ majorizes $(x_1, \cdots, x_n)$. Since $x_1, \cdots, x_2, \cdots, x_n \in \left[0, \frac{1}{2}\right]$ and since f is convex on $\left[0, \frac{1}{2}\right]$, by The Hardy-Littlewood-Pólya Inequality, the convexity of f on $[0, \frac{1}{2}]$ implies that

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(0) + \cdots + f(0) = \frac{1}{8}.$$

We now consider the case when $x_1 \geq \frac{1}{2}$. We find that $(1 - x_1, 0, \cdots 0)$ majorizes $(x_2, \cdots, x_n)$. Since $1 - x_1, x_2, \cdots, x_n \in \left[0, \frac{1}{2}\right]$ and since f is convex on $\left[0, \frac{1}{2}\right]$, by The Hardy-Littlewood-Pólya Inequality,

$$\sum_{i=2}^{n} f(x_i) \leq f\left(1 - x_1\right) + f(0) + \cdots + f(0) = f\left(1 - x_1\right).$$

Setting $x_1 = \frac{1}{2} + \epsilon$ for some $\epsilon \in \left[0, \frac{1}{2}\right]$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i) &\leq f(x_1) + f\left(1 - x_1\right) \\
&= x_1(1 - x_1)[{x_1}^2 + (1 - x_1)^2] \\
&= \left(\frac{1}{4} - \epsilon^2\right)\left(\frac{1}{2} + 2\epsilon^2\right) \\
&= 2\left(\frac{1}{16} - \epsilon^4\right) \\
&\leq \frac{1}{8}.
\end{aligned}$$

□

[24] I slightly modified his solution in [KL].

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9.2. **IMO Code.**

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AFG	Afghanistan	ALB	Albania	ALG	Algeria
ARG	Argentina	ARM	Armenia	AUS	Australia
AUT	Austria	AZE	Azerbaijan	BAH	Bahrain
BGD	Bangladesh	BLR	Belarus	BEL	Belgium
BEN	Benin	BOL	Bolivia	BIH	BIH
BRA	Brazil	BRU	Brunei	BGR	Bulgaria
KHM	Cambodia	CMR	Cameroon	CAN	Canada
CHI	Chile	CHN	CHN	COL	Colombia
CIS	CIS	CRI	Costa Rica	HRV	Croatia
CUB	Cuba	CYP	Cyprus	CZE	Czech Republic
CZS	Czechoslovakia	DEN	Denmark	DOM	Dominican Republic
ECU	Ecuador	EST	Estonia	FIN	Finland
FRA	France	GEO	Georgia	GDR	GDR
GER	Germany	HEL	Greece	GTM	Guatemala
HND	Honduras	HKG	Hong Kong	HUN	Hungary
ISL	Iceland	IND	India	IDN	Indonesia
IRN	Islamic Republic of Iran	IRL	Ireland	ISR	Israel
ITA	Italy	JPN	Japan	KAZ	Kazakhstan
PRK	PRK	KOR	Republic of Korea	KWT	Kuwait
KGZ	Kyrgyzstan	LVA	Latvia	LIE	Liechtenstein
LTU	Lithuania	LUX	Luxembourg	MAC	Macau
MKD	MKD	MAS	Malaysia	MLT	Malta
MRT	Mauritania	MEX	Mexico	MDA	Republic of Moldova
MNG	Mongolia	MNE	Montenegro	MAR	Morocco
MOZ	Mozambique	NLD	Netherlands	NZL	New Zealand
NIC	Nicaragua	NGA	Nigeria	NOR	Norway
PAK	Pakistan	PAN	Panama	PAR	Paraguay
PER	Peru	PHI	Philippines	POL	Poland
POR	Portugal	PRI	Puerto Rico	ROU	Romania
RUS	Russian Federation	SLV	El Salvador	SAU	Saudi Arabia
SEN	Senegal	SRB	Serbia	SCG	Serbia and Montenegro
SGP	Singapore	SVK	Slovakia	SVN	Slovenia
SAF	South Africa	ESP	Spain	LKA	Sri Lanka
SWE	Sweden	SUI	Switzerland	SYR	Syria
TWN	Taiwan	TJK	Tajikistan	THA	Thailand
TTO	Trinidad and Tobago	TUN	Tunisia	TUR	Turkey
NCY	NCY	TKM	Turkmenistan	UKR	Ukraine
UAE	United Arab Emirates	UNK	United Kingdom	USA	United States of America
URY	Uruguay	USS	USS	UZB	Uzbekistan
VEN	Venezuela	VNM	Vietnam	YUG	Yugoslavia

BIH	Bosnia and Herzegovina
CHN	People's Republic of China
CIS	Commonwealth of Independent States
FRG	Federal Republic of Germany
GDR	German Democratic Republic
MKD	The Former Yugoslav Republic of Macedonia
NCY	Turkish Republic of Northern Cyprus
PRK	Democratic People's Republic of Korea
USS	Union of the Soviet Socialist Republics